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Non-Boolean Classical Relevant Logics I

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In honor of Robert Meyer

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Abstract: Relevant logics have traditionally been viewed as *paraconsistent*. This paper shows that this view of relevant logics is wrong. It does so by showing forth a logic which extends classical logic, yet satisfies the Entailment Theorem as well as the variable sharing property. In addition it has the same **S4**-type modal feature as the original relevant logic **E** as well as the same enthymematical deduction theorem.
The variable sharing property was only ever regarded as a necessary property for a logic to have in order for it to not validate the so-called *paradoxes of implication*. The Entailment Theorem on the other hand was regarded as both necessary and sufficient. This paper shows that the latter theorem also holds for classical logic, and so cannot be regarded as a sufficient property for blocking the paradoxes. The concept of suppression is taken up, but shown to be properly weaker than that of variable sharing.

Keywords: disjunctive syllogism · entailment · modality · paraconsistency · relevant logics · suppression

> Of commandments, there is only one. Thou shalt not use the disjunctive syllogism. Or else one will be like the heathen.
>
> *The Relevantist*[1]

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1 Introduction

Received wisdom has it that relevant logics are inherently *paraconsistent*, where a logic is classified as paraconsistent just in case it does not validate the inference from A together with $\sim A$ to an arbitrary B. That this is so is testified to by the following quotes from both adherents and adversaries to the relevant school:

- **relevance logic,** any of a range of logics and philosophies of logic united by their insistence that the premises of a valid inference must be relevant to the conclusion. Standard, or classical, logic contains inferences that break this requirement, e.g., the *spread law,* that from a contradiction any proposition whatsoever follows. (**?**, p. 792)
- Relevant logic is paraconsistent, so we all count as paraconsistentists to the extent that we count as relevantists. (**?**, p. 19)
- All relevantists agree in rejecting *disjunctive syllogism* (DS):

$$\text{(DS)} \quad \frac{\begin{array}{l} p \vee q \\ \sim p \end{array}}{q}$$

  (**?**, p. 41)
- The most fully developed formal response to these 'paradoxes' consists of abandoning C, the principle of disjunctive syllogism. Logics which do this are called *relevance logics* [. . .] (**?**, p. 204f.)
- Recently Bob Meyer has claimed that relevant logic is mistaken in rejecting $\mathrm{DS}_{\vee}$[. . .]. In contrast I claim that the rejection of $\mathrm{DS}_{\vee}$ is central to the whole conception of relevant logic.[2] (**?**, p. 66)
- Relevance logic holds that disjunctive syllogism for truth functional disjunctions is an invalid argument form. (**?**, p. 131)

I make two substantive claims regarding relevant logics in this paper: the first is that paraconsistency does not follow from concerns over relevance in the traditional Anderson-Belnap understanding of this, where the two criteria of variable sharing and that of satisfying the Entailment Theorem were regarded as, respectively, necessary and necessary and sufficient, for avoiding the so-called *fallacies of relevance*. I will show this by showing forth a logic which I have called '$\Pi'_{\mathbf{E}}$' which has both these properties, yet is not a paraconsistent logic.

The Entailment Theorem, as stated by Anderson and Belnap, cuts away logics with more primitive rules than modus ponens and adjunction, and so does by definition cut away as non-relevant for instance Ackermann's logic Π' which has disjunctive syllogism as a primitive rule. In light of all the weaker logics which are regarded as relevant yet have more primitive rules than adjunction and modus ponens, this property might be thought to be in need of weakening. I will show, however, that it counts as relevant even *classical logic*. As a consequence one can't uphold the claim

[1] The scornfully intended passage is from Meyer's *A Farewell to Entailment* (**?**, p. 578). *The Relevantist* is the personification of the position portrayed in **?**.

[2] $\mathrm{DS}_{\vee}$ is identified as the inference schema *if A and* $\ulcorner \sim A \vee B \urcorner$ *are true, then so is B*.

that this property is both necessary and sufficient for relevance. The second substantive claim of this paper is therefore that the only properties currently available for defining what the extension of ‘relevant logic’ is, are merely necessary ones.

Although somewhat *ad hoc*, $\Pi'_{\mathbf{E}}$ is a fairly decent logic. However, it does not extend Anderson and Belnap’s favorite logic **E**. The sequel to this paper, *Non-Boolean Classical Relevant Logics II*, shows that both **E** and **R** can be extended to the non-paraconsistent logics **Æ** and **M**. Both these logics have interesting properties; both extend classical logic, and **Æ** even extends the classical modal logic **S4** and can be naturally extended so as to also extend **S5**. Both logics, however, heavily rely on the truth-constant known as the *Ackermann constant*, and so are better left to be dealt with in a separate paper. That paper also gives an interpretation of the two consequence relations commonly used and often confounded in debates regarding relevance and paraconsistency.

The plan for the paper is as follows: section **??** gives a historical account of relevant logics and sets the stage for the rest of the paper. Section **??** defines the Hilbert consequence relation and presents Ackermann’s logic Π', Anderson and Belnap’s **E** and **R** as well as my own $\Pi'_{\mathbf{E}}$. I show that $\Pi'_{\mathbf{E}}$ has the same modal properties as **E** and that the variable sharing property holds for all these four logics. Section **??** shows in what sense Π' and $\Pi'_{\mathbf{E}}$ are extensions of classical logic and proves that the deduction theorem for the material conditional holds for both Π' and $\Pi'_{\mathbf{E}}$, and that an enthymematical deduction theorem holds for $\Pi'_{\mathbf{E}}$, but fails for Π'. Section **??** then shows that the Entailment Theorem holds for $\Pi'_{\mathbf{E}}$. From the proof it will be evident that the Entailment Theorem holds for any axiomatic extension of $\Pi'_{\mathbf{E}}$, and therefore also for pure classical logic. Section **??** gives a brief discussion of how best to understand the concept of a relevant logic in light of this and also shows that the concept of *suppression freedom* is properly weaker than that of variable sharing. Section **??** then sums up by way of looking at what has been achieved in relation to Anderson and Belnap’s onslaught at disjunctive syllogism in the infamous section from **?** called *The Dog*.

In the event of the 10th anniversary of Meyer’s death, and in recognition of the importance of his ideas for the field of relevant logic in general and for this paper in particular, I dedicate this paper in his honor.

2 The birth of relevant logics

Entailment lies at the very heart of the philosophy of logic. The now agreed upon usage of the term ‘entailment’ derives from a section of G. E. Moores essay *External and Internal Relations*:

> Let us express the relation which we assert to hold between a particular proposition p and a particular proposition q, when we say that in this sense q “follows from” og “is deducible from” p, by the symbol “ent”; which I have chosen to express it, because it may be used as an abbreviation for “entails,” and because “p entails q” is a natural expression for “q follows from p,” *i.e.*, “entails” can naturally be used as the converse of “follows from.” (**?**, p. 53)

Given this elucidation, it would seem that 'entailment' is to be more or less synonymous with 'implication'. At the time of writing his essay, however, that term was used by Bertrand Russell to mean 'material implication'. The russellian backdrop of Moore's essay can be found in Russell's writings from the 1903 book *The Principles of Mathematics* and onward. Implication is therein viewed as a truth-functional relation holding between propositions the study of which being the main objective of propositional logic. That implication is equated with material implication can be seen from the following quote:

> the assertion that q is true or p false turns out to be strictly equivalent to "p implies q"; [...]. It follows from the above equivalence that of any two propositions there must be one which implies the other, that false propositions imply all propositions, and true propositions are implied by all propositions. (**?**, §16)

In formalism, these three consequences are rendered as

$$\begin{array}{ll} \text{(PM1)} & (A \supset B) \vee (B \supset A) \\ \text{(PM2)} & {\sim}A \supset (A \supset B) \\ \text{(PM3)} & A \supset (B \supset A) \end{array}$$

and are often called the *paradoxes of material implication* and leveled against the view that the material conditional might plausibly represent the relation of entailment.

Thus Russell anno 1903 seems to simply use 'implication' in places where readers both at the time and later would have rather used 'material implication' or 'material conditional'. Had he not also been careless in using other words normally reserved for contexts of entailment such as 'premise' or 'hypothesis' instead of 'antecedent', and used 'deduction', 'inference' and 'implication' more or less interchangeably, the matter might have been cleared up quickly. Although Russell quite explicitly differentiated between material implication and entailment, insisting that one indeed needed to in order to answer Lewis Carroll's puzzle,[3] some readers, Moore included, seem to have interpreted Russell as equating entailment and material implication. Two other such readers were the founders of modern modal logic, namely Hugh MacCall and Clarence Irving Lewis.

That any true proposition should be implied by any and every proposition, and furthermore that any false proposition should imply any and every proposition, or that the relation of implication should hold, in one way or the other, between any two propositions, seemed to both MacCall and Lewis abhorrent to both common sense and the ordinary meaning of 'implies'.[4] Whereas there need be no connection between antecedent and consequent in a true material conditional, there, according to MacColl and Lewis, need to be such a connection if the corresponding implication-relation is to hold. Both fleshed out this relation in terms of modality so that "A entails B" is to be reckoned true just if it be *impossible* that A be true and B false. Lewis

[3] See **?**, §38 where entailment is called "the notion of *therefore*". The distinction is also drawn by **?**, p. 9.

[4] See for instance **?** and **?**. According to Stephen Read, MacColl developed the modal system **T** decades before Fey and von Wright did. See his essay **?** for detail and also for information on how MacColl's work relate to that of Lewis'.

called this relation *strict implication* and introduced the the famous “fish-hook”, $\prec$, for it. MacColl simply called the relation *implication* and used the colon, as in $A : B$, to express it.

The first paradox of material implication—that between any two proposition, one implies the other—does not hold for strict implication. However, the strict implication has paradoxes akin to the latter two material ones; if the antecedent of a strict implication is necessarily false, then it will strictly imply every proposition. Likewise, if the consequent is necessarily true it will be strictly implied by every proposition. Thus just as there is no need for an extra connection between antecedent and consequent of material conditional beyond that afforded by the space of classical truth-functions, there seems to be no need of any such connection between antecedent and consequent of a true strict conditional beyond that allowed by the space of modality over such truth-functions. The following are instances of these paradoxes of *strict implication*:

$$\begin{array}{ll} \text{(PS1)} & A \wedge {\sim}A \prec B \\ \text{(PS2)} & A \prec B \vee {\sim}B \end{array}$$

One of the persons who thought this to show that strict implication is not the same thing as entailment, was Wilhelm Ackermann. Ackermann wrote in 1956 an essay entitled *Begründung einer Strengen Implikation*, which translates to *reasons for a rigorous implication*. Just as MacColl and Lewis, Ackermann also wanted his implication to express entailment. However, he does not make any attempt at reducing it to notions of modality as MacColl and Lewis did. Ackermann wanted modal notions to be expressible within the system, but entailment is not fashioned as a modal notion *per se*. The rigorous implication is rather to express a more intimate connection between the antecedent and consequent. Ackermann writes in the introduction to the essay:

> The rigorous implication, expressed as $A \to B$, expresses that there exists a logical connection between A and B; that the content of B is part of the content of A, or how now best to put it. That there exists such a connection has nothing to do with the correctness or falsity of A and B. This is why one ought to reject the validity of the formula $A \to (B \to A)$; it expresses that $B \to A$ can be inferred from A while it is obvious that the correctness of A has no bearing on whether there is a logical connection between B and A. The same reasons compel one to also reject the validity of the formulas $A \to (B \to A \,\&\, B)$, $A \to (\overline{A} \to B)$ and $A \to ((A \to B) \to B)$. The same holds for $B \to (A \to A)$, since the validity of $A \to A$ is independent of the correctness of B. My own *rigorous* implication differs from the strict one in that the latter formula is rejected as a universally valid formula—the same is true of $(A \,\&\, \overline{A}) \to B$—on account of the fact that the concept of implication—understood as a logical connection between two statements—does not encompass statements which implies or is implied by every other. A formula such as $(A \to B) \to ((B \to C) \to (A \to C))$, on the other hand, is to

> be recognized as valid since the inference from $B \rightarrow C$ to $A \rightarrow C$ is logically compelled given the assumption of $A \rightarrow B$. (**?**, p. 113)[5]

This leaves much to be desired in order to determine what type of connection there needs to be between A and B in order for A to entail B. Despite the title, the essay does not elaborate much more on the matter. Nor does the sequel-paper **?** clear up what the logical connection is supposed to consist in. In fact, **?**, p. 214 states that even Lewis wanted his strict implication to express such a logical connection, and that he merely developed his own because one may doubt that such a logical connection exists in the case of the paradoxes of strict implication.

What seems clear from Ackermann's two papers, however, is that his concept of rigorous implication is intended as a successor-concept of Lewis' strict implication—in fact, Ackermann showed in **?** that Lewis' **S2** is in fact *interpretable* in his own logic in that $\tau(A \multimap B)$ is provable in Ackermann's logic if $A \multimap B$ is provable in **S2**. Thus rigorous implication seems to be akin to a strict conditional, but with an added non-modal clause on the relationship between antecedent and consequent. One is, however, left none the wiser as to the content of this clause.

Anderson and Belnap took up Ackermann's ideas and turned them into a whole research program in which not only Ackermann's logic was scrutinized, but a whole field of logics—the initial field of *relevant logics*—were examined in order to probe the notion of entailment. The logical connection alluded to by Ackermann was rephrased as a connection of *relevance*, and failures to respect it branded *fallacies of relevance*. They realized that this, like Ackermann's, notion was quite obscure and came up with two formal conditions for making it precise, namely the *variable sharing property*, which they deemed to be only a necessary condition for relevance, and the *relevant deduction property*, also called the *Entailment Theorem*, which they deemed to be both necessary and sufficient (**?**, §5.1). Simply put, the variable sharing property is the property a logic has if $A \rightarrow B$ is a logical theorem only if A and B share a propositional variable. It is intended to give content to the idea that A and B need to share content, or there need to be some commonality of meaning, for $A \rightarrow B$ to be logically true. The relevant deduction theorem gives content to the idea that the premises of an argument need to be *used* in deriving the conclusion in order for the premises to be relevant to the conclusion. Only the first of these criteria apply to Ackermann's own logic, however, and so a significant schism erupted between Ackermann's approach and that of Anderson and Belnap in that Ackermann's logic has the variable sharing property without having the relevant deduction property.

Anderson and Belnap created the research field of relevant logics by, essentially, dropping disjunctive syllogism, the rule

$$(DS) \quad \{A, {\sim}A \vee B\} \Vdash B$$

[5]The translation is my own. Note that Ackermann seems to think that $B \multimap (A \multimap A)$ holds in Lewis' systems. This is true for *normal* modal logics such as **S4**, but it fails in Lewis' preferred systems **S2** and **S3**. Note, however, that Ackermann does not mention $B \multimap (A \multimap A)$ in the sequel-paper **?** in which he compares his logic to **S2**. See **?**, ch. 11 and **?**, ch. 4 for more on strict implication and its paradoxes.

from Ackermann's logic Π'.[6] Their reason for doing so was, simply put, that if $\rightarrow$ is to *express* entailment, then one ought to expect that a deduction theorem is forthcoming. However, since Ackermann's logic validates (DS), it also validates the explosion rule—that one may infer any B from a contradiction—and so no such simple deduction theorem can hold unless the logic also validates $A \wedge {\sim}A \rightarrow B$, one of the primordial examples of a relevant fallacy. Relevant logics have for more or less this reason been regarded as *paraconsistent* logics.

For purposes of this paper it will be sufficient to define the paraconsistent/explosive divide in the, to quote **?**, p. 344, "neither very restrictive nor substantive" way as follows:

Definition 1 A consequence relation $\Vdash$ is PARACONSISTENT just in case it is not the case that $\{A, {\sim}A\} \Vdash B$ holds for every A and B, and EXPLOSIVE if it does hold.

For the logics and their consequence relations presented in this paper it will be the case that $\{A, {\sim}A\} \Vdash B$ holds if and only if $\{A, {\sim}A \vee B\} \Vdash B$ holds, and so disjunctive syllogism will hold for a consequence relation just in case it is explosive.

Now any relevant logic will regard $A \rightarrow (B \rightarrow B)$ as incorrect since it violates the variable sharing principle, and so no relevant logic can have a simple deduction theorem for its Hilbert consequence relation since $A \vdash^h B \rightarrow B$ holds for even Anderson and Belnap's two favorite logics, **E** and **R**. The deduction theorem held forth by Anderson and Belnap was rather that which related their notion of a relevant deduction to that of being a logical theorem in either the Hilbert consequence relation, or, what Anderson and Belnap rather preferred, a Fitch-style natural deduction calculus. But if it's not a simple deduction theorem that is required, then why can't (DS) be a valid rule of inference for the Hilbert consequence relation?

One might, of course, have good reasons for opting for a paraconsistent logic. The main goal of this essay, however, is to show and argue that relevance does not *force* paraconsistency for the Hilbert consequence relation. The historical reason, however, for relevant logics being viewed as paraconsistent logics has to do with a shift in the notion of what counts as a Hilbert-style proof as a result of Anderson and Belnap's insistence upon a simple deduction theorem; in their own words, "In fact, the search for a suitable deduction theorem for Ackermann's systems [...] provided the initial impetus leading us to the research reported in this book." (**?**, p. 261). As a consequence of the shift, Anderson and Belnap argue not against (DS) understood in the Hilbert-tradition, but against an axiomatic version of it, namely $A \wedge ({\sim}A \vee B) \rightarrow B$. I will show, however, that it is possible to modify Ackermann's Π' in a slightly different manner than Anderson and Belnap did in such a way as to make (DS) derivable while at the same time retaining the variable sharing property and a relevant deduction theorem. Even though $A \wedge ({\sim}A \vee B) \rightarrow B$ will not be a theorem, there will be theorems on the form $(A \wedge ({\sim}A \vee B)) \wedge C \rightarrow B$, where C is a logical theorem, and so it will be the case that $\{A, {\sim}A \vee B, C\} \vdash^r B$, where $\vdash^r$ is the relevant consequence relation. Anderson and Belnap considered this, but tossed it aside as yet another confused idea by *The Man*.[7] This, then, was rather rash; I will show that there is a logic which validate

[6] Ackermann designated that rule by 'γ', and so much literature on relevant logics will refer to the rule, and variants of it, as precisely this.

[7] Read "the classical logician".

the Entailment Theorem, has the variable sharing property, has the modal features desired by Anderson and Belnap, as well as having both a deduction theorem for the material conditional, and hence is explosive with regards to the Hilbert consequence relation, as well has having an enthymematical deduction theorem. Thus relevance does not force paraconsistency for the Hilbert consequence relation even under these extra requirements.

Anderson and Belnap defined $\vdash^r$ by adding restrictions on what counts as a Hilbert derivation. The reason they defined it the way they did, was to tease out a notion of premise use. However, the consequence relation can more easily be defined as follows: $\Theta \vdash^i A =_{df} \varnothing \vdash^h \bigwedge \Theta_f \to A$ where Θ_f is some finite subset of Θ. Such a definition is in fact possible for any logic and any implication connective and often gives rise to a consequence relation quite different from the one from which it is defined. For instance, the Hilbert consequence relation of the logic **LP** is paraconsistent, but since **LP** is theorem-vise identical to classical logic (**?**, Thm. III.13) it follows that the consequence relation $\vdash^i_{\mathbf{LP}}$ is identical to the consequence relation of classical logic and therefore not paraconsistent. The case, as we shall see, is the opposite for the logic presented in this paper; $\Pi'_{\mathbf{E}}$'s Hilbert consequence relation is explosive, whereas its relevant consequence relation is paraconsistent.

3 Relevant logics: definitions and variable sharing

This section sets forth the axioms and rules of the logics Π', **E**, $\Pi'_{\mathbf{E}}$ and **R**, $\Pi'_{\mathbf{E}}$ being a mixture of Π' and **E**. I then show that all these logics have the variable sharing property before I give a brief discussion on the modal aspects of these logics.

Every logic in this paper will be thought of as a set of axioms together with a set of rules. Rules will be on form $\Gamma \Vdash A$ where Γ is a finite set. For each logic there are two different consequence relations which both will be important in this paper. The easiest one to specify is the Hilbert consequence relation for a logic:

Definition 2 [The Hilbert consequence relation of a logic] A HILBERT PROOF of a formula A from a set of formulas Γ in the logic **L** is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of **L**, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of **L**. The existential claim that there is such a proof is written $\Gamma \vdash^h_{\mathbf{L}} A$ and expressed as "there exists a Hilbert-derivation of A from Γ in the logic **L**", or more casually as "the rule $\Gamma \vdash^h A$ is *derivable* in **L**".

Definition 3 (Parenthesis conventions and defined connectives) Both $\vee$ and $\wedge$ are to bind tighter than $\to$, and so I'll usually drop parenthesis enclosing conjunctions and disjunctions whenever possible. The material conditional, relevant equivalence and the modal operator $\Box$ are defined as follows:

$$\begin{aligned} A \supset B &=_{df} {\sim}A \vee B \\ A \leftrightarrow B &=_{df} (A \to B) \wedge (B \to A) \\ \Box A &=_{df} (A \to A) \to A \end{aligned}$$

Definition 4 (Ackermann's Π') The following list of axioms and rules are a slightly more economical set than those given in **?** for the logic Π':

(Ax1)	$A \to A$
(Ax2)	$A \to A \vee B$ and $B \to A \vee B$
(Ax3)	$A \wedge B \to A$ and $A \wedge B \to B$
(Ax4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(Ax6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(Ax7)	$(A \to (A \to B)) \to (A \to B)$
(Ax8)	$(A \to B) \to ((C \to A) \to (C \to B))$
(Ax9)	$(A \to B) \to ((B \to C) \to (A \to C))$
(Ax10)	$\sim\sim A \to A$
(Ax11)	$(A \to \sim B) \to (B \to \sim A)$
(Ax12)	$(A \to \sim A) \to \sim A$
(α)	$\{A, A \to B\} \Vdash B$
(β)	$\{A, B\} \Vdash A \wedge B$
(γ)	$\{A, \sim A \vee B\} \Vdash B$
(δ)	$\{A \to (B \to C), B\} \Vdash A \to C$

In January 1959, two and a half years after the publication of **?**, Anderson and Belnap read a paper entitled *A modification of Ackermann's "rigorous implication"* (**?**) for the twenty-third annual meeting of the Association for Symbolic logic. The modification of Π' presented was the logic **E**:

Definition 5 (Anderson and Belnap's E) Anderson and Belnap's **E** consists of axioms (Ax1)–(Ax12), together with the rules (α) and (β) above, as well as the following two axioms:

(Ax13)	$((A \to A) \to B) \to B$
(Ax14)	$\Box A \wedge \Box B \to \Box(A \wedge B)$

Anderson and Belnap initially had $(((A \to A) \wedge (B \to B)) \to C) \to C$ instead of (Ax13) and (Ax14). These are equivalent axiomatizations, and so Anderson and Belnap later preferred the variant given here since it splits apart the conjunctive and implicational properties of the logic and makes the modal character of it more explicit; (Ax13) is a slightly generalized version of the modal **T** axiom $\Box A \to A$,[8] whereas (Ax14) is needed to enforce that $\Box$, on the definition given by Anderson and Belnap, interacts with conjunction in the way it should given the intended reading of $\Box$ as logical necessity. The nice feature of the simpler axiomatization, though, is that it makes it evident that **E** is a sublogic of Π'.

Ackermann augmented his logic Π' with a truth-constant, $\curlywedge$, which was to be read *das Absurde*, and defined $\Box A$ as $\sim A \to \curlywedge$. Anderson and Belnap then realized that $\Box A$ can in fact be equivalently defined as $(A \to A) \to A$ and show in **?** that $\Box$ defined this way has the modal features of a **S4** modality. Π' and **E** turn out to be theoremvise identical, and so **E** has this feature as well. Both Ackermann as well as Anderson

[8] The slight generalization is needed, as the proof below shows, for the proof of the admissibility of the necessitation rule to go through.

and Belnap wanted their logic to have modal features. Ackermann definitely wanted his logic to be able to *express* modality, whereas Anderson and Belnap thought of the *conditional* of **E** itself as a strict and relevant conditional, and therefore that the conditional itself had modal features. In fact, by adding the *Ackermann constant* **t**—by and large the negation of Ackermann's $\curlywedge$—it becomes evident that it is legitimate to read $\Box A$ not only as *A is logically necessary*, but also, and equivalently, as *A is entailed by logic*. This is further dealt with in the sequel to this paper, *Non-Boolean Classical Relevant Logics II*. The stronger logic **R**, on the other hand, was from the outset thought of as a non-modal logic.

Definition 6 (Anderson and Belnap's R) Anderson and Belnap's **R** is got by adding (Ax15), the axiom called *assertion*, to **E** and deleting the then superfluous (Ax13) and (Ax14).

$$\text{(Ax15)} \quad A \to ((A \to B) \to B)$$

It is easy to see that the modal reading of $\Box A$ is lost in **R** since $A \to \Box A$ is in fact an instance of the assertion axiom. This is why Anderson and Belnap differentiate between entailment, which they take to be a modal concept, and implication in that they talk of **E** as a (the) logic of entailment, whereas the **R** is merely a logic of relevant implication.[9]

Definition 7 ($\Pi'_{\mathbf{E}}$) My own mixture of Π' and **E**, $\Pi'_{\mathbf{E}}$, is the logic which is to be identified as axioms (Ax1)–(Ax7), (Ax10)–(Ax14) and the rules (α) and (β) above, as well as the following axioms:

$$\begin{array}{rl}
(\text{Ax}8^\flat) & (A \to B) \wedge (C \to C) \to ((C \to A) \to (C \to B)) \\
(\text{Ax}9^\flat) & (A \to B) \wedge (C \to C) \to ((B \to C) \to (A \to C)) \\
(\text{Ax}16) & (A \to B) \wedge (B \to C) \to (A \to C) \\
(\text{Ax}17) & (A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B \\
(\mathbf{K}) & \Box(A \to B) \to (\Box A \to \Box B) \\
(\mathbf{4}) & \Box A \to \Box\Box A
\end{array}$$

The motivation behind $\Pi'_{\mathbf{E}}$ is to preserve as much as possible of **E**, while making sure that (DS) becomes derivable. Most importantly it should preserve **E**'s deduction theorem, and so can't, as I will later show, simply be replaced by **E** strengthened by (γ). Adding further primitive rules beyond (α) and (β) tend to make such a deduction theorem impossible, and so the only plausible solution is to ensure that (DS) becomes a derivable rule. Thus one needs an explosive *axiom*—an axiom the addition of which suffices for making the consequence relation $\vdash^h$ explosive—which here is (Ax17). Notice that (Ax8$^\flat$) and (Ax9$^\flat$), as well as (Ax17), add an extra self-implication to the antecedent. This trick is well known amongst relevantists. It is, for instance, often pointed out that *Factor*, the formula $(A \to B) \to (A \wedge C \to B \wedge C)$ is not a theorem of **R**, but that $(A \to B) \wedge (C \to C) \to (A \wedge C \to B \wedge C)$ is. The motivation for weakening the pre- and suffixing axioms, (Ax8) and (Ax9), is purely technical in that

[9] Since this difference between entailment and implication is not a widely accepted one, I will continue to use these two concepts interchangeably.

I haven't been able to prove the variable sharing property without weakening them.[10] The conjunct $(B \to B)$ in (Ax17) is also there solely in order to ensure that the variable sharing property will hold. The modal axioms (**K**) and (**4**) are derivable in **E**, but are not so with only (Ax8$^\flat$) and (Ax9$^\flat$) available, and so are added as separate axioms to $\Pi'_{\mathbf{E}}$ in order to preserve **E**'s **S4**-modality. (Ax16) is yet another theorem of **E** for which the minor pre- and suffixing axioms of $\Pi'_{\mathbf{E}}$ prove insufficient to derive.[11]

$\Pi'_{\mathbf{E}}$ has several nice properties deemed important by Anderson and Belnap for an entailment logic to have—primarily (1) satisfying the variable sharing property, (2) having a definable **S4**-modality and (3) satisfying the relevant deduction/Entailment theorem. Despite this, $\Pi'_{\mathbf{E}}$ is rather ad hoc in that its modal features are simply superimposed and do not flow from the properties of the conditional. The foremost purpose of showing forth $\Pi'_{\mathbf{E}}$, however, is to show that it is *possible* for a truth-constant-free logic to have an explosive $\vdash^h$-relation, while also satisfying the three listed properties. In the sequel to this paper I'll show that the core idea of $\Pi'_{\mathbf{E}}$ is expressible in a much more natural way if Ackermann's truth constant **t** is available.

Before I show that $\Pi'_{\mathbf{E}}$ has the variable sharing property, I'll prove that both the necessitation rule and the (δ) rule are admissible for any extension of both **E** and $\Pi'_{\mathbf{E}}$, thus showing that the $\Box$ of $\Pi'_{\mathbf{E}}$ is, like the $\Box$ of **E**, a **S4**-modality.

Lemma 1 *Any logic considered in this essay will have the following derived rules:*

(transitivity)	$A \to B, B \to C \vdash^h A \to C$
(prefixing rule)	$A \to B \vdash^h (C \to A) \to (C \to B)$
(suffixing rule)	$A \to B \vdash^h (B \to C) \to (A \to C)$
(leftER)	$A \to (B \to C), D \to B \vdash^h A \to (D \to C)$

Proof *Left for the reader.* $\Box$

Theorem 1 *The necessitation rule $A \vdash^h \Box A$ is admissible in any $\to$-axiomatic extension $\mathbf{L}$ of either $\mathbf{E}$ or $\Pi'_{\mathbf{E}}$, i.e. if $\varnothing \vdash^h_{\mathbf{L}} A$, then $\varnothing \vdash^h_{\mathbf{L}} \Box A$.*

Proof The proof is by induction on the length of proof. In the base case, A is an axiom. Since all axioms of **L** *are $\to$-formulas we get $\Box A$ from the following general proof that $B \to C \vdash^h_{\mathbf{L}} \Box(B \to C)$:*

(1)	$B \to C$	*assumption*
(2)	$(B \to B) \to (B \to C)$	*1, prefixing rule*
(3)	$((B \to C) \to (B \to C)) \to ((B \to C) \to (B \to C))$	*Ax1*
(4)	$((B \to C) \to (B \to C)) \to ((B \to B) \to (B \to C))$	*2, 3, leftER*
(5)	$((B \to B) \to (B \to C)) \to (B \to C)$	*Ax13*
(6)	$((B \to C) \to (B \to C)) \to (B \to C)$	*4, 5, transitivity*
(7)	$\Box(B \to C)$	*6, def. of* $\Box$

[10] Incidentally, Lewis abandoned **S3** in favor of **S2** when it was pointed out to him that the suffixing axiom, (Ax9), was derivable in **S3** (see **?** for references). This is, presumably, why Ackermann mentions (Ax9) in the introduction to **?** quoted from above. It is, however, easy to verify that (Ax8$^\flat$) and (Ax9$^\flat$), with $\to$ of course replaced by $\multimap$, are theorems of **S2**.

[11] `MaGIC` finds a 12-element algebra which is a countermodel to (Ax16) for $\Pi'_{\mathbf{E}}$ minus (Ax16). `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—is an open source computer program created by John K. Slaney (**?**).

Now for the inductive part:

(α) : *Assume that A is got by applying (α), i.e. modus ponens. Thus for some formula B, both B and $B \to A$ are logical theorems. For inductive hypothesis we may assume that $\Box B$ is a logical theorem as well.*

(1)	$B \to A$	*assumed logical theorem*
(2)	$\Box B$	*assumed logical theorem*
(3)	$(B \to B) \to B$	*2, def. of* $\Box$
(4)	$(B \to A) \to ((B \to B) \to A)$	*3, suffixing-rule*
(5)	$((B \to B) \to A) \to A$	*Ax13*
(6)	$(A \to A) \to (B \to A)$	*1, suffixing-rule*
(7)	$(A \to A) \to A$	*4–6, transitivity*
(8)	$\Box A$	*7, def. of* $\Box$

(β) : *Assume that A is the formula $B \wedge C$ and is got by (β). For inductive hypothesis we may assume that $\Box B$ and $\Box C$ are logic theorems. Using (β) itself we get $\Box B \wedge \Box C$ which is the antecedent of (Ax14) which then yields $\Box(B \wedge C)$.*

$\Box$

Anderson and Belnap conceived of **E** as a logic of both necessity and relevance. The above theorem lends some support for the idea that the $\Box$ of both $\Pi'_{\mathbf{E}}$ and **E** may indeed be interpreted as a logical necessity. That interpretation is harder to justify for Ackermann's logic Π', since the necessitation rule, $A \vdash^h \Box A$ is not merely admissible, but *derivable* in Π' due to it having (δ) as a primitive rule. This was, however, precisely why Ackermann hastened to note (**?**, p. 120) that (δ) needs to be restricted so as not to apply to "non-logical" assumptions. Anderson and Belnap then realized that Ackermann's idea of restricting (δ) can be achieved by adding (Ax13) instead. In fact, (δ) is not only an admissible rule in **E**, but (Ax13) makes the variant $A \to (B \to C), \Box B \vdash^h_{\mathbf{L}} A \to C$ a derivable rule:

Theorem 2
1. *$A \to (B \to C), \Box B \vdash^h_{\mathbf{L}} A \to C$ is a derivable rule in both* **E** *and* $\Pi'_{\mathbf{E}}$.
2. *(δ) is an admissible rule in any $\to$-axiomatic extension of either* **E** *or* $\Pi'_{\mathbf{E}}$.

Proof
1.

(1)	$A \to (B \to C)$	*assumption*
(2)	$\Box B$	*assumption*
(3)	$(B \to B) \to B$	*2, def. of* $\Box$
(4)	$A \to ((B \to B) \to C)$	*1, 3, leftER*
(5)	$((B \to B) \to C) \to C$	*Ax13*
(6)	$A \to C$	*4, 5, transitivity*

2. *Assume that both $A \to (B \to C)$ and B are logical theorems. Thm.* **??** *entails that $\Box B$ is also a logical theorem, and so the above derivation shows that $A \to C$ is a logical theorem too.*

$\Box$

Neither **E** nor **R** has (γ), that is disjunctive syllogism, as a derivable rule, and so both count as *paraconsistent* logics. However, it was early realized that even though this is so, (γ) could still be *admissible* in these logics. This was one of the open problems stated in **?** and subsequently solved in the positive for both **E** and **R** in **?**. Since (Ax13) and (Ax14) of **E** are derivable in Π', and both (γ) and (δ) are admissible in **E**, it follows that these two logics are in fact theorem-vise identical.

3.1 Variable sharing

We saw earlier that Ackermann spoke of a logical connection which needs to hold between A and B in order for A to entail B. One way to specify this is as a claim about meanings, that there needs to be some connection between the meaning of A and that of B for the entailment to hold. Belnap, then, made the following suggestion in his 1960-essay *Entailment and Relevance*:

> Confining our attention to propositional logic, a partial solution becomes almost obvious once we note that in propositional logic, commonality of meaning is carried by identity of propositional variables. Thus, for A to be relevant to B in the required sense, a necessary condition is that A and B have some propositional variable in common. (**?**, p. 144)

Definition 8 A logic **L** without truth-constants has the VARIABLE SHARING PROPERTY just in case for every formula A and B, $\varnothing \vdash^h_{\mathbf{L}} A \rightarrow B$ only if A and B share a propositional parameter.

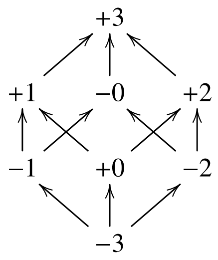
Belnap's goal was to show that **E** has this property, but also remarked (**?**, fn. 3) that the proof also covers Ackermann's Π'. It was later remarked that the model is also a model for **R**, and that therefore it as well has the variable sharing property.

To show this, Belnap constructed the 8-valued model shown in Fig. **??** in which $\mathcal{T}$ is the set of designated elements, $\sim$, $\Box$ and $\rightarrow$ are interpreted according to the displayed matrices and conjunction and disjunction are interpreted as infimum and supremum over the displayed ordering. Rules are regarded to hold in the model if they preserve designated values—if the premises are all evaluated to designated values, then the conclusion is also evaluated to some designated value.

Theorem 3 Π', **E** *and* **R** *all have the variable sharing property.*

Proof *Assume that A and B share no propositional variable. Assign to every propositional variable in A the value* $+1$, *and* $+2$ *to every variable in B. It is easy to check that both* $\{-1, +1\}$ *and* $\{-2, +2\}$ *are closed under the functions which interprets* $\sim, \Box, \wedge, \vee, \rightarrow$, *and so A will be assigned either* -1 *or* $+1$ *and B either* -2 *or* $+2$. *It is then easy to check that* $A \rightarrow B$ *will be assigned* -3. *Since the model is a model for all the axioms and rules of* Π', ***E*** *and* ***R*** *(left for the reader), it follows that* $A \rightarrow B$ *is not a theorem which then ends the proof.* □

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	−3	−2	−1	−0	+0	+1	+2	+3	$\sim$	$\Box$
−3	+3	+3	+3	+3	+3	+3	+3	+3	+3	−3
−2	−3	+2	−3	+2	−3	−3	+2	+3	+2	−2
−1	−3	−3	+1	+1	−3	+1	−3	+3	+1	−1
−0	−3	−3	−3	+0	−3	−3	−3	+3	+0	−0
+0	−3	−2	−1	−0	+0	+1	+2	+3	−0	+0
+1	−3	−3	−1	−1	−3	+1	−3	+3	−1	+1
+2	−3	−2	−3	−2	−3	−3	+2	+3	−2	+2
+3	−3	−3	−3	−3	−3	−3	−3	+3	−3	+3

Fig. 1 Belnap's model of relevance

So Π', **E** and **R** all have the variable sharing property.[12] Belnap's model validates Ackermann's (γ)-rule. However, it does not validate $\Pi'_{\mathbf{E}}$'s axiom $(A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B$, and so is not a model for $\Pi'_{\mathbf{E}}$. The following model, however, found by `MaGIC` (**?**), is a model for $\Pi'_{\mathbf{E}}$:

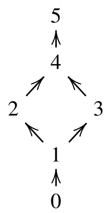
Theorem 4 $\Pi'_{\mathbf{E}}$ *has the variable sharing property.*

Proof Belnap's proof works also in this case: the model in Fig. **??** *validates all axioms and rules of* $\Pi'_{\mathbf{E}}$ *and* $\{-1, +1\}$ *and* $\{-2, +2\}$ *are still closed under all the propositional functions.* $\Box$

Belnap noted that his model of relevance validated Ackermann's (γ), i.e. disjunctive syllogism. Simply from this fact it follows that the concept of variable sharing does not entail paraconsistency. However, Anderson and Belnap were dissatisfied with Π' since they could not find a suitable deduction theorem for it. Such a deduction theorem typically holds provided the logic does not have primitive rules beyond (α) and (β) which, therefore, half-way at least, explains why Ackermann's two additional rules were cut. That $\Pi'_{\mathbf{E}}$ also has the variable-sharing property shows that an explosive logic can have this property despite having only (α) and (β) as primitive rules. I will later show that the deduction theorem they found for **E**, the *Entailment*

[12] Are there algebras with fewer than eight elements which can be used to show the same thing? The smallest such, according to Slaney's `MaGIC`, is the following six-element algebra for **R**. In it {2} and {3} do the same job as $\{-1, +1\}$ and $\{-2, +2\}$ do in Belnap's model:

$\mathcal{T} = \{1, 2, 3, 4, 5\}$

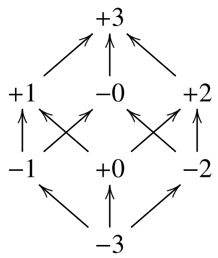


$\to$	0	1	2	3	4	5	$\sim$
0	5	5	5	5	5	5	5
1	0	1	2	3	4	5	4
2	0	0	2	0	2	5	2
3	0	0	0	3	3	5	3
4	0	0	0	0	1	5	1
5	0	0	0	0	0	5	0

FIGURE 2. A 6-element model for variable sharing

This model, however, is not a model for Π' since it does not validate (γ): $1 \wedge ({\sim}1 \vee 0) \in \mathcal{T}$, but $0 \notin \mathcal{T}$.

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	−3	−2	−1	−0	+0	+1	+2	+3	$\sim$	$\Box$
−3	+0	+2	+1	+3	+0	+1	+2	+3	+3	−3
−2	−3	+2	−3	+2	−3	−3	+2	+2	+2	−2
−1	−3	−3	+1	+1	−3	+1	−3	+1	+1	−1
−0	−3	−3	−3	+0	−3	−3	−3	+0	+0	−0
+0	−3	−2	−1	−0	+0	+1	+2	+3	−0	+0
+1	−3	−3	−1	−1	−3	+1	−3	+1	−1	+1
+2	−3	−2	−3	−2	−3	−3	+2	+2	−2	+2
+3	−3	−3	−3	−3	−3	−3	−3	+0	−3	+3

Fig. 3 $\Pi'_{\mathbf{E}}$'s model of relevance

Theorem, as well the so-called *enthymematical* deduction theorem, also holds for $\Pi'_{\mathbf{E}}$, thus putting it beyond doubt that Anderson and Belnap accepted the relevant implication of paraconsistency quite foolhardily.

Before I do so, however, let's look at the possibility of adding (Ax17) to **R**. I have not been able to decide whether $A \wedge \sim A \to B$ is derivable in this logic, only that it does not suffice for making the so-called *Mingle* axiom $A \to (A \to A)$ derivable, which when added to **R** yields an irrelevant logic since, for instance, the Kleene axiom $A \wedge \sim A \to B \vee \sim B$ is a theorem of **RM**.[13] However, if one adds the Church constant $\top$, axiomatized simply by $A \to \top$, then $A \wedge \sim A \to B$ does become derivable.[14] This shows also that the logic is at best such that the Church constant can only be added non-conservatively, which I believe is a rather unwelcomed feature. One could then rather weaken (Ax17) to

$$(A \wedge (\sim A \vee B)) \wedge ((A \to A) \wedge (B \to B)) \to B.$$

`MaGIC` verifies that one can add this to $\mathbf{R}^{\top}$ without incurring $A \wedge \sim A \to B$, but the logic would then fail to satisfy the variable sharing property as, on the assumption of $\top$'s presence, $(A \wedge \sim A) \wedge (A \to A) \to B$ becomes derivable. There might be other ways of weakening (Ax17) which could be added to $\mathbf{R}^{\top}$ without yielding an irrelevant logic, but I have so far not found one. As noted in **?**, p. 298, however, $(A \wedge (\sim A \vee B)) \wedge C \to B$, where C itself is any theorem of **R**, fails to hold in Belnap's model of relevance. Of course, that fact does not show, as Anderson and Belnap seem to imply, that there can't be any model which shows that **R** plus $(A \wedge (\sim A \vee B)) \wedge C \to B$ has the variable sharing property, where C itself is a logical theorem of the extended logic.[15] I'll come back to the this argument of Anderson and Belnap in the last section of this paper.

[13] That Mingle is not derivable in **R** plus (Ax17) is easily verified by `MaGIC`.

[14] This is easily seen by noting that $\top \to (\bot \to \bot)$ is a theorem, where $\bot =_{df} \sim \top$, and that therefore the instance $(A \wedge (\sim A \vee \bot)) \wedge (\bot \to \bot) \to \bot$ suffices for deriving $A \wedge \sim A \to B$.

[15] This is in fact in effect how the logics **Æ** and **M** in the sequel to this paper are defined for which even Belnap's original model suffice for proving the variable sharing property.

4 Relation to classical logic and deduction theorems

This section first explains the sense in which Π' and $\Pi'_{\mathbf{E}}$ extend classical logic, and how this is different from Anderson and Belnap's logics **E** and **R**. I then go on to prove that the deduction theorem for the material conditional holds for both Π' and $\Pi'_{\mathbf{E}}$ and that the enthymematic deduction theorem holds for $\Pi'_{\mathbf{E}}$, but fails for Π'.

4.1 Relation to classical logic

Let **TV** be classical logic. Both Π' and $\Pi'_{\mathbf{E}}$ extend classical logic in the sense that they are closed under the consequence relation of classical logic; intuitively, if one can derive $A_1, \ldots, A_n$ from Γ in $\Pi'(\Pi'_{\mathbf{E}})$ and B is a logical consequence of the A_i's in classical logic, then B is a logical consequence of Γ in $\Pi'(\Pi'_{\mathbf{E}})$. The following definition and theorem make this precise.

Definition 9 τ is the translation from the set FORM of formulas generated by the set of propositional variables $\{p_i \mid i \in \mathbb{N}\}$ and the connectives $\{\sim, \vee, \wedge, \to\}$ to the set of formulas generated by $\{p_i \mid i \in \mathbb{N}\} \cup \{q_{ij} \mid i, j \in \mathbb{N}\}$ over the set of connectives $\{\sim, \vee, \wedge\}$, determined by the following clauses:

- $\tau(p_i) =_{df} p_i$
- $\tau(\sim A) =_{df} \sim\tau(A)$
- $\tau(A \to B) =_{df} q_{\#_A \#_B}$
- $\tau(A \wedge B) =_{df} \tau(A) \wedge \tau(B)$
- $\tau(A \vee B) =_{df} \tau(A) \vee \tau(B)$
- $\Delta^\tau =_{df} \{\tau(A) \mid A \in \Delta\}$

where # is an enumeration of every formula in FORM and Δ is any subset of FORM.

Theorem 5 **(Classical extension)** *For* $\mathbf{L} \in \{\Pi', \Pi'_{\mathbf{E}}\}$*, if* $\Delta^\tau \vdash^h_{\mathbf{TV}} \tau(B)$*, then* $\Gamma \vdash^h_{\mathbf{L}} B$*, where* Δ *is any set of formulas such that* $\Delta \subseteq \{A \mid \Gamma \vdash^h_{\mathbf{L}} A\}$.

Proof *Left for the reader.* □

This sets Π' and $\Pi'_{\mathbf{E}}$ apart from Anderson and Belnap's **E** and **R** in which (γ) is only an *admissible* rule, which entails that, although the logical theorems of the latter two logics are extensions of that of classical logic, classical *reasoning* is not always supported by **E** and **R**. An important case where this is not so is Peano arithmetic: **?** showed that relevant arithmetic, $R^\#$, is not closed under (γ); even though $R^\# \vdash^h_{\mathbf{R}} 0 \neq 0 \vee A$ for every classical arithmetical sentence A provable in classical Peano arithmetic, Meyer and Friedman showed that there are theorems of classical Peano arithmetic which are not theorems of $R^\#$. Since $R^\# \vdash^h_{\mathbf{R}} 0 = 0$, it therefore follows that there are interesting cases where (γ) fails. Thus neither **E** nor **R** are extensions of classical logic in the way that Π' and $\Pi'_{\mathbf{E}}$ are.[16]

[16] In order for this to work for **E** one seems to need to add $A \Vdash 0 = 0 \to A$ as an additional *arithmetical* rule, where, then, $0 = 0$ gets the same logical properties as Restall's truth-constant t in **?**, namely that both $t \to A \vdash^h A$ and $A \vdash^h t \to A$ are derivable. It is shown in **?**, thm. 11.27 that $\mathbf{L}^\# \vdash^h_{\mathbf{L}} \sim t \vee A$ holds for classical Peano theorems A for a variety of contraction-free logics **L** and so easily carries over to to $\mathbf{E}^\#$ provided the extra arithmetical rule is added.

4.2 Deduction theorems

As I mentioned in section **??** there can't be a simple deduction theorem for relevant logics. There are, however, interesting weaker such theorems and I will now prove some of them.

Theorem 6 (Extensional deduction theorem) *For* $\mathbf{L} \in \{\Pi', \Pi'_{\mathbf{E}}\}$,

$$\Gamma \cup \{A\} \vdash^h_{\mathbf{L}} B \Longleftrightarrow \Gamma \vdash^h_{\mathbf{L}} A \supset B.$$

Proof The right to left direction is trivial since (DS) is a derived rule of both Π' *and* $\Pi'_{\mathbf{E}}$*. The other direction is an induction on the length of proof. So assume that* $B_1, \ldots, B_n$ *is a Hilbert proof of B from* $\Gamma \cup \{A\}$*. The proof is a simple induction showing that* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_i$ *for every i. Base case: if* B_i *is in* Γ*, then rather trivially we get* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_i$*. If* B_i *is A, then since* $A \vee {\sim}A$ *is a logical theorem, we also get that* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_i$*. Assume now for induction that* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_j$ *and* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_k$*.*

First assume that B_i *is got from* B_j *and* B_k *using adjunction (β). Using (β) we may infer that* $\Gamma \vdash^h_{\mathbf{L}} ({\sim}A \vee B_j) \wedge ({\sim}A \vee B_k)$*. Contraposing the distribution axiom (Ax4) and using (α) we get that* $\Gamma \vdash^h_{\mathbf{L}} {\sim}A \vee (B_j \wedge B_k)$*, and therefore* $\Gamma \vdash^h_{\mathbf{L}} A \supset (B_j \wedge B_k)$ *by the definition of* $\supset$*.*

B_i *obtained by (α): We may assume that* $B_k = B_j \to B_i$*. Again we get that* $\Gamma \vdash^h_{\mathbf{L}} A \supset (B_j \wedge (B_j \to B_i))$*. From the contraction axiom (Ax7) we get that* $\varnothing \vdash^h_{\mathbf{L}} B_j \wedge (B_j \to B_i) \to B_i$*, and so fiddling then yields that* $\varnothing \vdash^h_{\mathbf{L}} (A \supset (B_j \wedge (B_j \to B_i))) \to (A \supset B_i)$ *and therefore* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_i$*.*

B_i *obtained by (γ): We may assume that* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_j$ *and* $\Gamma \vdash^h_{\mathbf{L}} A \supset (B_j \supset B_i)$*. Since* $\varnothing \vdash^h_{\mathbf{L}} {\sim}B_j \vee B_j$*, we then easily get that* $\Gamma \vdash^h_{\mathbf{L}} A \supset B_i$ *which ends the proof.* □

Note that the above proof would also hold for **E** and **R** strengthened by (γ); thus all of Π', $\Pi'_{\mathbf{E}}$, $\mathbf{E}[\gamma]$ and $\mathbf{R}[\gamma]$ have the variable sharing property and a simple deduction theorem, although one using $\supset$, not $\to$. We have already seen why such a simple deduction theorem can't hold using $\to$. The following theorem shows, however, that an *enthymematical* deduction theorem holds for $\Pi'_{\mathbf{E}}$, **E** and **R**:[17]

Theorem 7 (Enthymematic deduction theorem) *For* $\mathbf{L} \in \{\Pi'_{\mathbf{E}}, \mathbf{E}, \mathbf{R}\}$,

$$\{A_1, \ldots, A_n\} \vdash^h_{\mathbf{L}} B \Longleftrightarrow \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} A_i \wedge Axioms \to B$$

where Axioms is some conjunction of axioms of the logic in question.

Proof The right to left direction is trivial. The other direction is an induction on the length of proof. So assume that $B_1, \ldots, B_n$ *is the Hilbert proof of B from* $\{A_1, \ldots, A_n\}$*. The goal is to prove by induction that* $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} A_i \wedge Axioms \to B_j$ *for every* $j \leq n$*.*

If B_j *is one of the* A_i*'s,* $\bigwedge_{i \leq n} A_i \wedge Axioms \to B_j$ *is obviously a logical theorem. If* B_j *is an axiom, then it may me assumed to be one of the conjuncts in Axioms, and so* $\bigwedge_{i \leq n} A_i \wedge Axioms \to B_j$ *is again an obvious logical theorem.*

[17] The following theorem is an easy consequence of Anderson and Belnap's Entailment Theorem which we'll get back to later. For more on deduction theorems in relevant logics, see **?**, §1.4.

Now assume that B_j is got from some B_k and B_l using (β). We may then assume for inductive hypothesis that both $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_k$ *and* $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_l$ *are theorems. Since*

$$(\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_k) \wedge (\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_l) \\ \rightarrow (\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_k \wedge B_l)$$

is an instance of (Ax5), we then easily get that $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_k \wedge B_l$*.*

Assume lastly that B_j is got from some B_k and B_l using (α), and let B_l therefore be the formula $B_k \rightarrow B_j$. From the inductive hypothesis that both $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_k$ *and* $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow (B_k \rightarrow B_j)$ *are theorems, one gets* $\bigwedge_{i\leq n} A_i \wedge Axioms \rightarrow B_j$ *using leftER and contraction (Ax7) which then ends the proof.* □

I noted above that adding (γ) to **E** or **R** suffices for yielding the ⊃-deduction theorem. That this deduction theorem fails for **E** and **R** is used by Anderson and Belnap as an argument against ⊃ being a proper conditional: "But of course $\overline{A} \vee B$ is no kind of conditional, since *modus ponens* fails for it, as we have remarked *ad nauseam* before." (**?**, p. 259). What is true at least is that they use such a petitio *ad nauseam*. An extenuating fact—one which they ought to have shown forth, but, to my knowledge at least, never did—is the fact that if one simply adds (γ) to either **E** or **R**, then the enthymematical deduction theorem will fail:

Theorem 8 *The enthymematic deduction theorem fails for* **E**[γ], Π′ *and* **R**[γ]

Proof It was noted earlier that Belnap's model of relevance (Fig. **??***) validates* (γ) *and so is a model for* **E**[γ], Π′ *and* **R**[γ]. *As* **?***, p. 298 notes, however,* $(A \wedge ({\sim}A \vee B)) \wedge Axioms \rightarrow B$ *can be made to fail in the model, where Axioms is any conjunction of axioms of* **R**.[18] *Let A and B be propositional variables, and assign* $+1$ *to A and let B, as well as every propositional variable in Axioms, be assigned to* -3. *Inspecting the model it is then easy to verify that* $\{-3, -1, +1, +3\}$ *is closed under all propositional functions, and therefore that* $(A \wedge ({\sim}A \vee B)) \wedge Axioms$ *will be assigned to* -1. *However,* $-1 \rightarrow -3 = -3$, *and so the the model is not a model for* $(A \wedge ({\sim}A \vee B)) \wedge Axioms \rightarrow B$. *Thus the enthymematical deduction theorem fails for* **E**[γ], Π′ *and* **R**[γ]. □

One might have thought that the cost of having (γ) was the enthymematical deduction theorem. The above theorem shows that the enthymematical deduction theorem is lost in some cases where (γ) is added as a primitive rule. The existence of $\Pi'_{\mathbf{E}}$ shows that this is not so in general, however. It also shows that the variable sharing property can not be used, not even in conjunction with the demand for an enthymematical deduction theorem and a **S4**-modality, to exclude ⊃ as a bona fide conditional. However, variable sharing was only deemed to be a necessary condition, whereas the Entailment Theorem was by Anderson and Belnap regarded as both necessary and sufficient.[19] Section **??** looks closer at this alleged necessary and sufficient property for relevance before section **??** takes a quick look at the concept of *suppression* before the last section finally sums up by way of looking at Anderson and Belnap's judgement over the formula $(A \wedge ({\sim}A \vee B)) \wedge Axioms \rightarrow B$ in the infamous part of their book, entitled *The Dog*.

[18] Their argument could in fact easily be extended to also cover any *theorem* of **R**.

[19] It is somewhat strange that they never tried to show that the variable sharing property follows from the Entailment Theorem. We'll see in the next section why this would have been doomed to fail.

5 Relevant deduction *aufgehoben*

In giving a proof that A is a logical consequence of the set of formulas Γ, one will typically state some of the Γ-formulas as premises, and then use the logic's resources—its axioms and rules—in order to derive A. However, a Hilbert derivation may use axioms and rules which are not conducive, so to speak, to the conclusion. For instance, "there exists a Hilbert-derivation of $A \to A$ from $B \to B$" will be a correct claim for all logics in this essay even though $B \to B$ is not *used*, in any familiar sense of the word at least, in obtaining the conclusion. The notion of a *relevant deduction*, however, is designed to tease out a more conducive notion of premise use.

Definition 10 [The relevant consequence relation of a logic] A RELEVANT DEDUCTION of a formula A from a set of formulas Γ in the logic $\mathbf{L}$ having only modus ponens, (α), and adjunction, (β), as primitive rules, is defined as a Hilbert proof $A_1, \ldots, A_n$ of A from Γ such that it is possible to mark the A_i's with #'s according to the following rules:

1. If $A_i \in \Gamma$, then A_i is marked.
2. If A_i is got from A_j and A_k using modus ponens, then A_i is marked if either or both of A_j and A_k are marked.
3. Adjunction is only used on premises which are either both marked or both unmarked.
4. If A_i is got from A_j and A_k using adjunction and both of A_j and A_k are marked, then A_i is marked.
5. No other formulas are marked.
6. As a consequence of (1–5), A_n is marked.

The existential claim that there is such a proof is written $\Gamma \vdash^r_{\mathbf{L}} A$ and expressed as "there exists a relevant derivation of A from Γ in the logic $\mathbf{L}$".

Thus a relevant derivation is a special case of a Hilbert proof. The #-markings are intended to give content to the idea that premises need to be used in arriving at the conclusion. Note that it does so both by restricting the use of adjunction and the use of axioms. An ordinary Hilbert proof does not differentiate between logical axioms and members of the premise set Γ, and so even though it is quite OK to put logical axioms into Γ, this will have no effect on the outcome. Quite the contrary is true for relevant deduction. If one thinks of the #-marked formulas as those which are, taken together,[20] conducive to the conclusion, then logical axioms are not conducive unless they are members of Γ or themselves relevantly derivable from Γ. And so one will often have to include in Γ logical axioms in order to be able to use them conducively to derive the conclusion.

Now the Entailment Theorem is a mere variant of the enthymematical deduction theorem. The latter theorem, however, states that $\Psi \vdash^h A$ if and only if the $\to$-conditional which has A as its consequent, and has its antecedent made up of the

[20] The notion of "taking together" here is the extensional-conjunctive one. There is also a stricter intensional-conjunctive sense of "taking together" which gives rise to a stricter notion of relevant deduction in which *all* the premises in Γ need to be used. Anderson and Belnap, however, prefer the notion where it is sufficient that *some* of the premises be used to obtain the conclusion (**?**, p. 36). The next subsection deals with this notion of premise use.

conjunction of some of the Ψ's together with *Axioms* is a logical truth in the sense of $\vdash^h$. This does not give any details on which axioms may enter into the conjunction *Axioms*. Even the way that *Axioms* is constructed in the proof of the theorem may produce conjunctions which are in a quite clear sense not used in the Hilbert proof of A from Ψ: for instance $A, B \to B, A$ is an acceptable Hilbert proof of A from A, but *Axioms* will in this case be the formula $B \to B$ even though no real use is made this axiom in the proof.

Note, then, that also relevant deductions can have non-conducive, as it were, formulas. For instance, $A^\#, B \to B, A^\#$ is an acceptable relevant deduction of A from A too, only with displayed #-markings. With such markings, $\vdash^r$ is supposed to hone in on a notion of premise use, and so the Entailment Theorem states that $\Psi \cup \Theta \vdash^r A$, where the formulas in Ψ are non-logical axioms, whereas Θ consists of only logical axioms, if and only if the $\to$-conditional which has A as its consequent and has its conjunctive antecedent made up of a conjunction of the Ψ's together with the conjunction of Θ's *actually used* in the relevant deduction of A from $\Psi \cup \Theta$, is a logical theorem in the sense of $\vdash^h$. To prove the theorem we first need a lemma which shows that all unmarked formulas in a relevant deduction are of a special kind:

Lemma 2 *Let* $\mathbf{L}$ *be any logic with only modus ponens and adjunction as primitive rules so that* $\vdash^r_{\mathbf{L}}$ *is defined. Assume that* $\Theta \vdash^r_{\mathbf{L}} A$ *and let* $A_1, \ldots, A_n$ *be a relevant deduction with #-markings according to the rules* (1)–(6) *in Def.* **??** *above. Claim: If* A_i *is unmarked, then* $\varnothing \vdash^h_{\mathbf{L}} A_i$.

Proof *By induction: in the base case* A_i *is a logical axiom not in* Θ, *and so* $\varnothing \vdash^h_{\mathbf{L}} A_i$. *If* A_i *is got by either adjunction or modus ponens, then since it is unmarked it is got from two unmarked premises according to* (2)–(5), *and so we may assume that the premises are logical theorems. However, since every rule is theorem-preserving, it follows that* A_i *is a logical theorem as well.*

Theorem 9 (Relevant Deduction/Entailment Theorem) *For* $\mathbf{L} \in \{\mathbf{E}, \mathbf{R}, \Pi'_{\mathbf{E}}\}$,

$$\Psi \cup \Theta \vdash^r_{\mathbf{L}} B \Longleftrightarrow \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} \psi_i \wedge \bigwedge_{i \leq m} \theta_i \to B$$

where $\{\theta_1, \ldots, \theta_m\} \subseteq \Theta \subseteq Axioms_{\mathbf{L}}$ *is the set of logical axioms of* $\mathbf{L}$ used *in the relevant deduction of B from* $\Psi \cup \Theta$, *and* $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$, *where* $\Psi \cap Axioms_{\mathbf{L}} = \varnothing$, *and the* ψ_i*'s make out the set of every such non-logical assumption* used *in the relevant deduction of B from* $\Psi \cup \Theta$.

Proof *Assume first that* $\Psi \cup \Theta \vdash^r_{\mathbf{L}} B$, *and let* $B_1, \ldots, B_k$ *be the relevant deduction of B from* $\Psi \cup \Theta$ *where the list is supplied with an analysis which legitimizes the steps according to the rules for relevant deducibility.*

Let $\psi_1, \ldots, \psi_n$ *and* $\theta_1, \ldots, \theta_m$ *be all the formulas from, respectively,* Ψ *and* Θ *which occur on the list* $B_1, \ldots, B_k$ *with justification (1). Such formulas are by definition marked by #. The rest of the proof is a simple induction to show that for all* $B_{j \leq k}$ *which are* marked *by #,* $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} \psi_i \wedge \bigwedge_{i \leq m} \theta_i \to B_j$.

Assume first that B_j *is obtained without using a rule. Since* B_j *is #-marked, it is amongst* $\{\psi_1, \ldots, \psi_n\} \cup \{\theta_1, \ldots, \theta_m\}$ *and so* $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} \psi_i \wedge \bigwedge_{i \leq m} \theta_i \to B_j$.

Now for the rules. Assume first that B_j is obtained from B_g and B_h using adjunction. Since B_j is by assumption marked, we can infer from (3)–(5) of Def. **??** *that both B_g and B_h are marked. As induction hypothesis we may therefore assume that both*

$$\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_g \quad \textit{and } \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_h.$$

Using (Ax5) one then gets that $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_g \wedge B_h$.

Lastly, assume that B_j is obtained from B_g and B_h using modus ponens, and let B_h be $B_g \to B_j$. The proof now splits into three parts since modus ponens can be use to obtain a marked formula in three different ways:

1. *Assume first that both premises are marked. We can then assume as induction hypothesis that both*

   $$\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_g \quad \textit{and } \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to (B_g \to B_j).$$

   $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_j$ now follows by using leftER and contraction (Ax7).
2. *Assume now that only B_g is marked. We may then assume as induction hypothesis that $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_g$. However, since $B_g \to B_j$ is not marked, Lem.* **??** *entails that $\varnothing \vdash^h_{\mathbf{L}} B_g \to B_j$, so $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_j$ follows using transitivity.*
3. *Lastly, assume that only $B_g \to Bj$ is marked. We may then assume as induction hypothesis that $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to (B_g \to B_j)$. However, since B_g is not marked, Lem.* **??** *entails that $\vdash^h_{\mathbf{L}} B_g$. The ($\delta$)-rule is admissible, according to Thm.* **??***.2, in any axiomatic extension of either* $\mathbf{E}$ *or $\Pi'_{\mathbf{E}}$, and so $\vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_j$.*

Thus $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B_j$ for all marked B_j's in the relevant deduction $B_1,\dots,B_k$ of B from $\Psi \cup \Theta$. However, since B is, according to (6) of Def. **??***, marked, it follows that also $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B$ which therefore ends the first half of the proof.*

Assume now that $\varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B$, where the θ's are logical axiom of $\mathbf{L}$ *and the ψ's are not. Let $H_1,\dots,H_j$ be a Hilbert derivation of $\bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B$. The following list, where square brackets are added for readability, is a Hilbert proof of B from any set $\Psi \cup \Theta$ where the ψ's are all members of Ψ and the θ's are all members of Θ:*

$$\begin{aligned}&[\psi_1],\dots,[\psi_n],[\psi_1 \wedge \psi_2],[(\psi_1 \wedge \psi_2)\wedge \psi_3],\dots,[\textstyle\bigwedge_{i\leq n}\psi_i],\\ &\quad [\theta_1],\dots,[\theta_m],[\theta_1 \wedge \theta_2],[(\theta_1 \wedge \theta_2)\wedge \theta_3],\dots,[\textstyle\bigwedge_{i\leq m}\theta_i],\\ &\qquad [\textstyle\bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i],[H_1],\dots,[\textstyle\bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i \to B],[B]\end{aligned}$$

Now all the ψ's and all the θ's and therefore any conjunction thereof can be #-marked according to (1) and (4) of Def. **??***. Thus $\bigwedge_{i\leq n}\psi_i \wedge \bigwedge_{i\leq m}\theta_i$ is marked. But then regardless of whether the H_i's are marked, we get that B is marked according to (2) of Def.* **??** *since it is got from H_j and a marked formula using modus ponens.* □

Thus $\to$ of all of $\mathbf{E}, \mathbf{R}$, and $\Pi'_{\mathbf{E}}$ expresses, in a sense, the relation of relevant deducibility. Let's look at how this plays out by looking at disjunctive syllogism for $\vdash^r_{\mathbf{L}}$. Is it the case for any of our logics that $\{A, {\sim}A \vee B\} \vdash^r_{\mathbf{L}} B$? Now we know that ($\gamma$) is not derivable in **E** or **R**, but it is in $\Pi'_{\mathbf{E}}$ as the following easy Hilbert derivation makes plain:

(1)	A	assumption
(2)	$\sim A \vee B$	assumption
(3)	$A \wedge (\sim A \vee B)$	1, 2, (β)
(4)	$B \to B$	logical axiom of $\Pi'_{\mathbf{E}}$
(5)	$(A \wedge (\sim A \vee B)) \wedge (B \to B)$	3, 4, (β)
(6)	$(A \wedge (\sim A \vee B)) \wedge (B \to B) \to B$	logical axiom of $\Pi'_{\mathbf{E}}$
(7)	B	5, 6, (α)

These lines can't, however, be marked by # according to the rules of a relevant derivation; lines 1–3 will all be marked, but line 4 can't be. The same, therefore, goes for line 5. Line 6 can't be marked, and so line 7 will not be either. However, if one only adds $B \to B$ as an additional assumption, then line 4 will be marked, and therefore line 5. Line 6 is still unmarked, but since modus ponens carries # forth provided at least one of the premises are marked, it follows that 7 will be marked as well. Thus despite $\{A, \sim A \vee B\} \nvdash^r_{\Pi'_{\mathbf{E}}} B$, we do have that $\{A, \sim A \vee B, B \to B\} \vdash^r_{\Pi'_{\mathbf{E}}} B$. In a similar vein we get that despite the fact that $\{A, \sim A\} \nvdash^r_{\Pi'_{\mathbf{E}}} B$, we do have that $\{A, \sim A, B \to B\} \vdash^r_{\Pi'_{\mathbf{E}}} B$. Thus $\vdash^r_{\mathbf{L}}$ is a paraconsistent consequence relation not only for **E** and **R**, but also for $\Pi'_{\mathbf{E}}$. Is that so for all logics? Definitely not. Let **RX** be **R** strengthened by the axiom form of disjunctive syllogism, $A \wedge (\sim A \vee B) \to B$.[21] The proof

(1)	A	assumption
(2)	$\sim A \vee B$	assumption
(3)	$A \wedge (\sim A \vee B)$	1, 2, (β)
(4)	$A \wedge (\sim A \vee B) \to B$	logical axiom of **RX**
(5)	B	3, 4, (α)

is easily seen to be a relevant proof of B from premises A and $\sim A \vee B$ in the logic **RX**. Thus $\{A, \sim A \vee B\} \vdash^r_{\mathbf{RX}} B$. Inspecting the proof of the Entailment Theorem it is easy to see that it holds for a wide class of logics: in fact any axiomatic extension of the positive fragment of $\Pi'_{\mathbf{E}}$ minus (Ax4), (Ax14), (Ax16), (**K**) and (**4**) will do.

Thus it also holds for **RX** and it even holds for classical logic. It is strange, then, that Anderson and Belnap think of the Entailment Theorem—in fact they even state the theorem only as the left to right part of what I did above—as a necessary and sufficient criterion for a logic to be truly relevant.

That the use criterion of relevance precisified by the concept of relevant deducibility is blatantly circular was noted even as early as 1974 by Meyer who in his axiomatization of **CR**—**R** with Boolean negation—points out:

[21]**?**, p. 176 claims that **R** so strengthened yields classical logic. This is not so: the weakening axiom $A \to (B \to A)$ fails in the following model for **RX**:

$\mathcal{T} = \{1, 3\}$
$[\![A]\!] = 1$
$[\![B]\!] = 3$

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] & & 2 \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	1	2	3	2
2	0	0	1	3	1
3	0	0	0	3	0

FIGURE 4. Counter-model to weakening for **RX**

> Eventually, even the worthy old *use criterion* has to go.[...] The point is rather that the use criterion motivates whatever system one wants to motivate, within very broad limits, depending on one's *antecedent* understanding of what constitutes a deduction; [...]. (Another reason why the use criterion has to go is its circularity; since the use in question is the use of *modus ponens* for $\rightarrow$, and since $\rightarrow$ is the very connective which we are using the criterion to explicate, our *pre-understanding* of $\rightarrow$ assumes an unwarranted importance. Of course, the classicist might say (though none have been *quite* so nasty), one can *use* A to get $B \vee \overline{B}$; plug A in as antecedent of the logical truth $A \rightarrow B \vee \overline{B}$. The circularity of the refutation, alas, exposes the circularity of what it refutes.) (**?**, pp. 56–57)

Now, finally someone had the guts to be that nasty. The definition of a relevant deduction makes it plain that the concept is secondary to that of the ordinary Hilbertian and its notion of a logical axiom. Anderson and Belnap preferred proof-system was a Fitch-style natural deduction calculus with an explicit premise-coding designed to keep track of dependencies. The concept of deduction in such a system is not dependent on the notion of a Hilbert proof, but Meyer finds this way to fare no better, however:

> These technical flaggings, whether or not we find pre-theoretic intuitions in which to ground them securely, do have their point; we keep track of assumptions in an E-valid deduction, in a way that causes these assumptions relevantly to entail their conclusions as the logic E determines that they shall. They have *another* point only if we can find some independent ground for the specific maneuvers. [...] Nor should it be overlooked that some of the Anderson-Belnap "natural deduction" rules are evidently cooked to motivate the corresponding E-thesis. (**?**, pp. 615–616)

This section has shown that Π'_{E} has the relevant deduction theorem. However, that concept was also shown to be both rather trivial, and to be parasitic on that of Hilbert derivability and unable to supply a justification of which logical axioms to include from the outset. The concept was intended to give the necessary and sufficient condition needed in order to avoid the so-called paradoxes of implication. Since, however, it also applies to classical logic, it evidently can not.

What it does point to, however, is the two different ways one can understand premises. The relevant sense of premise is one where nothing may be tacitly assumed; not even logical truths. The other sense is where logical truths may be suppressed. Logical consequence with the latter notion of premise can be enthymematical expressed in the object language due to the enthymematical deduction theorem, whereas the Entailment Theorem shows that the first is expressed by the relevant conditional $\rightarrow$. That Π'_{E} has both the variable sharing property, the enthymematical deduction theorem, as well as the the Entailment Theorem, shows that one can interpret explosive claims such as "everything follows from a contradiction", even when "follows from" is interpreted as an object-language conditional, charitably without committing the speaker to a fallacy of relevance. The charitable reading, then, is simply the enthymematical one, where the antecedent clause "in normal situations, where logic holds, ...” is assumed as tacitly implied. Sometimes, however, such assumptions should not be made, as when rebutting the claim by insistingly appending “that is true in *normal situations!*, but...”. This latter notion of consequence, then, forces one to make premises out of logical truths as well. This kind of *logical pluralism* is further expanded upon in the sequel to this paper.

The charitable reading alluded to here did occur to Anderson and Belnap, but they tossed it away as a confused idea of *The Man*. The last section sums up this paper by way of looking closer at the infamous section of **?** called *The Dog*. Before we get to the summary, however, I will first take a quick look at a different notion of relevant deducibility and the notion of *substructurality* as well as to look closer at the concept of suppression and the attempt at using it to ground relevance.

5.1 Substructurality

Relevant logics are often characterized as *substructural* logics. This short section gives a brief discussion of substructurality and how this relates to the notion of premise use.

Notice first of all that both $\vdash^h$ and $\vdash^r$ satisfy the three Tarski conditions of a *closure relation*, that is for $\Vdash\in\{\vdash^h,\vdash^r\}$,

$$\begin{array}{rl} \text{Reflexivity:} & \Theta \Vdash A \text{ if } A \in \Theta \\ \text{Weakening:} & \Theta \cup \Delta \Vdash A \text{ if } \Theta \Vdash A \\ \text{Cut:} & \Theta \cup \Delta \Vdash B \text{ if } \Theta \Vdash A \text{ and } \Delta \cup \{A\} \Vdash B. \end{array}$$

The three other structural properties of contraction, commutativity and associativity hold trivially since both $\vdash^h$ and $\vdash^r$ relate *sets* of formulas to a single formula.[22] A consequence relation is said to be *substructural* if at least one of these six structural properties fail. Since all hold for both $\vdash^h$ and $\vdash^r$, neither of them classifies as substructural.

Anderson and Belnap thought that the proper notion of entailment related a set of formulas, the correct premise combination of which was extensional conjunction, to a single formula. The consequence of this is that the notion of use appealed to by them is a weak one in which it is sufficient that

> *some*, but not necessarily all, of the premisses must be *used* in arriving at B. This guarantees that the conjunction of the premisses is relevant to the conclusion, which is what is required of a sensible account of entailment [. . .]. (**?**, p. 278)

They did consider the stricter notion of use which requires all premises to be used, but deem that

[22] Both reflexivity and weakening drop quite immediate from the definition of both $\vdash^h$ and $\vdash^r$. That cut holds is easily seen by the usual replacing proof: simply replace the assumption A in the proof of B from $\{A\} \cup \Delta$ by the proof of A from Θ. This will evidently be a $\vdash^h$-proof of A from $\Theta \cup \Delta$. That this also yields a $\vdash^r$-proof is seen by noting that since A is #-marked if it is used in the $\vdash^r$-proof of B from $\{A\} \cup \Delta$ and it is marked in the $\vdash^r$-proof of A from Θ, it follows that so replacing A allows every succeeding inference step to be justified by the same #-rules as the rest of the $\vdash^r$-proof of B from $\{A\} \cup \Delta$.

> if in fact a proof fails to use all the apparatus in the hypothesis, the argument is faulted on grounds of inelegance rather than logical incorrectness—and it is only the latter problem which is of overriding importance for **E**. (**?**, p. 279).

This is, essentially, the reason why their relevant consequence relation retains all the structural properties. However, there seems to be nothing essential within the relevant ideology for insisting on extensional premise combination. Relevant logics are often fitted with an intensional conjunction, the *fusion*-connective $\circ$, which then allows one to have a notion of intensional premise combination also in the object language.[23] The question, then, is whether the stricter notion of premise use can do the job of being a necessary and sufficient property for relevance.

This stricter notion of consequence can, in the case of **R**, be got by modifying the notion of a $\vdash^r$-proof slightly: use different tags for each non-logical premise and demand that each tag be preserved through the derivation to the conclusion, with uses of modus ponens carrying the tags of the premises forwards and adjunction restricted to premises with the same tags. This notion of consequence for **R** *is* substructural as weakening does not hold for it: $\{A\} \vdash^{r_2} A$ holds, but $\{A, B\} \nvdash^{r_2} A$. **?**, building on work by Moh and Church, proved that **T**, **E** and **R** all have deduction theorems on the form[24]

$$\{A_1, \ldots, A_n\} \vdash^{r_2} B \Longleftrightarrow \varnothing \vdash^h A_1 \to (A_2 \to \ldots (A_n \to B)).$$

Now the pressing question here is not if such a deduction theorem also holds for $\Pi'_{\mathbf{E}}$—a question that I will not address—but rather whether $\vdash^{r_2}$ hones in on a notion of premise use which fares better than the extensional one at play in $\vdash^r$. The answer is 'no': to see this, note that the notion of premise use also here is dependent on a prior notion of logically available resources: by simply adding the weakening axiom $A \to (B \to A)$ to **R** one gets classical logic. Here, again, it is possible to *use* both A and B to get, now in the $\vdash^{r_2}$ sense, A from the premise set $\{A, B\}$, as

$$A^{\#_1}, B^{\#_2}, A \to (B \to A), (B \to A)^{\#_1}, A^{\#_1,\#_2}$$

is a $\vdash^{r_2}$-proof of A from $\{A, B\}$ where both premises are used.

The way $\vdash^r$ and $\vdash^{r_2}$ are defined make them conceptually dependent on $\vdash^h$, something which might seem rather unsatisfactory. As already mentioned, Anderson and Belnap made use of a natural deduction calculus with indices to keep track of dependencies. If one is after a substructural notion of consequence, one might rather prefer a substructural proof theory, like the natural deduction calculus presented in **?**. Neither calculi require the prior notion of a Hilbert proof to make sense of consequence relations akin to $\vdash^r$ or $\vdash^{r_2}$. Maybe it is the link to $\vdash^h$ which so decidedly destroys the usefulness of the use criterion in the way that I have presented it. We have, however, seen that Meyer doubted the independence of the use criterion also in the case of Anderson and Belnap's favorite proof system. According to Restall, "For relevant logics the conditional $A \to B$ encodes the fact that we used A in a deduction of B. If

[23] The minimal conditions on $\circ$ are the residuation rules: $A \circ B \to C \Vdash A \to (B \to C)$ and $A \to (B \to C) \Vdash A \circ B \to C$. $A \circ B$ is not definable in **E**, but is definable as $\sim(A \to \sim B)$ in strong logics like **R**.

[24] In the case of **T** and **E**, further restrictions on the use of modus ponens are required.

we allow ourselves weakening, then we lose all sight of what was actually *used* in a deduction" (**?**, p. 26). Note, then, that also in the case of Restall's proof system for logics with weakening it is possible to use both A and B, in the intensional sense of 'and', to get A as the following proof using his system makes clear:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A; B \vdash A \circ B}\,(\circ I) \qquad \dfrac{A \vdash A}{A; B \vdash A}\,(\text{weakening})}{A; B \vdash A}\,(\circ E)$$

This section has shown that Anderson and Belnap's way of precisifying the intuitive notion of premise use, both its extensional and intensional variants, is unsuccessful in that it does not do the job Anderson and Belnap intended for it. This is not to say that it is impossible to make the notion of premise use precise and thus put it to good relevant use. At this point, however, it seems fair to claim that the burden of proof is on the adherent of the use criterion. Since this paper is dedicated in honor of Meyer, let me therefore end this section by yet another quote from his "A Farewell to Entailment":

> The pre-theoretical questions that one has about relevance—what sort of association is required for A to be relevant to B?—are also systematic questions. Answer them differently, and one will have different notions of what is relevant to what. While it is a great merit of Anderson and Belnap to have raised the question, the elusive notion will remain elusive so long as the human mind is capable of entertaining alternative views as to what entails what. [. . .] Relevance, as a component in Relevant motivation, has, over the years, been producing steadily diminishing philosophical and technical returns. While an occasional appeal thereto may still be useful in fixing the logical intuitions, nothing hangs on it, any more. To relevance, farewell! (**?**, pp. 618–619)

6 Anti-suppression to the rescue?

Anderson and Belnap proposed the variable sharing property as a necessary criterion for avoiding the paradoxes, whereas the Entailment Theorem was set forth as both necessary and sufficient. I have shown that the latter property also applies to classical logic. Since classical logic does not have the variable sharing property, however, the latter property can't be sufficient if the first is to be necessary. Giving up the variable sharing property seems out of the question, and so we are left with at best two necessary properties, but no suggestions for how to curtail the extension of 'relevant logic' beyond this.

There is another tradition within relevant logics going back to **?** which did not focused Anderson and Belnap's Entailment Theorem to the same extent. That tradition had a bent towards weaker logics than Anderson and Belnap's **E** and rather focused on the property of variable sharing as the more important property. They claim, however, that the implicational paradoxes are not simply due to violation of this property, but rather due to the more fundamental feature of *suppression*. To quote Priest's approval of their dictum: "the Routleys argue cogently that the failure of relevance, in the technical sense, is but a symptom of suppression, which is the fundamental malaise." (**?**,

p. 90). Maybe freedom from suppression, then, can be viewed as a sufficient property. The ANTI-SUPPRESSION PRINCIPLE is in **?** formulated as follows

> for every statement p there is some statement q such that the consequences of q are a proper subset of the joint consequences of p and q. There is no privileged class of statements which are generally suppressible. (**?**, p. 146)

Now 'consequences of' here can't mean Hilbert-derivability since logical theorems are suppressible for this notion of consequence . Let's therefore interpret it as a kind of relevant derivability. One suggestion, then, is to interpret the principle as follows:

Definition 11 A logic **L** is ANTI-SUPPRESSIVE just in case for every formula A, there exist formulas B and C such that $\vdash^h_{\mathbf{L}} A \wedge B \to C$, but $\nvdash^h_{\mathbf{L}} B \to C$.

Alas, as the next two results show, anti-suppresiveness turns out to be properly weaker than variable sharing, and so, contra the claims of **?**, cannot be the more fundamental phenomenon.[25]

Definition 12 A logic **L** without truth-constants has the QUASI VARIABLE SHARING PROPERTY (QVSP) just in case for every formula A and B, $\vdash_{\mathbf{L}} A \to B$ only if either A and B share a propositional parameter, or both $\vdash_{\mathbf{L}} {\sim}A$ and $\vdash_{\mathbf{L}} B$.

Theorem 10 *If a reasonable logic has the quasi variable sharing property, then it is anti-suppressive, where a logic is* reasonable *just in case (a)* $\vdash^h_{\mathbf{L}} A \wedge B \to A$ *for all formulas A and B, and (b)* $\nvdash^h_{\mathbf{L}} l$ *for all literals l.*

Proof Assume that **L** *has (QVSP). Let A be any formula. Let B be a propositional variable not in A, and let C be identical to A. Then* $\vdash_{\mathbf{L}} A \wedge B \to C$*. However, since B and C do not share any propositional variables, and furthermore,* $\nvdash_{\mathbf{L}} {\sim}B$*, it follows from the fact that* **L** *has (QVSP) that* $\nvdash_{\mathbf{L}} B \to C$*.*

Corollary 1 *The Anti-Suppressive principle is properly weaker than the variable sharing property.*

Proof The above theorem shows that Anti-Suppression is at least weaker than the variable sharing property. That it is properly weaker follows simply by noting that it is possible to satisfy the quasi variable sharing property, and therefore Anti-Suppression, without satisfying the variable sharing property. One such logic is **RM***,* **R** *with the* mingle *axiom,* $A \to (A \to A)$*, added (***?***, p. 417).*

«Parenthetical remark. Priest gives a slightly different account of suppression. Priest's claim is that strict implication is not the correct conditional

> since it allows the suppression of necessarily true antecedents and of necessarily false consequents. That is, the following are valid:
>
> $$\{L\alpha, \alpha \wedge \beta \to \gamma\} \vDash \beta \to \gamma \quad (1)$$
> $$\{L{\sim}\gamma, \alpha \to \beta \vee \gamma\} \vDash \alpha \to \beta \quad (2)$$
>
> (**?**, p. 90)

[25] There is much more that ought to be said about suppression and how it relates to intuitions behind relevant logics. This will, however, have to wait for another occasion.

'L' is here Priest's defined necessity operator. Let's translate Priests 'L' into '$\Box$' and replace $\vDash$ with $\vdash^h$ and call a logic $\Box$-SUPPRESSION FREE just in case it validates neither of (1) or (2) on this translation. Even though Priest's principle pertains to reasoning from premises which neither variable sharing nor the above anti-suppression principle does, it too is entailed by quasi variable sharing. The proof here is in fact quite similar to the one above, and so I leave it to the reader. *End parenthetical.*»

Relevant logics seem therefore to be an ill-defined family of logics. One way to solve this is to liberalize the concept to simply include any logic which has the variable sharing property for some binary connective or other. This seems to be the way that for instance Priest understands the term: "A propositional logic is *relevant* iff whenever $A \to B$ is logically valid, then A and B have a propositional parameter in common." (**?**, § 9.7.8).[26] This will then rule out classical logic as relevant, yet allow as relevant very uninteresting logics such as classical logic with only $A \to A$ added as a logic axiom, but with no rules governing $\to$. One could try to hone in on the more common usage as for instance "any extension of the first degree fragment of **E** having the variable sharing property", but that will rule out logics which do not have the extensional connectives such as the logics argued to be relevant in **?**. To also accommodate the tradition of weaker relevant logics with more rules than merely (α) and (β), it seems therefore that the concept of relevant logic is best taken to be coextensional with that of having the variable sharing property.

7 Summary: the revenge of The Man

I have in this paper shown that the two properties held forth by the Anderson-Belnap tradition of relevant logic to guard against the so-called implicational paradoxes do not force paraconsistency upon us. I showed this by showing forth $\Pi'_{\mathbf{E}}$, a logic which has both the variable sharing property and the Entailment Theorem—the two properties deemed by Anderson and Belnap to be, respectively, necessary and necessary and sufficient for avoiding the paradoxes. $\Pi'_{\mathbf{E}}$, despite having both these properties, as well as having the same modal features and enthymematical deduction theorem as Anderson and Belnap's favorite logic **E**, has disjunctive syllogism as a derivable rule for its Hilbert consequence relation. Even though the variable sharing property—as well as the weaker concept of suppression-freedom—is a substantive property, and does seem to do the curtailing job needed to block the implicational paradoxes with regards to the $\to$-formulas, it is not sufficient to block the logical truth of the material paradoxes, nor the strict paradoxes; all of

$$\text{(PM1)}\quad (A \supset B) \vee (B \supset A) \quad \text{(PM2)}\quad {\sim}A \supset (A \supset B) \quad \text{(PM3)}\quad A \supset (B \supset A)$$
$$\text{(PS1)}\quad A \wedge {\sim}A \prec B \quad \text{(PS2)}\quad A \prec B \vee {\sim}B$$

where the strict conditional is defined as usual, are in fact logically true in $\Pi'_{\mathbf{E}}$. Neither $\supset$, $\prec$, nor the enthymematically suppressed conditional expresses the relevant consequence relation, however, and it was only for this consequence relation that relevant

[26]This is also how **?**, p. 4 defines *relevant logic*.

restrictions were ever intended to apply. This holds despite the result shown in this paper that the Entailment Theorem can't be regarded as a necessary and sufficient condition for avoiding the paradoxes since it holds for pure classical logic.

$\Pi'_{\mathbf{E}}$ does not extend **E** and its modal features do not flow in a natural way from the properties of the conditional in the same way as is the case with **E**. This *ad hoc* feature might be reason to regarded $\Pi'_{\mathbf{E}}$ as a poor replacement for **E**. A natural explosive logic which does extends **E**—**Æ**—as well as a stronger sibling, **M**, which extends **R**—is presented in the sequel to this paper. The point of this one being first and foremost to bring the point home that considerations of relevance do no entail paraconsistency as commonly thought. This now being established, it is worth, then, to look a bit closer at the highly rhetorical—and plainly false as it turns out—claims made by Anderson and Belnap concerning disjunctive syllogism.

Anderson and Belnap picked up a story told by Sextus Empiricus in which one is told about a dog tracking an animal when coming to a fork in the road.

> after smelling at the two roads by which the quarry did not pass, he rushes off at once by the third without stopping to smell. For, says the old writer, The Dog implicitly reasons thus: "The creature went either by this road, or by that, or by the other: but it did not go by this road or by that: therefore it went by the other." (**?**, p. 296)

Now Anderson and Belnap claim that the *or* in this tale is intensional. If one defines the *fission* connective + as simply $\sim A \rightarrow B$, then surely disjunctive syllogism will be valid, since $A, \sim A + B \vdash^{h} B$ is simply modus ponens for $\rightarrow$ together with a little negation-fiddling. Thus The Dog makes no inferential mistake in using fission-disjunctive syllogism in this tale since it amount simply to modus ponens for $\rightarrow$. What they didn't accept, however, was that disjunctive syllogism should hold in conditional form with extensional conjunction and disjunction. It is in this context that they claim that in using disjunctive syllogism one is committing an inferential blunder on either of the two interpretations which assigns to the inferer the inference expressed by $A \wedge (\sim A \vee B) \rightarrow B$ or that expressed by $(A \wedge (\sim A \vee B)) \wedge Axioms \rightarrow B$:

> we do hold that the inference from $\overline{A}$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make (see §25.1, "The Dog"). Such an inference commits nothing less than a fallacy of relevance. (**?**, p. 165)

They back this claim, though, by pointing out that $A, \sim A \vee B \nvdash^{r}_{\mathbf{E}} B$. Even when given the charitable reading that what the inferer—derogatorily called *The Man* by Anderson and Belnap—intended was that A and $\sim A \vee B$, together with some conjunction of axioms, entail B, Anderson and Belnap retort that this is also an inferential blunder since $\nvdash^{h}_{\mathbf{E}} (A \wedge \sim (A \vee B)) \wedge Axioms \rightarrow B$. It is hard, then, not to simply view all of Anderson and Belnap' brawling claims of fallacy of relevance regarding disjunctive syllogism as a mere *petitio*. *Provided* that their favorite logic gives a correct account of entailment, then disjunctive syllogism is a hotbed for fallacies. But then again, any rule is in any logic in which it is not valid. I have shown, however, that Anderson and Belnap's two criteria for relevance—variable sharing and relevant deduction—do not in any way show that the inference licensed by disjunctive syllogism commits one to a fallacy of relevance. They are correct that if the claim was that expressed by the sentence $A \wedge ({\sim}A \vee B) \to B$, then that claim can't be logically true if variable sharing is to be upheld. This is simply because the logical truth of $A \wedge ({\sim}A \vee B) \to B$ suffices for the logical truth of $A \wedge {\sim}A \to B$ which clearly violates the variable sharing property. So if what "The Man" intended was that B follows from A and ${\sim}A \vee B$ in the pure $\vdash^r$-sense, then one might more plausibly charge him of committing a fallacy of relevance. Relevant logics have, however, two notions of *therefore*—the Hilbertian consequence relation and the relevant one. Because of the deduction theorems—the Entailment theorem with regards to $\vdash^r$ and the enthymematical deduction theorem in the case of $\vdash^h$—**E**, **R** as well as $\Pi'_{\mathbf{E}}$ can express these consequences within the object language. In the case of $\Pi'_{\mathbf{E}}$, $A \wedge ({\sim}A \vee B) \to B$ does express the, in general false, claim that B follows from, in the $\vdash^r$ sense of follows from, A and ${\sim}A \vee B$, whereas both $(A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B$ and $A \wedge ({\sim}A \vee B) \supset B$ express that B so follows, in the $\vdash^h$ sense of follows, but fall short of expressing that B follows from A and ${\sim}A \vee B$ in the stronger $\vdash^r$ sense. One may still claim that it is a fallacy of relevance to infer B from *only* A and ${\sim}A \vee B$, in the pure $\vdash^r$ sense of inferring. However, asserting either $A \wedge ({\sim}A \vee B) \supset B$ or $(A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B$, and therefore that B does follow from A and ${\sim}A \vee B$ in either the $\vdash^h$ sense of "follows from" or in the suppressed $\vdash^r$ sense, does not, as I've meticulously shown in this paper, commit one to a fallacy by any standard put forward by Anderson and Belnap, nor by the suppression-tradition of the Routleys. Thus no inferential blunder need be involved in inferring B from A and ${\sim}A \vee B$.

That this has gone more or less unnoticed by for so long should be cause enough rethink the value of the polemical tone initiated by Anderson and Belnap, which has, in my view, hindered their own research project and reception thereof considerably. To use the John Wisdom quote which Anderson and Belnap make use of themselves: "It's not the stuff, it's the style that stupefies." (**?**, p. 261).

And so, by rejecting error,
Dog finds the truth.

The Bestiarist[27]

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[27] The quote is from *The Book of Beasts*. Here quoted from **?**, p. 297.

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