The Stability and Instability of Solitary Waves

Philosophiae Doctor Thesis

Nguyet Thanh Nguyen

Department of Mathematics
University of Bergen
Norway

January 2010

Preface

This thesis contains works that have been carried out as part of Ph.D. program at the Department of Mathematics, University of Bergen, Norway. The studies started in January 2006, and I have conducted all of my works in Bergen University.

My thesis advisor is Prof. Henrik Kalisch at the Department of Mathematics, University of Bergen. Funding has been provided by Bergen University Research Fellowship.

The thesis research is numerical and theorical investigations of the stability of solitary-wave solutions of nonlinearly dispersive wave equations. In fact, numerical investigation play a crucial role in the study of how instabilities develop. Overall, we have produced five papers included in the thesis. All of them are written in collaboration with Prof. Kalisch, featuring numerical experiments done on a variety of nonlinear wave equations, and applications of the methods of Benjamin, Grillakis, Shatah, and Strauss to prove stability and instability of solitary waves in a variety of contexts.

Acknowledgements

First of all, I thank my supervisor, Prof. Henrik Kalisch at the Department of Mathematics, University of Bergen, for his substantial contribution (both as a thesis advisor and co-author of papers) to the thesis, including introduction of interested problems (which are reflected on papers in the thesis) and fruitful advice on difficult calculations that came up along the research.

The General Background is written while I'm in a preparation for an USA-NSF Postdoctoral Research Fellowship application. I would like to thank Prof. John Albert at the Department of Mathematics, University of Oklahoma, for his cooperation in writting a proposal (to the NSF) so that inspirations for future works are reflected throughout the Research section.

Overall, I must thank for the funding has been provided by Bergen University Research Fellowship 2006 – 2010.

Finally, I would like to thank Hung, my brother, who introduced Bergen University to me, and my two sisters, Phuong and Lien, who have maintained our mother during the time I have been away from home.

Nguyet Thanh Nguyen
Bergen, December 2009.

In memory of my father and to my mother.

Contents

I General Background **1**

1 Introduction **3**

2 The discovery of solitary waves **7**

3 Orbital stability and the general stability theory of solitary waves **11**

3.1 Orbital stability 11

3.2 The general stability theory 12

4 Research **15**

4.1 The stability and instability of negative solitary waves for the generalized BBM equation 15

4.2 The stability of solitary waves for the extended KdV equation . . . 18

4.3 Instability of solitary waves for a generalized version of the Camassa-Holm equation 20

A Appendix **23**

A.1 Derivation of the KdV and BBM equations 23

Bibliography **29**

II Papers **31**

A The stability of solitary waves of depression **33**

B Orbital stability of negative solitary waves **49**

C Stability of negative solitary waves **65**

D On the stability of internal waves **87**

Part I

General Background

Chapter 1

Introduction

The thesis research is on the qualitative properties of solitary-wave solutions, which have been long attracted interest because of their fundamental character, to one-dimensional nonlinear dispersive wave equations. Nonlinearly dispersive waves are important in many different fields of science and engineering, and it is often the case that the same mathematical equation will describe many different physical settings. Specific examples of nonlinear dispersive waves in nature are water waves, atmospheric waves, waves in elastic solids, pulses in optical fibers, and waves in plasmas, to name a few.

The type of equations being considered are those which arise as models for phenomena in which the wave amplitude is large enough that linearized approximations are no longer valid, but still small enough that simplifications are possible in the nonlinear terms of the underlying equations. In general, we restrict attention to one-dimensional phenomena (long-crested waves) and the equations have the form of an abstract Hamiltonian system

$$u_t = JE'(u), \tag{1.1}$$

where a solution $u(x,t)$ is a real-valued function of $x \in \mathbb{R}$ and $t \geq 0$. Here $E'(u)$ denotes the Fréchet derivative of E at u, where E is a certain functional; and J is a skew-symmetric linear operator, and thus, $E(u)$ is invariant with respect to time under the action of the evolutionary equation (1.1).

The balance of dispersion and nonlinearity in (1.1) makes possible solutions with coherent structure. Of particular interest among these solutions u are travelling waves, which have the form $u(x,t) = \Phi_c(x - ct)$, where the wavespeed $c \in \mathbb{R}$. If the profile function $\Phi_c(\xi)$ is that of a single hump which approaches zero as $|\xi| \to \infty$, the travelling-wave solution is called a *solitary wave*. On the other hand, $\Phi_c(\xi)$ could be a periodic function of ξ, in which case u is called a *periodic travelling wave.*

In this thesis, we are generally interested in the initial-valued problem for (1.1), where initial data $u(0,t) = g(x)$ is given which lies in some function space X and a solution $u(x,t)$ is sought which, as a function of x, lie in X for all $t > 0$. (Natural choices for X are often Sobolev spaces whose order is determined by Hamiltonians or energy invariants associated with the equation.) We aim to clarify the role of solitary waves in the evolutions of general solutions of the initial-valued problem.

It is remarked that, for the Korteweg-de Vries (KdV) equation, a typical example of an equation of type (1.1),

$$u_t + uu_x + u_{xxx} = 0, \tag{1.2}$$

where $J = \partial_x$ and $E(u) = \int_{-\infty}^{\infty}(-\frac{1}{6}u^3 + \frac{1}{2}u_x^2)dx$, powerful theoretical methods of solutions have been developed which lead one to expect that the general solution will consist of a finite number of solitary waves or periodic travelling waves interacting with each other, and not much else. In fact, results to this effect for KdV have been proved under the assumption that the solutions involved are smooth and decay rapidly at infinity for all time. However, for general solutions in Sobolev spaces such as H^1, where KdV is known to be globally well-posed, we are still far from being able to prove such a result (see, e.g., the survey article of Tao in [21]).

For other equations of type (1.1), although one expects solitary waves to play similar important role in general solutions as for KdV, even less is known. For example, even though the existence of smooth solitary waves for a generalized version of the Camassa-Holm (gCH) equation - an equation of type (1.1)*,

$$u_t + \omega u_x + 3uu_x - u_{xxt} = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{1.3}$$

is well-known (see, e.g, [14] and Paper E), but their explicit formulas remain unknown. What has been possible and is investigated in this thesis, though, for many equations of type (1.1), is to prove the *stability* (or, in some cases, instability) of the solitary waves; that is whether they are relatively unaffected (or affected) by perturbations. Such results give at least a good idea of the behavior of general solutions in the neighborhood of these important solutions in X. The given analytical proof of the stability (or instability) is mainly based on the general theory of Grillakis, Shatah, and Strauss in [11]. Numerical simulation is also performed, which aimed at determining the behavior of solutions once they leave the regions of X described by the stability theory.

Finally, we recall the story of how solitary waves were first appearing in the scientific scene in the next chapter. After that, we give a precise definition of the stability and briefly review of the general stability theory for the solitary waves

* Here $J = \frac{\partial_x}{1-\partial_x^2}$ and $E(u) = -\frac{1}{2}\int_{-\infty}^{\infty}(u^3 + \gamma u u_x^2 + \omega u^2)dx$.

to the evolutionary partial differential equations of type (1.1) due to Grillakis, Shatah, and Strauss [11] in Chapter 3. Then in Chapter 4, an introduction of the main contributions of the thesis are represented, while their results are appearing in five papers (Paper A, B, C, D, and E) in Part II. In addition, to close Part I and gain physical meaning of the equations, which are appearing in the thesis, a derivation of the KdV and regularized long-wave equations is also given in the Appendix A.1.

Chapter 2

The discovery of solitary waves

In this chapter, we will see how the solitary waves first appeared on the scientific scene. The work is based on the materials of Drazin and Johnson [10], and the Wikipedia website [23]. It will also be seen that the KdV equation not only is of mathematical interest but also is of practical one. In fact, KdV is indeed the relevant one for the solitary waves (and much more besides).

The solitary wave is the so-called wave because its shape occurs as a single hump and is localised, which was firstly observed by John Scott Russell (9 May 1808 – 8 June 1882) on the Edinburgh-Glasgow canal in 1834. He called it the 'great wave of translation' and reported his observations to the British Association in his 1844 'Report on Waves' in the following words:

I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300–450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [23 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

John Russell also performed some laboratory experiments. He built wave tanks at his home and generated solitary waves by dropping a weight at one end of a water channel. He was able to deduce empirically that the volume of water in the wave is equal to the volume of water displaced. Furthermore, the speed c of the solitary wave is obtained by the relation

$$c^2 = g(h+a), \tag{2.1}$$

where a is the amplitude of the wave, h is the undisturbed depth of water, and g is the acceleration of gravity. The solitary wave is therefore a *gravity wave*. It is immediately an important consequence of equation (2.1): higher waves travel faster (apply to waves of elevation).

In 1871, Joseph Bousinesq mentioned Scott Russell's name in his paper. And in 1876, Lord Rayleigh published a paper in Philosophical Magazine to support John Scott Russell's experimental observation with his mathematical theory. To put Russell's formula (2.1) on a firmer footing, both Bousinesq and Rayleigh assumed that a solitary wave has a length scale λ much greater than the depth h of the water, i.e, $h/\lambda \ll 1$. They deduced Russell's formula for c (from the equations of motion for an inviscid incompressible fluid). In fact, they also showed that the wave profile $z = \zeta(x,t)$ is given by

$$\zeta(x,t) = a\,\text{sech}^2[\beta(x-ct)], \tag{2.2}$$

where

$$\beta^{-2} = 4h^2(h+a)/3a \text{ for any } a > 0, \tag{2.3}$$

(although the sech^2 profile is strictly correct only if $a/h \ll 1$.) However, these authors did not write down a simple equation for $\zeta(x,t)$ which admits (2.2) as a solution. This final step was completed by Diederik Korteweg and Gustav de Vries in 1895. (However, Korteweg and de Vries did not mention John Russell's name at all in their 1895 paper but they did quote Boussinesq's paper in 1871 and Lord Rayleigh's paper in 1876. Although the paper by Korteweg & de Vries in 1895 was not the first theoretical treatment of this subject, it was a very important milestone in the history of the development of soliton theory.) They showed that, provided ε was small, then

$$\zeta_t = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\frac{2}{3}\varepsilon\zeta_\chi + \zeta\zeta_\chi + \frac{1}{3}\sigma\zeta_{\chi\chi\chi}\right), \tag{2.4}$$

where χ is a coordinate chosen to be moving (almost) with the wave, and the parameter σ incorporates the surface tension, τ, in the form

$$\sigma = \frac{1}{3}h^3 - \frac{\tau h}{g\rho}, \tag{2.5}$$

where ρ is the density of the liquid, and ε is an arbitrary parameter. For interests of how the equation (2.4) was derived (in dimensionless form and in the absence of surface tension), the reader is referred to see the Appendix A.1.

Finally, let us see one connection between the Korteweg-de Vries (KdV) equation (2.4), the Russell wavespeed formula (2.1), and the sech^2 profile (2.2), under the assumption $a/h \ll 1$, as follows. If the solution of equation (2.4) is stationary in the frame then $\zeta = \zeta(\chi)$ and so, the following ordinary differential equation appears

$$\frac{2}{3}\varepsilon\zeta' + \zeta\zeta' + \frac{1}{3}\sigma\zeta''' = 0,$$

where the prime denotes the derivative with respect to χ. Next, if we consider $\zeta \to 0$ as $|\chi| \to \infty$ (as in the case for the solitary wave) then this equation can be integrated to yield

$$\frac{2}{3}\varepsilon\zeta + \frac{1}{2}\zeta^2 + \frac{1}{3}\sigma\zeta'' = 0.$$

Then, integrate this equation once more with the integrating factor ζ', there appears

$$2\varepsilon\zeta^2 + \zeta^3 + \sigma(\zeta')^2 = 0.$$

It is elementary to verify directly by substitution that this equation admits a solitary-wave solution of the form

$$\zeta(\chi) = a\,\text{sech}^2(\beta\chi),$$

provided

$$a = 4\sigma\beta^2 \text{ and } \varepsilon = -2\sigma\beta^2. \tag{2.6}$$

Therefore, once the coordinate χ is defined (Korteweg & de Vries, 1895) as

$$\chi = x - \sqrt{gh}\left(1 - \tfrac{\varepsilon}{h}\right)t,$$

the solitary-wave solution becomes

$$\zeta(x,t) = a\,\text{sech}^2\left\{\tfrac{1}{2}\sqrt{\tfrac{a}{\sigma}}\left[x - \sqrt{gh}(1 + \tfrac{a}{2h})t\right]\right\}.$$

It immediately implies that the wavespeed has a form of

$$c \sim \sqrt{gh}\left(1 + \tfrac{a}{2h}\right).$$

Therefore,

$$c^2 \sim g(h+a) + O(a/h),$$

and this agreed with the c formula of Russell if we assume that $a/h \ll 1$. Besides, if we neglect the surface tension (so that $\sigma = \frac{1}{3}h^3$ [cf. (2.5)]), it can be seen from (2.6):

$$\beta \sim \frac{1}{2}\sqrt{\frac{3a}{h^3}},$$

and this also agreed with the work of Boussinesq and Lord Rayleigh. To see this, rewrite their formula (2.3) as

$$\beta^2 = \frac{3a}{4h^2(h+a)} = \frac{3a}{4h^3(1+a/h)} \sim \frac{3a}{4h^3} \text{ [cf. } a/h \ll 1].$$

Chapter 3

Orbital stability and the general stability theory of solitary waves

A precise definition of the stability together with its general theory for the solitary waves are explained in this Chapter.

3.1 Orbital stability

Now, we explain more precisely what we mean by stability. Consider the KdV equation (1.2) set in the space given by $X = H^1(\mathbb{R})$, the Sobolev space of functions in $L^2(\mathbb{R})$ whose first derivatives are also in $L^2(\mathbb{R})$. It is known that solitary-wave solutions to KdV exist if and only if $c > 0$, and are unique (up to translation) for given wavespeeds c. That is, $u(x,t) = \Phi_c(x - ct)$ is a solution of KdV in $H^1(\mathbb{R})$ if and only if $c > 0$ and $\Phi_c(\xi)$ belongs to the set $\mathbf{M}$ of *translates* of the fixed function $3c\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}\,\xi\right)$.

It is also known that the set $\mathbf{M}$ is a *stable* set for the initial-value problem for KdV in $H^1(\mathbb{R})$. This means that for every $\varepsilon > 0$, there exist $\delta > 0$ such that if $g \in H^1(\mathbb{R})$ and g is within δ of $\mathbf{M}$ in the H^1-norm, then the solution $u(\cdot,t)$ of the initial-value problem with data $u(x,0) = g(x)$ will exist, and will stay within ε of $\mathbf{M}$ in H^1-norm for all $t > 0$. This result, first proved by Benjamin and Bona [1, 4] (for the stability part) and Kenig, Ponce and Vega [17] (for the existence part).

In general, for any equation of type (1.1), we say that a solitary-wave solution Φ_c is stable if the set $\mathbf{M}$ of its translates is stable in the above sense. Otherwise, it is *unstable* if $\mathbf{M}$ is not stable. Of course, such a result is more useful if $\mathbf{M}$ is of minimal size: we would like to identify the smallest possible set containing the orbit of Φ_c.

3.2 The general stability theory

The main points of the general stability theory for the solitary waves to the evolutionary partial differential equations of type (1.1) due to Grillakis, Shatah, and Strauss [11] is reviewed in this section. (For illustration of the theory apparently in most of the details, the reader is referred to read Paper C, where we applied the theory on the stability and instability of negative solitary waves for the generalized BBM equation explicitly.)

First of all, when the ansatz $u(x,t) = \Phi_c(x-ct)$ is substituted into equation (1.1), there would appear an ordinary differential equation, which turns out to have a form of the Larange's equation

$$E'(\Phi_c) = -cV'(\Phi_c), \tag{3.1}$$

where V is also another constant of the motion for (1.1) which controls the X-norm, and which makes it easy to prove global existence of solutions in X once local existence is known. Here $E'(\Phi_c)$ and $V'(\Phi_c)$ are Fréchet derivative at Φ_c of E and V, respectively. In other words, the solitary waves can be characterized as critical points of the energy E restricted to the level sets of the momentum V, with $-c$ acting as a Lagrange multiplier.

Preliminaries. Now, determining the optimal translation τ_α for a given solitary wave Φ_c and a perturbation u can be achieved by choosing $\alpha \in \mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \Big\{ u(\xi + \alpha(u)) - \Phi_c(\xi) \Big\}^2 d\xi = \inf_{a\in\mathbb{R}} \int_{-\infty}^{\infty} \Big\{ u(\xi + a) - \Phi_c(\xi) \Big\}^2 d\xi,$$

where $\xi = x - ct$ and if this infimum exists. Now, if the integral on the right is a differentiable function of a, and if u is sufficiently closed enough to a translation of Φ_c; that is u is in a sufficiently small ε-neighborhood of $\mathbf{M}$ in the X-norm, then $\alpha(u)$ can be determined by solving the equation

$$\big\langle u(\cdot + \alpha(u)), \Phi' \big\rangle = 0. \tag{3.2}$$

In fact, this result is obtained by using the implicit function theorem. Next, we pay close attention to the following linear operator

$$\mathscr{L}_c = E''(\Phi_c) + cV''(\Phi_c),$$

which is assumed to have only *one simple negative eigenvalue whose corresponding eigenfunction* χ_c, *one simple zero eigenvalue with* Φ_c' *as its corresponding eigenfunction (which could be seen by equation* (3.1)*), and the rest of its spectrum is positive, continuous, and bounded away from zero.* Now for a given wavespeed

c, the stability of the corresponding solitary wave Φ_c is determined by the convexity of the scalar function

$$d(c) = E(\Phi_c) + cV(\Phi_c).$$

In particular if $d''(c) > 0$, then it can often be shown that the solitary wave is stable, while if $d''(c) < 0$, the solitary wave is expected to be unstable. We remark that at a critical speed c^* such that $d(c^*) = 0$, the methods of Grillakis/Shatah/Strauss give no information about the stability of Φ_c. Otherwise, the analysis of the stability and instability theory is described as follows.

The stability theory. Now, for a given wave speed c, if the scalar function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is a convex function, that is if $d''(c) > 0$, then it can be derived that: There exists a positive constant β such that a conditional coercivity of the bilinear form $\langle \mathscr{L}_c z, z\rangle \geq \beta \|z\|_X^2$ holds for all nonzero $z \in X$ for which z is orthogonal both $V'(\Phi_c)$ and Φ_c'. Consequently, it can be shown that the functional $E(u)$ attains its local minimum at Φ_c subject to the constancy of $V(u)$. (Note that the latter condition could be removed by an additional scaling argument). And this fact turns out to imply the unconditional orbital stability of Φ_c (which would follow from a standard Lyapunov function argument) with respect to small but finite perturbations.

The instability theory. Recall that χ_c is an eigenfunction of $\mathscr{L}_c$ with the corresponding negative eigenvalue. Now for a given wavespeed c, if a scalar function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is concave, that is if $d''(c) < 0$, then it can be shown that:

> *There exists a curve $\nu \mapsto \Psi_\nu \equiv \Phi_\nu + s(\nu)\chi_c$ in a neighborhood of c such that $\Psi_c = \Phi_c$ $\big(s(c) = 0\big)$, $V(\Psi_\nu) = V(\Phi_c)$ for all ν, and where the functional E attains its local maximum at Φ_c, i.e, $E(\Psi_\nu) < E(\Phi_c)$ for all $\nu \neq c$.*

Note that the existence of the curve $\nu \mapsto \Psi_\nu \equiv \Phi_\nu + s(\nu)\chi_c$ in a neighborhood of c will be demonstrated by solving the following equation for a C^1-function $s(\nu)$:

$$V(\Phi_\nu + s\chi_c) - V(\Phi_c) = 0.$$

We remark that this process is allowed theorically by using the implicit function theorem if the following condition

$$\partial_s V(\Phi_\nu + s\chi_c)\Big|_{\nu=c,s=0} = \langle V'(\Phi_c), \chi_c\rangle \neq 0.$$

is assumed to be satisfied by the given evolutionary equation.

Recall that **M** is the minimal set containing translates of the solitary wave. Now, let $\varepsilon > 0$ sufficiently small be given. For a solution $u(x,t)$ with its initial data $u(x,0) = u_0$ stay within ε of **M**, denote $[0,t_1)$ as the maximal time interval for which $u(\cdot,t)$ stay within ε of **M**. Instability will be proved by showing that $t_1 < \infty$. This will be done via a real Lyapunov functional L, which is defined on ε of **M** by the following integral

$$L(t) = \int_{-\infty}^{\infty} Y(x - \alpha(u(t))u(x,t)dx, \tag{3.3}$$

where $\alpha(u)$ is defined in equation (3.2), and Y is defined for which $JY = y \equiv \frac{d\Psi_\nu}{d\nu}\Big|_{\nu=c} = \frac{d\Phi_c}{dc} + s'(c)\chi_c$. Firstly, observe that the integral in (3.3) converges. Indeed, an upper bound on the growth of $L(t)$ can be given as follows:

> *There is a positive constant D such that $|L(t)| \le D(1+t^{\zeta})$, for $0 \le t < t_1$, and where $0 < \zeta < 1$.*

We remark that the key elements in this estimate is that fact of

$$\sup_{z\in\mathbb{R}} \left| \int_z^{\infty} u(x,t)dx \right| \le C(1+t^{\zeta}),^*$$

for a positive constant C and $0 < \zeta < 1$, and also by an assumption of $y = \frac{d\Phi_c}{dc} + s'(c)\chi_c \in L^2(\mathbb{R})$ together with $(1+\sqrt{|\xi|})y \in L^1(\mathbb{R})$, and thus an assumption of both $\frac{d\Phi_c}{dc}$ and χ_c have exponential decay as $|\xi| \to \infty$ will be sufficiently satisfied.

On the other hand, base on the concavity of d, and thus the fact of E attains its local maximum at Φ_c on the curve Ψ_ν, a lower bound of $L(t)$ can also be obtained by giving an estimate of its derivative:

$$\left|L'(t)\right| > m,$$

for a positive constant m. Therefore,

$$2D(1+t^{\zeta}) \ge |L(t)| + |L(0)| \ge \int_0^t |L'(s)|ds > \int_0^t m\,ds = mt,$$

for $t \in [0,t_1)$. However, since $\zeta < 1$, the rate of growth of the curve $f(t) = 2D(1+t^{\zeta})$ is less than the rate of growth of the line $l(t) = mt$. Consequently, t_1 must be the point where these two curves meet, and thus $t_1 < \infty$. □

* for a particular proof of this fact in the case of the gCH equation (1.3), the reader is referred to read Theorem 2.1 of the Paper E.

Chapter 4

Research

This chapter contains the thesis research, which is classified into three categories as describing in the following three sections. The first one is a numerical study and theorical investigation of the stability and instability of the *negative* solitary-wave solutions of the generalized BBM equation. The results are appearing in Paper A, B, and C, and in which Paper B presents numerical simulation, whereas theorical results are shown in the other two papers.

4.1 The stability and instability of negative solitary waves for the generalized BBM equation

The generalized Benjamin-Bona-Mahony (gBBM) equation

$$u_t + u_x + (u^p)_x - u_{xxt} = 0$$

where $p \geq 2$ is an integer, and which is a model equation for nonlinear waves. For the case $p = 2$, it is the so-called Benjamin-Bona-Mahony (BBM) equation, which is closely related to the KdV equation, and where the equation is used as a model for surface water wave of small amplitude and long wavelength on shallow water. For interests of how the BBM equation was derived from the KdV equation, the reader is referred to see the Appendix A.1. Besides, the gBBM equation also arises in some physical situations for other integral values of p.

Like KdV, gBBM has solitary waves with positive profiles for all wavespeeds $c > 1$. However, unlike KdV, gBBM also has solitary waves with *negative* profiles, which are corresponding to either *negative* wavespeeds if p is *even* or, *positive* wavespeeds if p is odd. Naturally, the questions arise what happens for those wavespeeds $c \in (0,1]$, and for those wavespeeds $c < 0$ with p is *odd*. In Paper C, we attempt to answer these questions, and it turns out that there are *no* non-trivial solitary waves for these considered cases. In fact, like KdV solitary waves, the apparently explicit expressions of all gBBM solitary waves (both positive and negative) are well-known. Overall, Figure 4.1 summarizes all of the explicit existence of gBBM negative and positive solitary waves, which propagate with both negative and positive velocities.

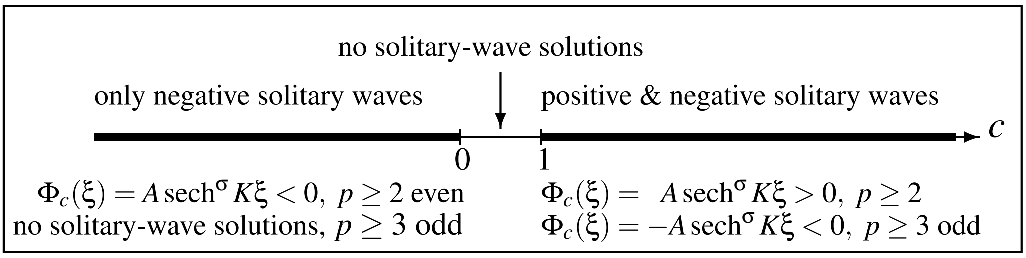

Figure 4.1: Solitary-wave solutions of gBBM equation. Here $A = [\frac{(p+1)(c-1)}{2}]^{1/(p-1)}$, $K = \frac{p-1}{2}\sqrt{\frac{c-1}{c}}$, $\sigma = \frac{2}{p-1}$, and $\xi = x - ct$.

It is remarked that positive solitary wave solutions of these equations are well-studied because they are common in physical situations, but less is known about negative solitary-wave solutions, though they do arise physically. Indeed, solitary waves with *positive* propagation velocity are always stable if $p \leq 5$. However, if $p > 5$, there is an explicit critical speed c_p^+ such that positive solitary waves are *stable* for those wavespeeds $c > c_p^+$, and they are unstable for $1 < c < c_p^+$. This result was proved by Souganidis and Strauss in [20] using the general theory of Grillakis, Shatah, and Strauss [11]. For a thorough review of the results, and a numerical study of the stability of positive solitary waves, the reader may consult the work of Bona, Mckinney, and Restrepo [5].

Now contrary to what one may expect, negative gBBM solitary waves with negative propagation velocity can be unstable even if $p \leq 5$. One of the contributions of the thesis is numerical and theorical proofs of the stability and instability of these negative solitary waves for both subcritical and supercritical p. In Paper C, we prove that such negative solitary waves are stable for $c < c_p^-$ and unstable for $c_p^- < c < 0$, where c_p^- is an explicitly known number (which is calculated in Paper B). The proof is an application of the stability theory of Grillakis, Shatah, and Strauss [11], with some modifications made necessary by the negativity of the solitary waves. Figure 4.2 summarizes the stable and unstable regimes for both negative and positive wavespeeds of gBBM solitary waves.

Note that the proof of stability for the case $p = 2$ is also treated separately in Paper A. However, in this Paper, we found a critical speed $c^* = -\frac{1}{6} = -0.1667$, which is not a sharp result as in the later found critical speed $c_2^- = -0.0969$ in Paper B. The reason was: we showed in Paper A the following statement:

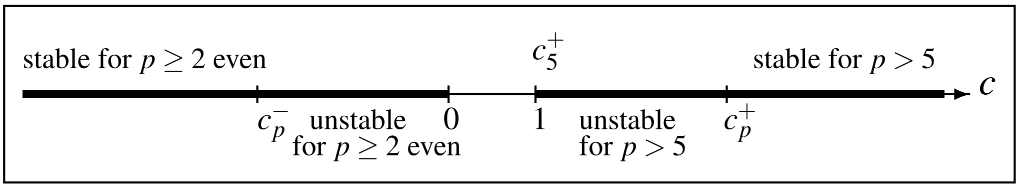

Figure 4.2: The stable and unstable regimes of the solitary waves for both negative and positive speed c. Here, $c_p^+ = \frac{1+\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, and $c_p^- = \frac{1-\sqrt{2+\sigma^{-1}}}{2(\sigma+1)}$, where $\sigma = \frac{2}{p-1}$. Note that $c_5^+ = 1$.

> *If a wavespeed $c > c^*$, then the corresponding solitary wave minimize $E(u)$ locally subject to the constancy of $V(u)$.*

The proof of this statement is based on the method of Benjamin in [1], which is *not* relying on the convexity property of the function $d(c)$. As can be seen by the stability theory of Grillakis/Shatah/Strauss, the fact of solitary waves happen to be critical points for the minimizing $E(u)$ when V is held constant exactly as a consequence of the situation where the scalar function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is convex, i.e, $d''(c) > 0$. Therefore, a sharp critical speed c_2^- is found by being as a negative root of $d''(c)$. In other words, the above statement is still valid if we replace c^* by c_2^-. Nevertheless, the proof in Paper A is more elementary and easier to follow than than the abstract general theory of Grillakis/Shatah/Strauss, which make full use of the close spectral analysis of the waves.

Further works. We remark that there remains the interesting and natural question of whether solitary waves with critical wavespeed $c = c_p^-$ or $c = c_p^+$ are stable. The results of Comech et al. for KdV in [8], suggest that they may not be. A further work could be: determining whether their technique would be applied to the gBBM equation.

Now, a more general question concerns the evolution of unstable solitary waves. Note that, the current state of theory generally only allow us to prove that solitary waves are unstable, without telling us what unstable solitary waves evolve into when perturbed. Our best information on this question comes, at present, from numerical computations. In paper B, we investigated this question for gBBM in the cases $p = 4, 6, 8$. (Note that the case $p = 2$ has been treated by Kalisch in [15].) We showed that when an unstable solitary wave is perturbed by *increasing* its amplitude, it can evolve into a larger, stable solitary wave; whereas when perturbed by *decreasing* its amplitude it can disperse completely, becoming a highly oscillatory wavetrain.

Further works. This study could be continue by examining the effects of more general perturbations (including, for example, wavelength perturbations), and the dependence of the instability mechanism on p. Such numerical study would also be of particular interest in the case of critical wavespeeds, mentioned in the preceding paragraph, where one would expect more sensitivity to the type of initial perturbation. Finally, interaction of gBBM (stable) solitary waves could also be an interesting open topic. For example, what happens when a left-moving negative solitary wave collides with a right-moving solitary waves? or how the overtaking collision between two negative solitary waves or two positive ones (with different wavespeeds). A joint work with Kalisch and H.Y. Nguyen has been begun in this direction. In a preprint [16], we found that when a left-moving negative BBM solitary wave interacts with a right-moving positive BBM solitary wave, small secondary solitary waves appear after the collision if the speed of the positive wave is related to the speed of the negative wave in a certain way. A further work could be interested to generalize this result to gBBM.

The second research is a theorical investigation of the stability for all solitary-wave solutions, both negative and positive, of the extended KdV equation. The result is appearing in Paper D. Besides, a proposed numerical investigation has also been planned to confirm the result.

4.2 The stability of solitary waves for the extended KdV equation

The consideration in this section is the *extended* Korteweg-de Vries (eKdV) equation

$$u_t + uu_x + \alpha u^2 u_x + u_{xxx} = 0,$$

which arises as a model for internal waves in stratified fluids. Like KdV, eKdV is completely integrable and admits only solitary-wave solutions with wavespeed $c > 0$, but unlike KdV, it has both positive and negative solitary-wave solutions, which turn out capture the typical broadening effect seen in internal waves [12].

Note that, as for gBBM and KdV solitary waves, the apparently explicit eKdV solitary waves are found. Indeed, it is showing in Paper D that, eKdV solitary waves may be expressed in the following form

$$\Phi_c(\xi) = \frac{A}{b + (1-b)\cosh^2 K(\xi + \xi_0)}, \tag{4.1}$$

where A is the wave amplitude, which is either equal to A^+ or A^-, and the arbitrary parameter ξ_0 represents the center of the wave. Here,

$$A^+ = \frac{-1+\sqrt{1+6c\alpha}}{\alpha},$$
$$A^- = \frac{-1-\sqrt{1+6c\alpha}}{\alpha},$$
$$K = \frac{\sqrt{c}}{2},$$
$$b = -\frac{\alpha A^2}{6c},$$

which require $c > 0,\ b \neq 1$, and a nonzero $\alpha \geq -1/6c$. In fact, when the wave amplitudes have the form of A^+, resulting *positive* solitary waves, which are given by the formula (4.1) for all nonzero $\alpha \geq -1/6c$. On the other hand, when the wave amplitudes are given by A^-, the expression (4.1) represents *negative* solitary waves only for those $\alpha > 0$. Otherwise, if $\alpha < 0$, there does *not* exist any solitary wave with amplitude A^-. Overall, the dependence of negative and positive solitary waves on the cubic nonlinear coefficient α and the wavespeed c are summarized in Figured 4.3.

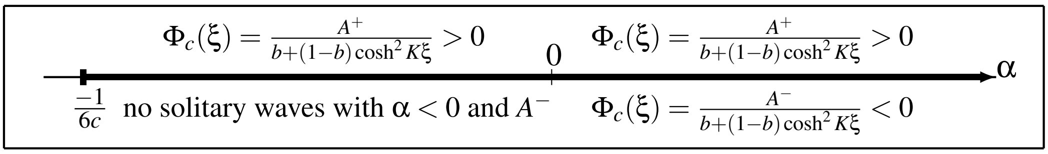

Figure 4.3: eKdV solitary waves. Here $\xi = x - ct$, and $K = \frac{1}{2}\sqrt{c}$ for a wave speed $c > 0$. The wave amplitude A is equal to either $A^+ = \frac{-1+\sqrt{1+6c\alpha}}{\alpha} > 0$ $(\alpha \geq -1/6c)$ or $A^- = \frac{-1-\sqrt{1+6c\alpha}}{\alpha} < 0$ $(\alpha > 0)$, $b = -\frac{\alpha A^2}{6c} < 1$.

Again, like gBBM, eKdV positive solitary waves are well-known, but less is known about eKdV negative solitary waves, though they do arise in physical situations. Another contribution of the thesis is a theorical proof of the stability for all these solitary-wave solutions, both negative and positive, of eKdV. The proof proceeds in Paper D by verifying the hypothesis of the stability theory of Grillakis, Shatah, and Strauss [11] for equations of the form $u = JE'(u)$, which have two conserved integrals $E(u)$ and $V(u)$. Here, for eKdV, $J = \partial_x$,

$$E(u) = \int_{-\infty}^{\infty} \left\{-\tfrac{1}{6}u^3 - \tfrac{\alpha}{12}u^4 + \tfrac{1}{2}u_x^2\right\}\, dx,$$

and

$$V(u) = \int_{-\infty}^{\infty} \tfrac{1}{2}u^2\, dx.$$

A spectral analysis of the linear operator

$$\mathscr{L}_c = E''(\Phi_c) + cV''(\Phi_c) = -\partial_x^2 + c - \Phi_c - \alpha\Phi_c^2$$

reduces the question of orbital stability to the question of whether or not the scalar function

$$d(c) = E(\Phi_c) + cV(\Phi_c)$$

is convex. To prove the stability, an explicit calculation proving the convexity is performed.

Further works. It could be an interesting open topic to do a numerical study of eKdV solitary waves (e.g., for broad waves) which, besides confirming the stability result, will also illuminate the question of how solitary waves (both positive and negative) with different wavespeeds will interact. As is well-known, equations such as eKdV and KdV are more difficult to solve numerically than gBBM. The numerical simulation could use a spectral discretization in space coupled with a high-order time-integration scheme.

The final research of the thesis is a theorical investigation of the instability of the solitary waves for a Camassa-Holm equation type. The result is appearing in Paper E.

4.3 Instability of solitary waves for a generalized version of the Camassa-Holm equation

A final research consideration is a generalized version of the Camassa-Holm (gCH) equation

$$u_t + \omega u_x + 3uu_x - u_{xxt} = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{4.2}$$

where $\omega \geq 0$ and $\gamma \in \mathbb{R}$. When $\omega = 0$ and the range of the parameter γ is roughly from -29.5 to 3.4, the equation (4.2) has been derived as a model for waves in elastic rods [9]. On the other hand, if $\gamma = 1$, the equation is used as a model for surface water waves, and in which case it is the so-called Camassa-Holm (CH) equation: being equivalent, up to the order of approximation, to KdV and BBM equation [13].

Solitary-wave solutions of (4.2) with profiles in H^1 are known to exist for all wavespeeds $c > \omega$ if $\gamma < 1$, and all $c \in \left(\omega, \frac{\omega\gamma}{\gamma-1}\right]$ if $\gamma > 1$. Stability of all solitary waves was proved by Constantin and Strauss in the case $\gamma = 1$ [7], and by Kalisch in the case $\gamma < 1$ [14]. In addition, stability of solitary waves with wavespeeds c

sufficiently close to the lower limit ω, for the case $\gamma > 1$, is also proved by Kalisch in [14]. Now, on the contrary with these stability results for gCH is another contribution of the thesis; that is the *instability* for gCH solitary waves with those wavespeeds c sufficiently close, but not equal to the upper limit $\frac{\omega\gamma}{\gamma-1}$, for $\gamma > 1$. The proof is represented in Paper E, which is also an application of Grillakis, Shatah, and Strauss theory [11], but non-trivial modifications have to be made to the argument of the estimate on a Lyapunov function.

It is remarked that in the case $\gamma = 1$ and $\omega = 0$: the explicit peaked solitary wave is known [7]. Similarly, for the case $\gamma > 1$: at the upper limit wavespeed $c = \frac{\omega\gamma}{\gamma-1}$, the explicit peakon solutions are known, and this fact are also mentioned in Paper E. However, for all other cases, apparently explicit smooth solitary waves are not known.

Overall, Figure 4.4 summarizes the stability properties of the gCH solitary waves together with the explicit existence of the peaked solitary waves for γ in the range $(1, \infty)$.

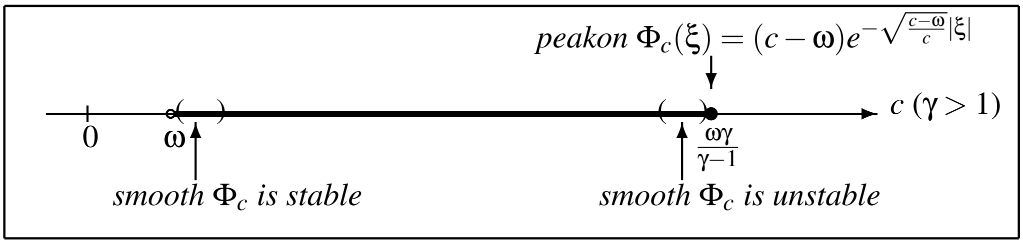

Figure 4.4: If $\gamma > 1$, solitary waves exist only in the range $\omega < c \leq \frac{\omega\gamma}{\gamma-1}$. The peaked solitary wave occurs at the maximum value $c = \frac{\omega\gamma}{\gamma-1}$. For c close to the lower limit ω, solitary waves Φ_c are stable. On the other hand, for c close enough, but not equal to the upper limit $\frac{\omega\gamma}{\gamma-1}$, solitary waves Φ_c are unstable.

Further works. In the case $\gamma > 1$, there remains interesting and natural question of what happens to solitary-wave solutions of the gCH equation with wavespeeds c that are neither close to ω or to $\frac{\omega\gamma}{\gamma-1}$, or that are equal to $\frac{\omega\gamma}{\gamma-1}$: whether such solitary waves are stable or unstable, and in the case of instability, numerical study could further investigate how unstable solitary waves evolve into when perturbed. Such numerical study would also be particular interest in the above instability case mentioned in the preceding paragraph. Besides, a further numerical study could answer the question of how stable gCH solitary waves with different wavespeeds will interact.

Appendix A

Appendix

A.1 Derivation of the KdV and BBM equations

The derivation of the KdV and regularized long-wave (or BBM) equations will be shown. The work is based on the ideas of Bejamin, Bona, Mahony, and Whitham in [3, 22].

Water surface wave problem.
Let $\vec{X} = (X,Y,Z)$ connote a standard Cartesian coordinate system with Z be the vertical direction and $Z = 0$ located at the surface of water in a long narrow channel of depth h. It is assumed that a typical wave amplitude is a, and a typical wavelength is λ, and that the quantities $\alpha = a/h$ and $\beta = h/\lambda$. Consideration is given to waves on the surface whose do not vary significantly in the Y-direction, and for which the effects of surface tension and viscosity may be safely ignored. And thus, the flow is irrotational so that a velocity potential function $\phi = \phi(X,Z,T)$ can be defined for which $\partial\phi/\partial X$ and $\partial\phi/\partial Z$ are the horizontal and vertical components of the fluid velocity vector, respectively.

Let the function $\eta(X,T)$ describes the vertical deviation of the surface from its rest position at the point X and time T. Inside the flow, for $-\infty < X < +\infty$ and $-h < Z < \eta$, reading from the continuity equation, which is based on the conservation law of mass, for an *incompressible* fluid, there appears

$$\Delta\phi = \phi_{XX} + \phi_{ZZ} = 0.$$

Note that this equation is valid for the water-wave problem since water could be regarded as an incompressible fluid, i.e, its density ρ is constant with respect to $\vec{X}$ and time T.

Now at the free surface $Z = \eta$, dynamic and kinematic boundary conditions are required. For the dynamic boundary condition, atmosphere pressure can be ignored, i.e, $p_{atm} = 0$. Consequently, using Bernoulli's equation for incompressible, no viscosity, and irrotational flows, there appears the following equation

$$\phi_T + \tfrac{1}{2}\phi_X^2 + \tfrac{1}{2}\phi_Z^2 + g\eta = 0,$$

where g is the acceleration of gravity. Note that, this equation originally can be derived from Newton's second law or the conservation law of momentum [18]. On the other hand, a kinematic boundary condition is that the fluid particle never leaves the surface; that is

$$\frac{D\eta}{DT} = \phi_Z \Longleftrightarrow \eta_T + \phi_X\eta_X - \phi_Z = 0.$$

Finally, at $Z = -h$, it is supposed that there is no flow through the bottom, that means zero normal velocity:

$$\phi_Z = 0.$$

In summary, we stated the following water-wave boundary value problem

$$\begin{cases} \phi_T + \frac{1}{2}\phi_X^2 + \frac{1}{2}\phi_Z^2 + g\eta = 0 & \text{if } -\infty < X < +\infty,\ Z = \eta, \\ \eta_T + \phi_X\eta_X - \phi_Z = 0 & \text{if } -\infty < X < +\infty,\ Z = \eta, \\ \phi_{XX} + \phi_{ZZ} = 0 & \text{if } -\infty < X < +\infty,\ -h < Z < \eta, \\ \phi_Z = 0 & \text{if } -\infty < X < +\infty,\ Z = -h. \end{cases}$$

Next, make a change of variable $\tilde{Z} = Z + h$, i.e, move up this system by the amount of h. Then, in term of $(X, Y, \tilde{Z})$, we have the following system

$$\begin{cases} \phi_T + \frac{1}{2}\phi_X^2 + \frac{1}{2}\phi_{\tilde{Z}}^2 + g\eta = 0 & \text{if } -\infty < X < +\infty,\ \tilde{Z} = \eta + h, \\ \eta_T + \phi_X\eta_X - \phi_{\tilde{Z}} = 0 & \text{if } -\infty < X < +\infty,\ \tilde{Z} = \eta + h, \\ \phi_{XX} + \phi_{\tilde{Z}\tilde{Z}} = 0 & \text{if } -\infty < X < +\infty,\ 0 < \tilde{Z} < \eta + h, \\ \phi_{\tilde{Z}} = 0 & \text{if } -\infty < X < +\infty,\ \tilde{Z} = 0. \end{cases}$$

Working with dimensionless variables.
From now on, we will work with dimensionless variables by letting

$$X' = \frac{1}{\lambda}X,\ Z' = \frac{1}{h}\tilde{Z},\ T' = \frac{c_0}{\lambda}T,\ \eta' = \frac{1}{a}\eta, \text{ and } \phi' = \frac{c_0}{g\lambda a}\phi,$$

where $c_0 = \sqrt{gh}$. This will tranfer the above system to

$$\begin{cases} \phi'_{T'} + \frac{1}{2}\alpha\phi'^2_{X'} + \frac{1}{2}\frac{\alpha}{\beta^2}\phi'^2_{Z'} + \eta' = 0 & \text{if } -\infty < X' < +\infty,\ Z' = 1 + \alpha\eta', \\ \eta'_{T'} + \alpha\phi'_{X'}\eta'_{X'} - \frac{1}{\beta^2}\phi'_{Z'} = 0 & \text{if } -\infty < X' < +\infty, Z' = 1 + \alpha\eta', \\ \beta^2\phi'_{X'X'} + \phi'_{Z'Z'} = 0 & \text{if } -\infty < X' < +\infty,\ 0 < Z' < 1 + \alpha\eta', \\ \phi'_{Z'} = 0 & \text{if } -\infty < X' < +\infty, Z' = 0. \end{cases} \tag{A.1}$$

It is elementary to check directly that a solution of the last two equations in the system has the form of seperated variables as follows

$$\phi'(X',Z') = \sum_{m=0}^{\infty} (-1)^m \frac{Z'^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial X'^{2m}} \beta^{2m}, \tag{A.2}$$

for a function $f = f(X',T')$. Then, insert this solution into the two first equations in the system (A.1) (the two boundary conditions at the surface) and let $w = f_{X'}$, there appears the following system of two equations

$$\begin{cases} \eta'_{T'} + w_{X'} + \alpha(\eta' w)_{X'} - \frac{1}{6} w_{X'X'X'} \beta^2 = O(\alpha\beta^2, \beta^4, \alpha^2) \\ \eta'_{X'} + w_{T'} + \alpha w w_{X'} - \frac{1}{2} w_{X'X'T'} \beta^2 = O(\alpha\beta^2, \beta^4, \alpha^2). \end{cases} \tag{A.3}$$

Remark 1 *$w = f_{X'}$ can be regarded as the horizontal velocity (in dimensionless variables) up to the order of $O(\beta^2)$ because in light of equation* (A.2), $\phi'_{X'} = f_{X'} + O(\beta^2)$.

Observe that the lowest order of the system (A.3) is

$$\begin{cases} \eta'_{T'} + w_{X'} = O(\alpha, \beta^2) \\ \eta'_{X'} + w_{T'} = O(\alpha, \beta^2). \end{cases} \tag{A.4}$$

Remark 2 *Korteweg-de Vries (KdV) or Benjamin-Bona-Mahoney (BBM) equation will be derived so that it is based on this system* (A.4)*, which is assumed that the quantities $\alpha = \frac{a}{h}$ and $\beta^2 = \left(\frac{h}{\lambda}\right)^2$ are of comparable magnitudes and small, i.e, $\beta^2 = O(\alpha) \ll 1$. And thus, the waves could be regarded as shallow and long waves with respect to the height h.*

Corollary 1 *It is appearing in Paper B that: the amplitude of negative BBM solitary waves with negative wavespeeds is of order* 1*, and thus these solutions do not fall into the regime of physical validity of the equation as a model of small amplitude surface waves. This concurs with the fact that the solitary waves of depression do not occur on the surface of fluids unless surface tension is very strong [2, 15].*

It turns out that the system (A.4) has a solution of the form

$$\begin{cases} w = \eta' + O(\alpha, \beta^2) \\ \eta'_{T'} + \eta'_{X'} = O(\alpha, \beta^2). \end{cases} \tag{A.5}$$

Remark 3 *In light of Remark 1, it can be seen by the first equation in this system* (A.5) *that KdV or BBM equation is derived by spectializing to a wave moving to the right/left (in the direction of increasing/decreasing value of X') if the surface wave η' is positive/negative.*

Corollary 2 *Consequently, the case of negative solitary waves with positive speeds, which is mentioned in Paper C, though they are mathematical-existence solutions to BBM equation, they also do not fall into the regime of physical validity of solutions for the equation as a long-wave model.*

However, in order to correct the solution up to the oder of $O(\alpha\beta^2, \beta^4, \alpha^2)$, we write

$$w = \eta' + \alpha A + \beta^2 B + O(\alpha\beta^2, \beta^4, \alpha^2), \tag{A.6}$$

where functions $A = A(\eta', \eta'_{X'}, \eta'_{T'}, \cdots)$ and $B = B(\eta', \eta'_{X'}, \eta'_{T'}, \cdots)$. Note that since the last equation in the system (A.5), it can be seen that

$$\begin{cases} A_{T'} = -A_{X'} + O(\alpha, \beta^2) \\ B_{T'} = -B_{X'} + O(\alpha, \beta^2). \end{cases}$$

Now, to get rid of w, substitude the form (A.6) back into the system (A.3), the two boundary conditions at the surface becomes

$$\begin{cases} \eta'_{T'} + \eta'_{X'} + \alpha\big(A_{X'} + 2\eta'_{X'}\eta'\big) - \beta^2\big(B_{X'} - \frac{1}{6}\eta'_{X'X'X'}\big) = O(\alpha\beta^2, \beta^4, \alpha^2) \\ \eta'_{T'} + \eta_{X'} + \alpha\big(-A_{X'} + \eta'_{X'}\eta'\big) - \beta^2\big(-B_{X'} + \frac{1}{2}\eta'_{X'X'X'}\big) = O(\alpha\beta^2, \beta^4, \alpha^2). \end{cases} \tag{A.7}$$

Compare parentheses in these two equations, it can be seen that a solution for A and B is

$$A = -\frac{1}{4}\eta'^2 \text{ and } B = \frac{1}{3}\eta'_{X'X'}.$$

KdV and BBM equations.
Then, plug back these values for A and B into either one of equations in the system (A.7), there appears the following equation

$$\eta'_{T'} + \eta'_{X'} + \frac{3}{2}\alpha\eta'\eta'_{X'} + \frac{1}{6}\beta^2\eta'_{X'X'X'} = O(\alpha\beta^2, \beta^4, \alpha^2).$$

This equation is the so-called KdV equation (in dimensionless form), which was derived in 1895. On the other hand, in light of the last equation in the system (A.5), an alternative model of KdV equation for the surface water-wave problem is

$$\eta'_{T'} + \eta'_{X'} + \frac{3}{2}\alpha\eta'\eta'_{X'} - \frac{1}{6}\beta^2\eta'_{X'X'T'} = O(\alpha\beta^2, \beta^4, \alpha^2).$$

Now, by changing of variables

$$x = \frac{\sqrt{6}}{\beta}X', \; t = \frac{\sqrt{6}}{\beta}T', \text{ and } u = \frac{3}{4}\alpha\eta',$$

this equation can be simplified further to

$$u_t + u_x + (u^2)_x - u_{xxt} = 0.$$

Here, for α and β^2 are sufficiently small, we considered $O(\alpha\beta^2, \beta^4, \alpha^2) = 0$. And this equation is the so-called regularized long-wave equation or it is well-known as the BBM equation, which was introduced by Benjamin, Bona, and Mahony in 1972 [3]. Moreover, a mathematical arising-problem leads to the generalized BBM (gBBM) equation or the generalized regularized long-wave equation appears as follow

$$u_t + u_x + (u^p)_x - u_{xxt} = 0,$$

for $p \geq 2$ is an integer. And the stability of the negative gBBM solitary waves is one of the considerations in the thesis.

Bibliography

[1] T.B. Benjamin, *The stability of solitary waves.* Proc. Roy. Soc. London A **328** (1972), 153–183.

[2] T.B Benjamin, *The solitary wave with surface tension.* Quart. Appl. Math. **40** (1982), 231–234

[3] T.B. Benjamin, J.B. Bona, and J.J. Mahony, *Model equations for long waves in nonlinear dispersive systems.* Philos. Trans. Roy. Soc. London A **272** (1972), 47–78.

[4] J.L. Bona, *On the stability theory of solitary waves.* Proc. Roy. Soc. London A **344** (1975), 363–374.

[5] J.L. Bona, W.R. McKinney and J.M. Restrepo, *Stable and unstable solitary-wave solutions of the generalized regularized long-wave equation*, J. Nonlinear Sci. **10** (2000), 603–638.

[6] J.L. Bona, P.E. Souganidis and W.A. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type.* Proc. Roy. Soc. London A **411** (1987), 395–412.

[7] A. Constantin and W.A.Strauss, *Stability of the Camassa-Holm solitons.* J. Nonlinear Sci. **12** (2002), 415–422.

[8] A. Comech, S. Cuccagna, and D.E. Pelinovsky, *Nonlinear instability of a critical traveling wave in the generalized Korteweg-de Vries equation.* SIAM J. Math. Anal. **39** (2007), 1–33

[9] H.H. Dai and Y. Huo, *Solitary wave shock waves and other travelling waves in a general compressible hyperelastic rod*, Proc. Roy. Soc. London A **456** (2000), 331–363.

[10] P.G. Drazin and R.S. Johnson, *Solitons: an Introduction.* Cambridge University Press, 1989.

[11] M. Grillakis, J. Shatah and W.A. Strauss, *Stability theory of solitary waves in the presence of symmetry.* J. Funct. Anal. **74** (1987), 160–197.

[12] K.R. Helfrich and K. Melville, *Long nonlinear internal waves.* Annu. Rev. Fluid Mechanics. **38** (2006) 395–425.

[13] R.S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid Mech. **455** (2002), 63–82.

[14] H. Kalisch, *Stability of solitary wave for a nonlinearly dispersive equation*, Discrete And Continuous Dynamic Systems **10** (2004), 709–717

[15] H. Kalisch, *Solitary waves of depression*, J. Comput. Anal. Appl. **8** (2006), 5–24.

[16] H. Kalisch, H.Y. Nguyen, and N.T. Nguyen, *Interaction of BBM solitary waves*, work in progress.

[17] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991) 323–347.

[18] P.K. Kundu, I.M. Cohen *Fluid Mechanic.* Elsevier, 2004.

[19] P.G. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech. **25** (1966), 321–330.

[20] P.E. Souganidis and W.A. Strauss, *Instability of a class of dispersive solitary waves*, Proc. Roy. Soc. Edinburgh **114A** (1990), 195–212.

[21] T.Tao, *Why are solitons stable?*, Bull. Amer. Math. Soc. **46** (2009), 1–33.

[22] G.B. Whitham, *Linear and Nonlinear Waves*. Wiley, New York, 1974.

[23] http://en.wikipedia.org/wiki/John_Scott_Russell.