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ON THE POSTAGE STAMP PROBLEM WITH THREE STAMP DENOMINATIONS, III

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The present paper is an immediate continuation of Selmer [7] and Selmer - Rödne [8]. All references to theorems and formulas from sections 1-13 are automatically to [7] or [8].

14. The sets of h_0- and $(h_0 - 1)$-representable numbers.

Let $A'_k = A_k \cup \{0\}$. The set (1.2) of h-representable numbers (at most h addends) may then in standard terminology be denoted by hA'_k. Our aim in the present section is to determine the sets $h_0A'_3$ and $(h_0 - 1)A'_3$.

We shall rely heavily on the results in Rödseth [6], and use his notation, with one exception: He operates with an integer r, $0 \leqq r < a_3$. To avoid confusion with our use of r, we shall replace his r by ℓ.

Rödseth's Lemma 4 states that

$$t^*_{-\ell} = x_v(a_3 - 1) + y_v(a_3 - a_2),\ (x_v, y_v) \in X_v \cup Y_v .$$

We consider the numbers (all $\equiv \ell \pmod{a_3}$):

$$(14.1)\quad (h_0 - t)a_3 - t^*_{-\ell} = (h_0 - t - x_v - y_v)a_3 + y_va_2 + x_v \geqq 0 ,$$

and claim that these belong to $h_0A'_3$ for $t \geqq 0$. This is trivial if $h_0 - t - x_v - y_v \geqq 0$, since the coefficient sum $\sum = h_0 - t \leqq h_0$. It remains to show that the set

$$S_\ell = \{\ell,\ \ell + a_3,\ \ell + 2a_3,\ \ldots,\ y_v a_2 + x_v - a_3\} \subset h_0 A_3' .$$

And this is proved by Rödseth, since S_ℓ is just the sequence (4.1) of [6].

On the other hand, the numbers (14.1) do <u>not</u> belong to $h_0 A_3'$ if $t = -t' < 0$. Assume to the contrary that

$$(h_0 + t')a_3 - t^*_{-\ell} = x_3 a_3 + x_2 a_2 + x_1,\ \sum x_i = h' \leqq h_0 .$$

As in section 3, we conclude that

$$t^*_{-\ell} - t'a_3 = (h_0 - h')a_3 + x_1(a_3 - 1) + x_2(a_3 - a_2)$$

has a representation by $\bar{A}_3 = \{a_3 - a_2,\ a_3 - 1,\ a_3\}$, cf. (2.15). (Rödseth uses $A_3^* = \bar{A}_3 \cup \{0\}$.) But this is a contradiction, since $t^*_{-\ell}$ is defined as the smallest integer in its residue class (mod a_3) with a representation by $\bar{A}_3$.

Letting (x_v, y_v) run through all lattice points of $X_v \cup Y_v$, we get all residue classes ℓ (mod a_3), and have the following

THEOREM 14.1.

$$h_0 A_3' = \bigcup_{(x_v, y_v) \in X_v \cup Y_v} \{(h_0 - t - x_v - y_v)a_3 + y_v a_2 + x_v \geqq 0,\ t = 0, 1, \ldots\} .$$

For use in the next section, we shall also determine $(h_0 - 1)A_3'$. Clearly

$$(h_0 - 1)A_3' \subset h_0 A_3' - a_3 = \{n - a_3 \mid n \in h_0 A_3'\} .$$

If A_3 is <u>pleasant</u>, it suffices to use regular representations, and clearly

$$(h_0 - 1)A_3' = (h_0A_3' - a_3) \cap \mathbb{N}_0$$

(where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) . For non-pleasant A_3 , however, we get problems with the number n_0 of (11.13):

(14.2) $n_0 = a_3 - r - 1 = (f - 1)a_2 + a_2 - 1 = n_{h_0-1}(A_3) + 1 \notin (h_0 - 1)A_3'$,

where $n_0 + a_3 = 2fa_2 + r - 1 \in h_0A_3'$, since $1 \leqq r \leqq a_2 - f - 1$ by (4.3). (For pleasant A_3 , it follows from (2.8) that $n_0 + a_3 = n_{h_0}(A_3) + 1 \notin h_0A_3'$.)

We shall show that n_0 is usually the only exception:

THEOREM 14.2. For A_3 non-pleasant, with $r \neq 1$ and $s \neq q$, we have

(14.3) $\quad (h_0 - 1)A_3' = (h_0A_3' - a_3) \cap \mathbb{N}_0 \setminus \{a_3 - r - 1\}$.

To prove this, we replace h_0 by $h_0 - 1$ in the arguments leading to Theorem 14.1. The only critical point is whether now $S_\ell \subset (h_0 - 1)A_3'$.

To show that $S_\ell \subset h_0A_3'$, Rödseth used his Lemma 5, which states that for $1 \leqq i \leqq v$, we have

(14.4) $\quad x_{i-1} + y_{i-1} + Q_i - 1 \leqq h_0 \quad$ if $P_i \leqq s_i$

(14.5) $\quad x_i + y_i + R_i - 1 \leqq h_0 \quad$ if $P_i > s_i$.

If these relations hold with strict inequalities, it follows that $S_\ell \subset (h_0 - 1)A_3'$.

We note that Rödseth's division algorithm for a_3/a_2 is the same as the one leading to our Theorem 6.1. In particular, we have

$a_3 = q_1a_2 - s_1$, hence $q_1 = q$, $s_1 = s$, and $v > 0$ for a non-pleasant A_3 , when $s \geqq q$ by (2.10).

Studying Rödseth's proof of his Lemma 5, we observe the following facts:

1) For $i = 1$, when $P_1 = q_1 \leqq s_1$, we have equality in (14.4) if and only if (x_0, y_0) is the upper right corner of Y_0 :

(14.6) $(x_0, y_0) = (s_0 - 1, P_1 - P_0 - 1) = (a_2 - 1, f - 1)$.

Then $y_0a_2 + x_0$ is just the number n_0 of (14.2).

2) For $i > 1$, hence $Q_i > 1$, a necessary condition for equality in (14.4) or (14.5) is $s_i = s_{i-1} - 1$ or $s_{i+1} = s_i - 1$, respectively. But then such a relation must hold from the start:

$$s = s_1 = s_0 - 1 = a_2 - 1 \text{ , hence } r = 1$$

(cf. the recurrence relation $s_{j+1} = q_{j+1}s_j - s_{j-1}$, $q_{j+1} \geqq 2$). If $r \neq 1$, we thus have strict inequalities in (14.4-5) for all $i > 1$.

In Rödseth's proof of $S_\ell \subset h_0A_3'$, he divides S_ℓ into "subsequences" between $y_{i-1}a_2 + x_{i-1}$ and

$$y_ia_2 + x_i = y_{i-1}a_2 + x_{i-1} + Q_i\left[\frac{x_{i-1}}{s_i}\right]a_3 .$$

We have noted that the case $i = 1$ needs a special treatment. Since $s_1 = s$, $Q_1 = 1$, we must consider the numbers $za_3 + y_0a_2 + x_0$, $0 \leqq z < [x_0/s]$. Using the "a_3-transfer" $a_3 = qa_2 - s$ of section 11, this may be written as

(14.7) $$za_3 + y_0a_2 + x_0 = (y_0 + zq)a_2 + x_0 - zs ,$$

with positive constant term, and a coefficient sum

$$\sum = x_0 + y_0 - z(s - q) \leqq x_0 + y_0 .$$

If $x_0 + y_0 < h_0$, then also $\sum < h_0$ for all z. If $x_0 + y_0 = h_0$, corresponding to the corner (14.6), then $\sum < h_0$ for $z > 0$ if $s > q$, but $\sum = h_0$ for all z when $s = q$.

If $s = q$, then $v = 1$ by Theorem 7.1, and the "subsequence" just completed covers the whole of S_ℓ. If $v > 1$, we have seen that the remaining subsequences yield no problems if $r \neq 1$.

This completes the proof of (14.3), and also shows that if $s = q$, then

$$(14.8) \quad (h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap \mathbb{N}_0 \setminus \left\{ta_3 - r - 1 \;\middle|\; t = 1, 2, \ldots, \left[\frac{a_2 - 1}{s}\right]\right\}.$$

Here $ta_3 - r - 1 = n_0 + (t - 1)a_3 = n_0 + za_3$, with $0 \leqq z < [x_0/s] = [(a_2 - 1)/s]$. Note that we may use also $z = [x_0/s]$ in (14.7), but the resulting number is then contained in $h_0 A_3'$ but not in $h_0 A_3' - a_3$.

We finally treat the case $r = 1$. A modification of Rödseth's method then seems to become rather complicated. However, we can settle the case directly by a more elementary application of a_3-transfers. With $r = 1$, the only such transfers which may reduce the coefficient sum are of the form

$$(14.9) \quad ea_3 = (ef + 1)a_2 - (a_2 - e), \; e = 1, 2, \ldots .$$

As in section 11, we start with the regular representations

$$(14.10) \quad n = e_3 a_3 + e_2 a_2 + e_1, \; e_1 \leqq a_2 - 1, \; e_2 \leqq f - 1 .$$

For $r = 1$, it is unnecessary to consider $e_2 = f$, since already $fa_2 + 1$ gives a new a_3.

For the n of (14.10), we shall decide if $n \in h_0 A_3'$. If $\sum_e = \sum e_i \leqq h_0$, we are finished. If $\sum_e > h_0$, we must try a transfer (14.9) with $e \leqq e_3$. The transfer is possible only if it yields a non-negative constant term, that is, if $e_1 \geqq a_2 - e$.

Similarly, we shall decide if $n' \in (h_0 - 1)A_3'$, where

$$(14.11) \qquad n' = n - a_3 = (e_3 - 1)a_3 + e_2 a_2 + e_1 \qquad (e_3 > 0),$$

with $\sum_e' = \sum_e - 1$, hence no problem if $\sum_e \leqq h_0$. If an a_3-transfer (14.9) is necessary and possible in (14.10), and yields a new $\sum \leqq h_0$, then the <u>same</u> transfer gives $\sum' \leqq h_0 - 1$ in (14.11), provided it is possible, that is, if $e \leqq e_3 - 1$. It is easily seen that this combination of conditions <u>fails</u> only in the case

$$(14.12) \qquad n = e_3 a_3 + (f - 1)a_2 + a_2 - e_3, \ \sum = h_0 + 1 .$$

Thus $n' = n - a_3 \notin (h_0 - 1)A_3'$ if $n' = e_3 a_3 - e_3 - 1$.

For the n of (14.12), we must use $e = e_3$ in (14.9), and get $n = (e_3 + 1)fa_2$, hence

$$n \in h_0 A_3 \Leftrightarrow (e_3 + 1)f \leqq h_0 = a_2 + f - 2 \Leftrightarrow e_3 \leqq \left[\frac{a_2 - 2}{f}\right].$$

We have thus shown that if $r = 1$, then

$$(14.13) \quad (h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap \mathbb{N}_0 \setminus \left\{ t(a_3 - 1) - 1 \;\middle|\; t = 1, 2, \ldots, \left[\frac{a_2 - 2}{f}\right] \right\}.$$

For $t = 1$, we get $t(a_3 - 1) - 1 = a_3 - 2 = n_0$.

No problems arise if we have $s = q$ and $r = 1$ <u>simultaneously</u>. Then $s = q = a_2 - 1$, $f = q - 1 = a_2 - 2$, and the "subtrahends" $\{\ \}$ in (14.8) and (14.13) both consist of n_0 only.

The results (14.3), (14.8) and (14.13) imply that, but for the specified exceptions with $t > 1$ for $r = 1$ or $s = q$, <u>the integers $\geq a_3$ with a representation in at most h_0 addends from A_3 have such a representation containing</u> a_3.

In particular, $[0, n_{h_0}(A_3)] \subset h_0 A_3'$. It then follows from (14.3) that

$$(14.14) \qquad r \neq 1,\ s \neq q \Rightarrow [0, n_{h_0}(A_3) - a_3] \setminus \{a_3 - r - 1\} \subset (h_0 - 1)A_3' .$$

This was first observed numerically for a large number of bases A_3, and gave the impetus for the investigations in this section.

As in Rödseth [6], let $\Lambda(n)$ denote the number of addends in a <u>minimal</u> representation of n by a given basis A_k. We clearly have

$$\Lambda\big(n_h(A_k) - (x + 1)a_k + 1\big) \geq h - x,\ \Lambda\big(n_h(A_k) - xa_k\big) \geq h - x ,$$

since otherwise addition of $(x + 1)a_k$ or xa_k would yield a contradiction. This raises the question whether there are integers $x > 0$ such that for the interval of length a_k:

$$(14.15) \qquad \big[n_h(A_k) - (x + 1)a_k + 1,\ n_h(A_k) - xa_k\big] \subset (h - x)A_k' .$$

We have just seen that this holds with $x = 1$ if $k = 3, h = h_0, A_3$ non-pleasant, $r \neq 1$, $s \neq q$. Already for $x = 2$, however, it is easy to find counterexamples:

$$A_3 = \{1, 7, 11\},\ h_0 = 6,\ n_6(A_3) = 48\ ;\ \Lambda(17) = 5 .$$

We have made the interesting observation that for <u>Frobenius-dependent</u> A_3 with $r > 1$, (14.15) holds also with larger x :

PROPOSITION 14.1. Let A_3 be Frobenius-dependent, with $r > 1$. In the notation (5.8), let

$$(p-1)a_2 \leq n \leq n_{h_0}(A_3), \quad x = \left\lceil \frac{n_{h_0}(A_3) - n}{a_3} \right\rceil .$$

Then

$$n \in (h_0 - x)A_3' .$$

A proof will be published elsewhere.

15. The cases with $n_h(A_4) = n_h(A_3)$.

In (3.3), we raised the question of basis extensions which do not increase the h-range. We shall solve this question completely in the case

(15.1) $$n_h(A_4) = n_h(A_3 \cup \{a_4\}) = n_h(A_3), \quad a_4 > a_3 .$$

Even if A_4 enters the formulation, the results depend entirely on the properties of A_3.

We see from (3.4) that the regular h-range g_h always increases by a basis extension (assuming admissible bases). The same argument shows that if A_3 is pleasant, then

$$n_h(A_4) \geq g_h(A_4) > g_h(A_3) = n_h(A_3) ,$$

so that we may assume non-pleasant A_3 in (15.1).

If $a_4 > n_{h_0}(A_3) + 1$, then A_4 is not admissible for $h = h_0$ (where $h_0 = a_2 + f - 2$ refers to A_3). If then $h = h_0' > h_0$ is the smallest h for which A_4 is admissible, we trivially have

$n_h(A_4) = n_h(A_3)$ for $h < h_0'$. On the other hand, it follows from (2.14) that

$$n_{h_0'}(A_4) \geqq a_4 + n_{h_0'-1}(A_3) = a_4 + n_{h_0'}(A_3) - a_3 > n_{h_0'}(A_3) .$$

Similarly, it follows from from (2.13-14) that

$$n_{h'}(A_4) \geqq n_{h'}(A_3),\ h' \geqq h_0' \Rightarrow n_h(A_4) > n_h(A_3),\ h > h' .$$

We may therefore restrict the problem (15.1) to the case

$$(15.2) \qquad n_{h_0}(A_4) = n_{h_0}(A_3),\ a_3 < a_4 \leqq n_{h_0}(A_3) + 1 .$$

Note that a similar simplification does not apply to larger bases, since the analogue of (2.14) does not necessarily hold for $k > 3$.

We already know one case of (15.2), resulting from the basis A_{h+2} of section 3:

$$(15.3) \qquad a_2 = h_0 + 1,\ a_3 = h_0 + 2,\ a_4 = \alpha a_2 + a_3,\ 1 \leqq \alpha \leqq h_0 - 1 .$$

To solve the general problem, we note that

$$n_{h_0}(A_4) = n_{h_0}(A_3) \Leftrightarrow n_{h_0}(A_3) + 1 \notin h_0 A_4'$$

$$(15.4) \qquad \Leftrightarrow n_{h_0}(A_3) + 1 - \delta a_4 \notin (h_0 - \delta) A_3',\ \delta = 1, 2, \ldots, h_0 .$$

In most cases, it suffices to consider $\delta = 1$. Since

$$N = n_{h_0}(A_3) + 1 - a_4 \in [0, n_{h_0}(A_3) - a_3] \subset (h_0 A_3' - a_3) \cap \mathbb{N}_0 ,$$

(15.4) <u>fails</u> already for $\delta = 1$ if N does not belong to the exceptions in (14.3), (14.8) or (14.13). These cases have the

common exception n_0 of (14.2), and $N = n_0$ does in fact lead to a general solution of (15.2):

(15.5) $\quad a_4 = \hat{a}_4 = n_{h_0}(A_3) - a_3 + r + 2 = n_{h_0}(A_3) - n_{h_0-1}(A_3) \Rightarrow n_{h_0}(A_4) = n_{h_0}(A_3)$.

This is clear since we cannot use $\delta \geqq 2$ in (15.4):

$$2\hat{a}_4 > n_{h_0}(A_3) + 1 \Leftrightarrow n_{h_0}(A_3) > 2a_3 - 2r - 3 ,$$

which always holds by (2.8). - Note that $\hat{a}_4 = a_3$ if A_3 is pleasant.

If $a_4 \neq \hat{a}_4$, a necessary condition for (15.2) is that N equals one of the exceptions in (14.8) or (14.13), with $t > 1$ (since $t = 1$ corresponds to n_0).

We start with (14.13), hence $r = 1$. Then $n_{h_0}(A_3)$ is given by (2.28), and we find that we must choose

(15.6) $\quad a_4 = a_3 + \tau(a_3 - 1), \ \tau = 1, 2, \ldots, \left[\frac{a_2 - 2}{f}\right] - 1$

(while $\tau = [(a_2 - 2)/f]$ corresponds to $\hat{a}_4$). We shall see that this is also <u>sufficient</u> for (15.2) to hold.

We consider a representation

(15.7) $\quad n_{h_0}(A_3) + 1 = x_4a_4 + x_3a_3 + x_2a_2 + x_1$,

and must show that $\sum x_i > h_0$. This is trivial if $x_4 = 0$, so we can assume $x_4 > 0$, and observe that

$$n_{h_0}(A_3) + 1 \equiv 0 , \ a_4 \equiv a_3 \equiv 1 \quad (\mathrm{mod}\ a_3 - 1 = fa_2) .$$

With $x_2 = \kappa f + x_2'$, $0 \leqq x_2' < f$, (15.7) then gives

$$x_4 + x_3 + x_1 \equiv (f - x_2')a_2 \ , \text{ hence } \ x_4 + x_3 + x_1 \geqq (f - x_2')a_2$$

$$x_4 + x_3 + x_2 + x_1 \geqq x_4 + x_3 + x_2' + x_1 \geqq (f - x_2')a_2 + x_2' \geqq a_2 + f - 1 = h_0 + 1 \ ,$$

as required. - In particular, we get the known case (15.3) from (15.5-6) with $f = 1$.

We next consider (14.8), hence $s = q$, $a_3 = q(a_2 - 1)$. By (2.29), we now have two possibilities for $n_{h_0}(A_3)$:

$$n_{h_0}(A_3) = \left(\left[\frac{a_2-1}{s}\right] + 2\right)a_3 - r - \begin{cases} 2 \ , \text{ if } \ s \nmid (a_2 - 1) \\ 3 \ , \text{ if } \ s \mid (a_2 - 1) \ . \end{cases}$$

These two cases must be considered separately.

If $s \nmid (a_2 - 1)$, we find that we must choose

(15.8) $\qquad a_4 = (\tau + 1)a_3 \ , \quad \tau = 1,2,\ldots,\left[\frac{a_2-1}{s}\right] - 1$

(while $\tau = [(a_2 - 1)/s]$ corresponds to $\hat{a}_4$). Again, this is also sufficient for (15.2) to hold:

We consider a representation (15.7). Since $a_3 \mid a_4$, we get

$$x_2a_2 + x_1 \equiv n_{h_0}(A_3) + 1 \equiv - r - 1 = - a_2 + f \quad (\text{mod } a_3 = q(a_2-1)) \ ,$$

from which we draw two conclusions:

1) $\quad x_2a_2 + x_1 \geqq a_3 - r - 1$

2) $\quad x_2a_2 + x_1 \equiv x_2 + x_1 \equiv f - 1 \quad (\text{mod } a_2 - 1)$.

Assuming $\sum x_i \leqq h_0$ in (15.7), hence $x_4 > 0$, we get $x_2 + x_1 < h_0 = (f - 1) + (a_2 - 1)$, so $x_2 + x_1 = f - 1$, and

$$x_2a_2 + x_1 \leqq (f-1)a_2 = a_3 - r - a_2 ,$$

contradicting the first conclusion.

If $s \mid (a_2 - 1)$, hence $m = (a_2 - 1)/s$ an integer, we find that we must choose

$$a_4 = (\tau + 1)a_3 - 1 , \quad \tau = 1,2,\ldots,\frac{a_2-1}{s} - 1 = m - 1 .$$

Now (15.4) holds for $\delta = 1$, and we examine $\delta = 2$:

$$n_{h_0}(A_3) + 1 - 2a_4 = (m - 2\tau)a_3 - r = (m - 2\tau - 1)a_3 + fa_2 .$$

If $\tau \geqq [\frac{1}{2}(m+1)]$, this expression is negative, and an examination of (15.4) for $\delta \geqq 2$ is unnecessary, so (15.2) holds. If $\tau < [\frac{1}{2}(m+1)]$, however, the right hand side belongs to $(h_0 - 2)A_3'$, and (15.4) fails for $\delta = 2$. Thus (15.2) is satisfied only if

$$(15.9) \quad a_4 = (\tau + 1)a_3 - 1 , \quad \tau = [\tfrac{1}{2}(m+1)],\ldots,m-1 ; \quad m = \frac{a_2-1}{s} .$$

Summing up, we have the following

THEOREM 15.1. <u>For non-pleasant</u> A_3 , <u>the equality</u> (15.2) <u>holds if and only if we have one of the cases</u>:

(15.5) <u>for arbitrary</u> A_3 ,
(15.6) <u>for</u> $r = 1$,
(15.8-9) <u>for</u> $s = q$.

Based on computations by Mossige, this result was conjectured long before a proof was found. The cases $r = 1$ or $s = q$ are also proved in Krätzig-Berle [4, Kap.4], the "if" part along the lines above, the "only if" part by explicit representations for $n_{h_0}(A_3) + 1$ from h_0A_4' in the remaining cases.

16. The cases with $n_h(A_3 \cup \{a\}) = n_h(A_3)$, $a < a_3$.

In analogy with (3.3), it is quite natural to ask for cases when

$$(16.1)\quad n_h(A_k^*) = n_h(A_{k-1} \cup \{a\}) = n_h(A_{k-1}) \ , \quad 1 < a < a_{k-1} \ , \quad a \notin A_{k-1} \ ,$$

assuming admissible bases.

We need a particular result for the similar problem regarding regular h-ranges:

$$(16.2)\qquad 1 < a < a_2 \Rightarrow g_h(A_k^*) > g_h(A_{k-1}) \ .$$

The proof is simple: It follows from Hofmeister [1, Satz 1] that the constant term of the regular representation for $g_h(A_\kappa)$ equals $a_2 - 2$ for all admissible A_κ . We conclude that the constant term $a_2 - 1$ of $g_h(A_{k-1}) + 1$ has a regular representation in at most $a_2 - 2$ addends 1 and $a \leq a_2 - 1$.

In particular, $g_h(A_3^*) > g_h(A_2)$, and hence also $n_h(A_3^*) > n_h(A_2)$. The first possibility for (16.1) thus occurs when $k = 4$:

$$(16.3)\quad n_h(A_4^*) = n_h(A_3 \cup \{a\}) = n_h(A_3) \ , \quad 1 < a < a_3 \ , \quad a \neq a_2 \ .$$

As in the preceding section, a study of this equality depends entirely on the properties of A_3 .

If $h = h_0^*$ is the smallest h for which A_4^* is admissible, we clearly have $h_0^* \leq h_0$ (where again $h_0 = a_2 + f - 2$ refers to A_3). To be "fair" to A_3 , we restrict the examination of (16.3) to $h \geq h_0$.

Before doing this, we just mention the analogous problem for <u>regular</u> h-ranges. By (16.2), we must then assume $a_2 < a < a_3$, and it is not difficult to prove that for $h \geq h_0$:

(16.4) $\quad g_h(A_4^*) = g_h(A_3) \Longleftrightarrow a = fa_2 + \rho \ , \ 0 \leqq \rho < r \ .$

(My original proof is reproduced in Krätzig-Berle [4, p.27].)

Similar arguments show that (16.3) is impossible with pleasant A_3 . With $n_h(A_4^*) \geqq g_h(A_4^*)$ and $n_h(A_3) = g_h(A_3)$, equality in (16.3) could only occur under the conditions of (16.4). But by (2.8-9), we then have

$$n_h(A_3) + 1 = (h - h_0 + 2)a_3 - r - 1 = (h - h_0)a_3 + 1\cdot a + fa_2 + r - \rho - 1 \ ,$$

with a coefficient sum $\leqq h$ except in the one case $r = a_2 - 1$, $\rho = 0$, hence $f \geqq 2$. But then

$$n_h(A_3) + 1 = (h - h_0)a_3 + 2a + a_2 - 2 \ , \quad \sum \leqq h \ .$$

In what follows, we may thus assume <u>non-pleasant</u> A_3 in (16.3).

Since A_3 and A_4^* have a <u>common largest element</u> a_3 , it is possible to use Meures' result (2.16), which in combination with (2.13) shows that for $h \geqq h_0 - 1$:

$$n_h(A_k) \leqq ha_k - g(\bar{A}_k) - 1 \ ,$$

with equality if $h \geqq h_1$ ("stabilization", cf. section 3). For non-pleasant A_3 , we know that $h_1 = h_0$. For A_4^* , we put $h_1 = h_1^*$. With

$$\bar{A}_3 = \{a_3 - a_2, \ a_3 - 1, \ a_3\} \ , \ \bar{A}_4^* = \bar{A}_3 \cup \{a_3 - a\} \ ,$$

we thus get, for $h \geqq h_0$:

$$n_h(A_3) = ha_3 - g(\bar{A}_3) - 1 \ , \quad n_h(A_4^*) \leqq ha_3 - g(\bar{A}_4^*) - 1 \ .$$

Since trivially $n_h(A_4^*) \geq n_h(A_3)$, this shows that

(16.5) $\qquad g(\bar{A}_4^*) = g(\bar{A}_3) \Rightarrow n_h(A_4^*) = n_h(A_3) \quad \text{for} \quad h \geq h_0$

(16.6) $\qquad h \geq h_1^* : n_h(A_4^*) = n_h(A_3) \Rightarrow g(\bar{A}_4^*) = g(\bar{A}_3)$.

We obviously have $g(\bar{A}_4^*) \leq g(\bar{A}_3)$. With strict inequality, $g(\bar{A}_3)$ has a representation by $\bar{A}_4^*$:

$$g(\bar{A}_3) = x_1(a_3 - a) + x_2(a_3 - a_2) + x_3(a_3 - 1) + x_4 a_3 .$$

It follows that

$$n_{h_0}(A_3) + 1 = h_0 a_3 - g(\bar{A}_3) = (h_0 - \sum x_i) a_3 + x_1 a + x_2 a_2 + x_3$$

has a representation by A_4^* with coefficient sum $h_0 - x_4 \leq h_0$, provided that $\sum x_i \leq h_0$. We thus have the following partial converse of (16.5):

(16.7) $\qquad g(\bar{A}_3) \in h_0 \bar{A}_4^* \Rightarrow n_h(A_4^*) > n_h(A_3) \quad \text{for} \quad h \geq h_0$.

We only proved this for $h = h_0$ above, but the general result with $h \geq h_0$ then follows immediately from (2.13-14).

There is one trivial case of equality in (16.3):

(16.8) $\qquad f = 1$, $a_2 = h_0 + 1$, $a_3 = h_0 + r + 1$, $a = a_2 - tr \geq 2$

(16.9) $\qquad \Rightarrow n_h(A_4^*) = n_h(A_3) \quad \text{for} \quad h \geq h_0$.

This follows from (16.5), since $\bar{A}_3$ and $\bar{A}_4^*$ are "equivalent" as Frobenius bases:

$$\bar{A}_3 = \{r, a_3 - 1, a_3\} , \quad \bar{A}_4^* = \{r, (t+1)r, a_3 - 1, a_3\} .$$

The second element of $\bar{A}_4^*$ is a multiple of the first one.

We assume that A_3 is non-pleasant. If it is also non-dependent, it follows from Theorem 10.1 that

$$n_{h_0}(A_4^*) \geqq n_{h_0}(A_3) \geqq (h_0+1)a_2 - a_3 .$$

Let $1 < a < a_2$. We then get $h_1^* \leqq h_0$ by Theorem 3.1, and can combine (16.5-6) to an equivalence for non-dependent A_3. And for Frobenius-dependent A_3, Krätzig-Berle [4, p.23] shows very simply that we always have $n_h(A_4^*) > n_h(A_3)$ except in the already settled cases (16.8), hence

(16.10) $\quad 1 < a < a_2 \; : \; g(\bar{A}_4^*) = g(\bar{A}_3) \iff n_h(A_4^*) = n_h(A_3)$.

Based on extensive computations by Mossige, I conjectured the following results:

THEOREM 16.1. Let $a_2 < a < a_3$. Then

$$n_h(A_4^*) > n_h(A_3) \quad \text{for} \quad h \geqq h_0 .$$

THEOREM 16.2. Let $1 < a < a_2$. In addition to (16.8), there is one more case of equality in (16.9):

$$f = 1 \; , \; a_2 = h_0 + 1 \; , \; a_3 = h_0 + r + 1 \; , \; a = tr + 1$$

$$h_0 = \tau r + \rho \; , \; 0 \leqq \rho < r - 1 \; , \; \tau \geqq \rho$$

$$r \equiv -1 \pmod{\rho + 1} \; , \; t = 1, 2, \ldots, \left[\frac{\tau+1}{\rho+1}\right] .$$

Both theorems were proved in the Master's thesis [2] of my student Kirfel. He used the methods of Rödseth [5] for determining

the Frobenius number $g(\overline{A}_3)$. A shortened version [3] is submitted for publication.

Another student of mine, Krätzig-Berle, gave an independent and very elegant proof of Theorem 16.1 in her Diplomarbeit [4, Satz 3.1]. Using the inequalities of Theorems 10.2-5, she could determine a h_0-representation by A_4^* of $n_{h_0}(A_3) + 1$.

We note that the bases A_3 of Theorem 16.2 satisfy the conditions (8.1-2), and so $n_h(A_3)$ can be determined explicitly by (8.3). It is fairly straightforward (cf. [4, Satz 2.3]) to show that this h-range is not increased when extending the basis with $a = tr + 1$. The hard problem is of course to show that all <u>other</u> cases (except (16.8)) lead to an increase of the h-range.

REFERENCES

1. G. Hofmeister, Über eine Menge von Abschnittsbasen, J. Reine Angew. Math. 213 (1963), 43-57.
2. C. Kirfel, Erweiterung dreielementiger Basen bei konstanter Frobeniuszahl und Reichweite, Master's thesis, Dept. of Math., Univ. of Bergen, 1982.
3. C. Kirfel, Erweiterung dreielementiger Basen bei konstanter Frobeniuszahl, II, to appear.
4. E. Krätzig-Berle, Zum Reichweitenproblem für dreielementige Basen, Diplomarbeit, Mainz, 1983.
5. Ö. Rödseth, On a linear diophantine problem of Frobenius, J. Reine Angew. Math. 301 (1978), 171-178.
6. Ö. Rödseth, On h-bases for n, Math. Scand. 48 (1981), 165-183.
7. E.S. Selmer, On the postage stamp problem with three stamp denominations, Math. Scand. 47 (1980), 29-71.
8. E.S. Selmer and A. Rödne, On the postage stamp problem with three stamp denominations, II, Math. Scand. 53 (1983), 145-156.

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