Solving System of Nonlinear Equations Using Methods in the Halley Class

by

Sara T. M. Suleiman

Thesis for the degree of Master of Science

Faculty of Mathematics and Natural Sciences
Department of Informatics
Optimization Group

May 2009

UNIVERSITY OF BERGEN

Abstract

Faculty of Mathematics and Natural Sciences
Department of Informatics

Master of Science

by

Sara T. M. Suleiman

In this thesis a new iterative frame work to solve the nonlinear system of equations $F(x) = 0$ in n-dimensional real space is established. This iterative frame work is based on a quadratic model of the function $F(x)$ at the current point. The convergence analysis shows that this frame work has Q-third rate of convergence.

The main advantages of this frame work that the system of nonlinear equations simplified to quadratic system of equations which hopefully has less computational complexity than the original system. It is shown that the Halley class inherits it's convergence properties from the quadratic model.

In practice, for the large-scale problems, the inexact Halley class methods is used to solve $F(x) = 0$, and has Q-third rate of convergence.

Acknowledgements

First of all, I would like to express my deepest sense of gratitude to my supervisor Professor Trond Steihaug for his patience, guidance, encouragement and excellent advice throughout this study. I appreciate his assistance in writing the thesis.

I would like to thank the administration staff at the department of Informatics.

I am enormously grateful to the Norwegian state educational loan fund for financing my master study.

I am deeply and forever thankful to my parents Mariam and Tagelsir for their love, support and encouragement throughout my entire life. Many thanks to all my friends for their constant support. Finally, I take this opportunity to express my profound gratitude to my husband Mohammed for his support, advice, help and patience during this work.

Contents

List of Algorithms

List of Figures

List of Tables

to my

parents Mariam and Tagelsir, and to my husband Mohamed

with my love

Chapter 1

Introduction

Nonlinearity is of interest to physicists and mathematicians, since most physical systems are inherently nonlinear in nature. One of the most important problem in Optimization and computational mathematics is to solve nonlinear system of equations. For example in nonlinear optimization where the aim is to find a minimum or maximum of a given nonlinear function. The first subproblem that we face is to solve system of nonlinear equations.

System of nonlinear equations are difficult to solve in general. The best way to solve these equations is by iterative methods. One of the classical method to solve the system of nonlinear equations is Newton method which has second order rate of convergence. This speed is low when we compare to third order method.

Our work concentrate on solving the system of nonlinear equations $F(x) = 0$ in the real n-dimensional linear space. $F(x)$ is sufficiently smooth. We are interested in solving the system using the Halley class methods with a real constant α and an adequate starting point $x_0 \in \mathbb{R}^n$. Halley's class defined by

$$x_{k+1} = x_k - \left[I + \frac{1}{2} L(x_k) \Big(I - \alpha L(x_k) \Big)^{-1} \right] F'(x_k)^{-1} F(x_k),\ k = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$L(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \tag{1.2}$$

and I is the identity matrix in $\mathbb{R}^{n \times n}$. Provided that F', F'' and $F'(x_k)^{-1}$ are defined. Halley class includes well known third order methods such as Chebyshev's method and Halley's method. We will show that Halley class has third order rate of convergence.

In this thesis, we will introduce a new iterative framework for sufficient starting point to solve the system of nonlinear equations $F(x) = 0$. The basis of this framework is approximately finding the root of the quadratic model around the current iteration. We will approximately solve the quadratic equations using one iteration of the methods in a Halley class. One iteration of Halley class is equivalent to solve two linear systems. Then for every outer iteration in the framework, we will solve two linear systems of equations. Moreover, if these systems are solved using a direct method, then we have a Halley class. And if solved using an iterative method, we have an inexact methods in Halley class. Convergence analysis will show that this framework is cubic convergent.

The structure of the thesis is:

Chapter 2 drives Chebyshev and Halley method in the scalar case, presents Super-Halley method for function in one variable and reviews the convergence properties of these methods.

Chapter 3 introduces the Halley class methods, and the relation with other classes such as Schwetlick class. Analyses the convergence properties of this class and makes numerical experiment to some methods in the class compared with Newton method.

Chapter 4 introduces a new iterative framework based on a quadratic model. We will solve the model approximately using one step of Halley class methods. We will study the convergence properties of this framework.

Chapter 5 solves the model approximately using one step of inexact Halley class methods. Shows the rate of convergence of the inexact Halley class methods is Q-third order.

Chapter 6 drives the Schröder method, talks about the case when we have singularity at the solution through a numerical example.

Chapter 7 summarizes the results and makes suggestion for future work.

Chapter 2

Scalar Equation

This chapter introduces in details the derivation of the Chebyshev and Halley methods in the case of scalar function of one variable, and presents Super-Halley method in the scalar case. Reviews the essential convergence properties of these methods [24].

2.1 Derivation of the methods

Consider the nonlinear equation

$$f(x) = 0, \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is two times continuously differentiable. To derive the Chebyshev method [26] for (2.1), consider the parabola on the form

$$m(x, y) = ay^2 + y + bx + c = 0. \tag{2.2}$$

and we solve the following equations

$$m(x, y(x)) = 0, \ \frac{d}{dx}m(x, y(x)) = 0 \text{ and } \frac{d^2}{dx^2}m(x, y(x)) = 0, \tag{2.3}$$

to find the unknown variables a and b , so we have

$$\frac{d}{dx}m(x, y(x)) = 2ay(x)y'(x) + y'(x) + b = 0, \tag{2.4}$$

since $\frac{d^2}{dx^2}m(x, y(x)) = \frac{d}{dx}\left[\frac{d}{dx}m(x, y(x))\right]$, then

$$\frac{d^2}{dx^2}m(x, y(x)) = 2ay(x)y''(x) + 2ay'(x)^2 + y''(x) = 0. \tag{2.5}$$

Then, imposing the conditions

$$y(x) = f(x),\ y'(x) = f'(x) \text{ and } y''(x) = f''(x). \tag{2.6}$$

Then the parabola equation (2.2) can be written as

$$m(x, f(x)) = af(x)^2 + f(x) + bx + c = 0, \tag{2.7}$$

so by solving (2.5), (2.4), we obtain

$$a = -\frac{f''(x)}{2(f(x)f''(x) + f'(x)^2)},\ b = -\frac{f'(x)^3}{f(x)f''(x) + f'(x)^2}. \tag{2.8}$$

To find the unknown variable c, substitute a and b in (2.8) into (2.7), so we get

$$c = \frac{f''(x)}{2(f(x)f''(x) + f'(x)^2)}f(x)^2 - f(x) + \frac{f'(x)^3}{f(x)f''(x) + f'(x)^2}x,$$

by collecting the above equation, we get

$$c = \frac{2f'(x)^3x - f(x)^2f''(x) - 2f(x)f'(x)^2}{2(f(x)f''(x) + f'(x)^2)}. \tag{2.9}$$

And then taking the intersection of the parabola (2.7) with the x-axis at the next iterate x_+ i.e. $m(x_+, 0) = 0$ we obtain that,

$$bx_+ + c = 0, \tag{2.10}$$

substitute the value of b in (2.8) and c in (2.9) into the equation (2.10), so we have

$$-\frac{f'(x)^3}{f(x)f''(x) + f'(x)^2}x_+ + \frac{2f'(x)^3x - f(x)^2f''(x) - 2f(x)f'(x)^2}{2(f(x)f''(x) + f'(x)^2)} = 0,$$

multiply the whole equation with the inverse of $\frac{f'(x)^3}{f(x)f''(x)+f'(x)^2}$, we obtain

$$-x_+ + \frac{2f'(x)^3 x - f(x)^2 f''(x) - 2f(x)f'(x)^2}{2f'(x)^3} = 0,$$

rearrange the above equation, we get

$$-x_+ + x - \frac{f(x)^2 f''(x)}{2f'(x)^3} - \frac{f(x)}{f'(x)} = 0.$$

Then we get the Chebyshev method

$$x_+ = x - \left(1 + \frac{1}{2}\frac{f''(x)f(x)}{f'(x)^2}\right)\frac{f(x)}{f'(x)},$$

therefore we called it the method of tangent parabola.

In the same way we can derive the Halley method [9, 26] by considering the hyperbola

$$m(x, y) = axy + y + bx + c = 0, \tag{2.11}$$

and solving the equations (2.3) for the unknown variables a and b, so we have that

$$\frac{d}{dx}m(x, y(x)) = 0 \Rightarrow ay(x) + b + (ax+1)y'(x) = 0 \tag{2.12}$$

$$\frac{d^2}{dx^2}m(x, y(x)) = 0 \Rightarrow ay'(x) + (ax+1)y''(x) + ay'(x) = 0. \tag{2.13}$$

By imposing the conditions (2.6) and solving (2.12) and (2.13), we get

$$a = \frac{-f''(x)}{xf''(x) + 2f'(x)}, \quad b = \frac{f''(x)f(x) - 2f'(x)^2}{xf''(x) + 2f'(x)}, \tag{2.14}$$

substitute a and b in (2.14) and use the conditions (2.6), then (2.11) can be written as,

$$m(x, f(x)) = \frac{-xf''(x)f(x)}{xf''(x) + 2f'(x)} + f(x) + \frac{x\left(f''(x)f(x) - 2f'(x)^2\right)}{xf''(x) + 2f'(x)} + c = 0. \tag{2.15}$$

Then from (2.15), the unknown variable c is

$$c = \frac{xf''(x)f(x)}{xf''(x)+2f'(x)} - f(x) - \frac{x\big(f''(x)f(x)-2f'(x)^2\big)}{xf''(x)+2f'(x)},$$

rearrange the equation, we get

$$c = \frac{xf''(x)f(x) - 2f'(x)f(x) - xf''(x)f(x) - xf''(x)f(x) + 2xf'(x)^2}{xf''(x)+2f'(x)},$$

collect the terms, we get

$$c = \frac{-2f'(x)f(x) - xf''(x)f(x) + 2xf'(x)^2}{xf''(x)+2f'(x)}. \tag{2.16}$$

And then by taking the intersection of the hyperbola (2.11) with the x-axis at the point x_+, so we have

$$bx_+ + c = 0,$$

substitute b in (2.14) and c in (2.16) into the above equation, we get

$$\frac{\big(f''(x)f(x)-2f'(x)^2\big)x_+}{xf''(x)+2f'(x)} + \frac{-2f'(x)f(x) - xf''(x)f(x) + 2xf'(x)^2}{xf''(x)+2f'(x)} = 0,$$

multiply the whole equation with the inverse of $\frac{f''(x)f(x)-2f'(x)^2}{xf''(x)+2f'(x)}$, we get

$$x_+ + \frac{-2f'(x)f(x) - xf''(x)f(x) + 2xf'(x)^2}{f''(x)f(x)-2f'(x)^2} = 0,$$

rearrange the above equation, we have

$$x_+ + \frac{-2f'(x)f(x)}{f''(x)f(x)-2f'(x)^2} - \frac{(f''(x)f(x)-2f'(x)^2)x}{f''(x)f(x)-2f'(x)^2} = 0,$$

the above equation can be rewritten as

$$x_+ + \frac{-2f'(x)f(x)}{2f'(x)^2(\frac{1}{2}f'(x)^{-2}f''(x)f(x)-1)} - x = 0,$$

rearrange the above equation, then we obtain the Halley method

$$x_+ = x - \left[\frac{1}{1 - \frac{1}{2} f'(x)^{-1} f''(x) f'(x)^{-1} f(x)}\right] \frac{f(x)}{f'(x)}, \tag{2.17}$$

so the Halley method is called the method of tangent hyperbola.

Super-Halley method or Convex Acceleration of Newton's method [9, 13] for (2.1) is

$$x_+ = x - \left[1 + \frac{1}{2} \frac{f'(x)^{-1} f''(x) f'(x)^{-1} f(x)}{I - f'(x)^{-1} f''(x) f'(x)^{-1} f(x)}\right] \frac{f(x)}{f'(x)}, \tag{2.18}$$

Sharma [24] generalized a quadratic equation $m(x, y)$ after imposing the conditions (2.3) that includes Chebyshev, Halley and Super-Halley for specific values.

2.2 Convergence Property

Theorem 2.1. *[24] Let $f : \mathbb{R} \to \mathbb{R}$ three times continuously differentiable in a neighborhood of the solution x^*. Assume that $f'(x^*)^{-1} \neq 0$. Let the starting point x_0 to be close to the solution. Then Chebyshev, Halley and Super-Halley methods are cubically convergent to the solution.*

Chapter 3

Halley Class

This chapter introduces the parametric class called the Halley Class, which is a class of iterative methods to solve the system of nonlinear equations $F(x) = 0$. The chapter also includes the convergence properties of the methods and the analysis of some numerical experiments.

3.1 Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \tag{3.1}$$

where the function $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following basic assumptions.

Let x^* be the solution of the problem (3.1), $F'(x)$ and $F''(x)$ be the first and second derivative of the function F at the point x (for the definition see (A.3) and (A.4) in the Appendix). Let $\mathcal{N}(x^*, r)$ be the ball with radius r defined by

$$\mathcal{N}(x^*, r) = \{x : \|x - x^*\|_2 \leq r\}.$$

The following basic assumptions is used in the whole thesis.

Assumption 3.1.

1. *$F(x)$ is two times continuously differentiable on the ball $\mathcal{N}(x^*, r)$.*
2. *$F'(x^*)$ is nonsingular.*
3. *$F''(x)$ is Lipschitz continuous on the ball $\mathcal{N}(x^*, r)$ (see (A.1) in the Appendix).*

Hernández and Gutiérrez [15] introduced a class for a parameter $\alpha \in [0, 1]$, where x_k is the current iteration and $F'(x_k)$ is invertible. This class is refering to the Halley class. Given an initial guess x_0 close to a solution of (3.1), the Halley class is

$$x_{k+1} = x_k - \left[I + \frac{1}{2} L(x_k) \Big(I - \alpha L(x_k) \Big)^{-1} \right] F'(x_k)^{-1} F(x_k), \; k = 0, 1, 2, \ldots, \tag{3.2}$$

where

$$L(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \tag{3.3}$$

and I is the identity matrix in $\mathbb{R}^{n \times n}$.

The Halley class was introduced by Hernández and Salanova [16] for a scalar function and later extended by Hernández and Gutiérrez to Banach spaces. In other paper, Hernández and Gutiérrez [12] defined Halley class (3.2) in Banach spaces where α lies in an interval bigger than $[0, 1]$, provided that the second derivative is bounded. In this thesis we will study the Halley class for all values of α. Convergence properties of this class will be discussed in Section 3.4.

Halley class includes the Chebyshev method [3, 15, 16] when $\alpha = 0$, the Halley method [2, 4, 15, 16] when $\alpha = \frac{1}{2}$ and Super-Halley method [14, 15] when $\alpha = 1$.

3.2 Halley class and Schwetlick class

Schwetlick [23] defined a class for a real scalar α, which we call the Schwetlick class. This class is obtained by solving the following equation for $y_k^{(i+1)}$, $i = 1, 2, 3, \ldots$.

$$\begin{aligned} F(x_k) + F'(x_k) \left(y_k^{(i+1)} - x_k \right) + \alpha F''(x_k) \left(y_k^{(i)} - x_k \right) \left(y_k^{(i+1)} - x_k \right) \\ + \left(\frac{1}{2} - \alpha \right) F''(x_k) \left(y_k^{(i)} - x_k \right) \left(y_k^{(i)} - x_k \right) = 0, \end{aligned} \tag{3.4}$$

where $y_k^{(0)} = x_k$ and $y_k^{(i+1)} = x_{k+1}$.

Lemma 3.1 will illustrate the relation between the Halley (3.2) and Schwetlick (3.4).

Lemma 3.1. *[11] Let $i = 1$ in Schwetlick class (3.4), then the Halley class (3.2) and Schwetlick class (3.4) are equivalent for any real α.*

Proof. Consider the Schwetlick class (3.4), put $i = 1$, i.e. solve (3.4) for $y_k^{(2)}$, where $y_k^{(0)} = x_k$ and $y_k^{(2)} = x_{k+1}$, we have

$$\begin{aligned} F(x_k) + F'(x_k)\left(y_k^{(2)} - x_k\right) + \alpha F''(x_k)\left(y_k^{(1)} - x_k\right)\left(y_k^{(2)} - x_k\right) \\ + \left(\frac{1}{2} - \alpha\right) F''(x_k)\left(y_k^{(1)} - x_k\right)\left(y_k^{(1)} - x_k\right) = 0. \end{aligned} \tag{3.5}$$

To solve (3.5), we need first to solve (3.4) for $y_k^{(1)}$, that means $i = 0$, since $y_k^{(0)} = x_k$, we get

$$F(x_k) + F'(x_k)\left(y_k^{(1)} - x_k\right) = 0,$$

then $y_k^{(1)}$ is given by

$$y_k^{(1)} = x_k - F'(x_k)^{-1}F(x_k). \tag{3.6}$$

By substituting the value of $y_k^{(1)}$ (3.6) into (3.5), we obtain

$$\begin{aligned} F(x_k) + F'(x_k)\left(y_k^{(2)} - x_k\right) - \alpha F''(x_k)F'(x_k)^{-1}F(x_k)\left(y_k^{(2)} - x_k\right) \\ + \left(\frac{1}{2} - \alpha\right) F''(x_k)\left(F'(x_k)^{-1}F(x_k)\right)\left(F'(x_k)^{-1}F(x_k)\right) = 0, \end{aligned} \tag{3.7}$$

collecting the terms with $\left(y_k^{(2)} - x_k\right)$ and rearranging the equation, we have

$$\begin{aligned} \left(F'(x_k) - \alpha F''(x_k)F'(x_k)^{-1}F(x_k)\right)\left(y_k^{(2)} - x_k\right) = \\ - F(x_k) - \left(\frac{1}{2} - \alpha\right) F''(x_k)\left(F'(x_k)^{-1}F(x_k)\right)\left(F'(x_k)^{-1}F(x_k)\right), \end{aligned} \tag{3.8}$$

therefore, by putting $x_{k+1} = y_k^{(2)}$ and defining $L(x)$ as we did before in (3.3), we have

$$F'(x_k)\big(I - \alpha L(x_k)\big)\,(x_{k+1} - x_k) = \\ - F(x_k) - \left(\frac{1}{2} - \alpha\right) F'(x_k)L(x_k)F'(x_k)^{-1}F(x_k),$$

multiplying by $\big(I - \alpha L(x_k)\big)^{-1} F'(x_k)^{-1}$ and rearranging the last equation, we have

$$x_{k+1} = x_k - (I - \alpha L(x_k))^{-1}\left[I + \left(\frac{1}{2} - \alpha\right) L(x_k)\right] F'(x_k)^{-1}F(x_k),$$

by using Lemma A.6 in Appendix A, then we get the Halley class

$$x_{k+1} = x_k - \left[I + \frac{1}{2}L(x_k)\,(I - \alpha L(x_k))^{-1}\right] F'(x_k)^{-1}F(x_k).$$

□

3.3 Practical form of Halley class

One step of Halley class methods (3.2) can be written in terms of a two steps method [11]. Which can be more practical than using the original method.

Consider the Halley class iteration (3.2) and since $F'(x_k)$ is nonsingular, then the two vectors $s^{(1)}$ and $s^{(2)}$ – the index k indicate to the iteration number k – are defined as the following,

$$s^{(1)} = -F'(x_k)^{-1}F(x_k), \quad s^{(2)} = x_{k+1} - \left(x_k + s^{(1)}\right), \tag{3.9}$$

where x_k is the current iteration and x_{k+1} is the next iteration of the Halley class (3.2). Then by using the definition of $s^{(2)}$ in (3.9) and the definition of $L(x)$ in (3.3), we have that

$$\big(I - \alpha L(x_k)\big)\big(s^{(1)} + s^{(2)}\big) = \big(I - \alpha L(x_k)\big)\big(x_{k+1} - x_k\big), \tag{3.10}$$

since the iterate x_{k+1} is defined in (3.2), then equation (3.10) becomes

$$\begin{aligned}&\big(I-\alpha L(x_k)\big)\big(s^{(1)}+s^{(2)}\big)=\\&\qquad -\big(I-\alpha L(x_k)\big)\left[I+\frac{1}{2}L(x_k)\big(I-\alpha L(x_k)\big)^{-1}\right]F'(x_k)^{-1}F(x_k),\end{aligned}$$

by using Lemma A.6 in Appendix A and definition of $s^{(1)}$, we get

$$\begin{aligned}\big(I-\alpha L(x_k)\big)\big(s^{(1)}+s^{(2)}\big)&=\big(I-\alpha L(x_k)\big)\big(I-\alpha L(x_k)\big)^{-1}\left[I+(\frac{1}{2}-\alpha)L(x_k)\right]s^{(1)}\\&=\left[I+(\frac{1}{2}-\alpha)L(x_k)\right]s^{(1)},\end{aligned}$$

so by rearranging the last equation, we get

$$\big(I-\alpha L(x_k)\big)s^{(1)}+\big(I-\alpha L(x_k)\big)s^{(2)}=\big(I-\alpha L(x_k)\big)s^{(1)}+\frac{1}{2}L(x_k)s^{(1)},$$

therefore, we get

$$\big(I-\alpha L(x_k)\big)s^{(2)}=\frac{1}{2}L(x_k)s^{(1)}, \tag{3.11}$$

by multiplying the both sides of equation (3.11) with $F'(x_k)$, substituting the value of $L(x_k)$ in (3.3) into (3.11) and using definition of $s^{(1)}$ in (3.9), we obtain that

$$\Big(F'(x_k)+\alpha F''(x_k)s^{(1)}\Big)s^{(2)}=-\frac{1}{2}F''(x_k)s^{(1)}s^{(1)}. \tag{3.12}$$

From equations (3.9) and (3.12), then we get the two steps method

$$\begin{aligned}F'(x_k)s^{(1)}&=-F(x_k),\\\Big(F'(x_k)+\alpha F''(x_k)s^{(1)}\Big)s^{(2)}&=-\frac{1}{2}F''(x_k)s^{(1)}s^{(1)},\\x_{k+1}&=x_k+s^{(1)}+s^{(2)},\end{aligned} \tag{3.13}$$

which is equivalent to the Halley class (3.2).

The two steps method (3.13) for a quadratic function and for $\alpha=1$ is equivalent to the two steps Newton method. Let $T_k(s)$ be the quadratic approximation to the function $F(x)$ at the point $x_k + s$, as

$$T_k(s) = F(x_k) + F'(x_k)s + \frac{1}{2}F''(x_k)ss \tag{3.14}$$

Consider the quadratic equation with unknown s

$$T_k(s) = 0, \tag{3.15}$$

where $T_k(s)$ is the approximation function at iteration k (3.14). problem (3.15) can be solved by the two steps method (3.13) with $\alpha = 1$. Let $s^{(0)} = 0$ and from definition of $T_k(s)$ (3.14), we observe that

$$F(x_k) = T_k(s^{(0)}),\ F'(x_k) = T_k'(s^{(0)}), \tag{3.16}$$

and

$$\frac{1}{2}F''(x_k)s^{(1)}s^{(1)} = T_k(s^{(1)}) - F(x_k) - F'(x_k)s^{(1)}, \tag{3.17}$$

from equation (3.13) the term $-F(x_k)-F'(x_k)s^{(1)}$ will vanish, so (3.17) will become

$$\frac{1}{2}F''(x_k)s^{(1)}s^{(1)} = T_k(s^{(1)}), \tag{3.18}$$

since $\alpha = 1$, then

$$F'(x_k) + F''(x_k)s^{(1)} = T_k'(s^{(1)}). \tag{3.19}$$

By substituting the observations (3.16), (3.18) and (3.19) into the two steps method (3.13) for $\alpha = 1$, we get that

$$\begin{aligned} s^{(0)} &= 0 \\ T_k'(s^{(0)})s^{(1)} &= -T_k(s^{(0)}) \\ T_k'(s^{(0)} + s^{(1)})s^{(2)} &= -T_k(s^{(0)} + s^{(1)}) \\ x_{k+1} &= x_k + s^{(0)} + s^{(1)} + s^{(2)}, \end{aligned} \tag{3.20}$$

which is a two steps Newton method [11]. Then we conclude that the Super-Halley method (Halley class when $\alpha = 1$) for a quadratic function is two steps Newton method, since Newton method of order two, therefore, Super-Halley method is of order four which is proved in [4, 14].

Consider Chebyshev's method ($\alpha = 0$)

$$\begin{aligned} F'(x_k)s^{(1)} &= -F(x_k), \\ F'(x_k)s^{(2)} &= -\frac{1}{2}F''(x_k)s^{(1)}s^{(1)}, \\ x_{k+1} &= x_k + s^{(1)} + s^{(2)}, \end{aligned}$$

let $s(0) = 0$ and $T_k(s)$ defined in (3.14), by using equations (3.16) and (3.18) in the Chebyshev method, we get

$$\begin{aligned} s^{(0)} &= 0 \\ T'_k(s^{(0)})s^{(1)} &= -T_k(s^{(0)}) \\ T'_k(s^{(0)})s^{(2)} &= -T_k(s^{(0)} + s^{(1)}) \\ x_{k+1} &= x_k + s^{(0)} + s^{(1)} + s^{(2)}, \end{aligned} \tag{3.21}$$

this is two steps of Simplified Newton method [19].

3.4 Convergence Property

This section discusses the local convergence property of the sequence $\{x_k\}$ that is given by (3.2) to the solution of (3.1) under some assumptions and error estimates.

Local convergence means that the initial iterate x_0 is close to a local solution x^* at which the sufficient conditions hold Kelley [18].

In Lemma (3.1) the equivalence between the Halley class (3.2) and Schwetlick class (3.4) was proved. Schwetlick [23] proved that the Schwetlick class (3.4) is cubic convergent (Chapter 5 Theorem 5.7.5). Then theorem 5.7.5 in [23] can be used to prove the convergence for the Halley class. Then the convergence theorem for the Halley class (all real values of α) stated as the following.

Theorem 3.2. *Assume that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is two times continuously differentiable on the neighborhood $\mathcal{N}(r, x^*)$ of a point x^* with radius r, where $F(x^*) = 0$ and*

$F'(x^*)$ is nonsingular. Assume that F'' is Lipschitz continuous on $\mathcal{N}(r, x^*)$. For each α there exists $\varepsilon > 0$, so that for all x_0 satisfying $\|x_0 - x^*\| \leq \varepsilon$ the iterates $\{x_k\}$ in the Halley class (3.2) are defined, and $x_0 \in \mathcal{N}$ and so $\|x_k - x^*\| \leq \varepsilon$, Moreover the iterates $\{x_k\}$ converges to x^* with at least Q-order 3 (see A.3 in the Appendix).

Many authors analysed the convergence theorem to the Chebyshev method ($\alpha = 0$), Halley method ($\alpha = \frac{1}{2}$) and Super-Halley method ($\alpha = 1$). Candela and Marquina [3] proved the convergence of Chebyshev method under the assumptions (1) and (3) in Assumption 3.1 and the following are satisfied

(1) $F'(x_0)^{-1}$ exists and $\|F'(x_0)^{-1}\| \leq B$ for some $B > 0$.

(2) There exists $K_1 > 0$ such that $\|F''(x)^{-1}\| \leq K_1$ for $x \in \mathcal{N}(r, x^*)$.

(3) There exists $a > 0$ such that $\|F'(x_0)^{-1}F(x_0)\| \leq a$.

Hernández and Gutiérrez [15] proved, by assuming Kantorovich-like conditions, the convergence of the Halley class iteration (3.2) for $\alpha \in [0, 1]$. In fact, together with the previous assumption (3), they assume the following ones:

(4) There exists $k \geq 0$ such that $\|F'(x_0)^{-1}(F''(x) - F''(y))\| \leq k\|x - y\|$ for $x, y \in \mathcal{N}(r, x^*)$.

(5) There exists $b > 0$ such that $\|F'(x_0)^{-1}F''(x_0)\| \leq b$.

(6) The polynomial $p(t) = \frac{k}{6}t^3 + \frac{b}{2}t^2 - t + a = 0$, for $k = 0$ has two positive roots r_1 and r_2, and has three roots one negative and two positive r_1 and r_2 for $k > 0$.

And they defined the majorising sequence $\{t_n\}$ as

$$t_0 = 0, t_{n+1} = t_n - \left[1 + \frac{L_p(t_n)}{2(1 - \alpha L_p(t_n))}\right] \frac{p(t_n)}{p'(t_n)}, \ n \geq 0 \tag{3.22}$$

where $L_p(t) = \frac{p(t)p''(t)}{p'(t)^2}$. They proved the majorising sequence is convergent and the convergence is third order. So they used this sequence to prove the convergence for the iteration $\{x_k\}$.

They are other authors analysed the convergence property for these methods (see [1, 6, 12, 25]).

3.5 Region of Convergence

Region of convergence consists of all starting points such that the iteration converges to the solution. Polyak [20] defined a subset of the region of convergence called the attraction basins as the following. There exist a ball $\mathcal{N}(r, x^*)$ of a solution x^* such that every starting point $x_0 \in \mathcal{N}$ implies convergence to the solution.

The region of convergence of Newton method, Chebyshev method, Halley method and Super-Halley method will be studied.

The numerical example taken from Cira [5] problem (1.1). This example is solving the nonlinear equations $F(x) = 0$, where the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} 3x_1^2 - x_2^2 \\ \frac{1}{2}\cosh(\frac{5x_1}{3}) + \frac{3}{5}x_1 - x_2 - \frac{3}{5} \end{bmatrix}. \tag{3.23}$$

Figure 3.1 plots the function's norm of (3.23). Observe that the function's norm

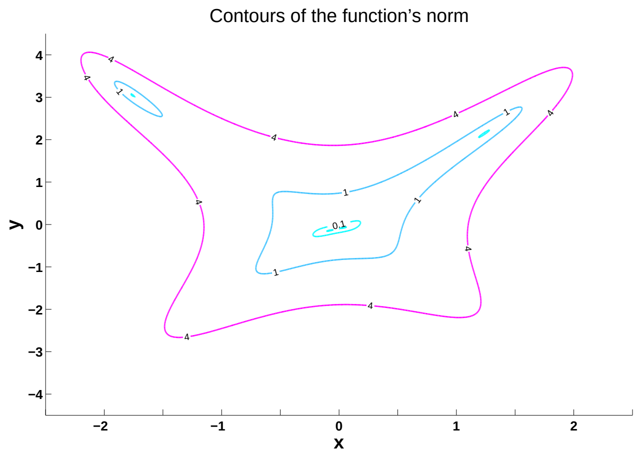


FIGURE 3.1: The figure shows the level curves of the function's norm and the four roots.

has four solutions. The solutions are:

(1) $x^* = (1.238800915\ldots,\ 2.145666126\ldots)$

(2) $x^* = (0.423464990\ldots,\ -0.733462878\ldots)$

(3) $x^* = (-0.0839997787\ldots,\ -0.145491885\ldots)$

(4) $x^* = (-1.758822925\ldots, 3.046370667\ldots)$

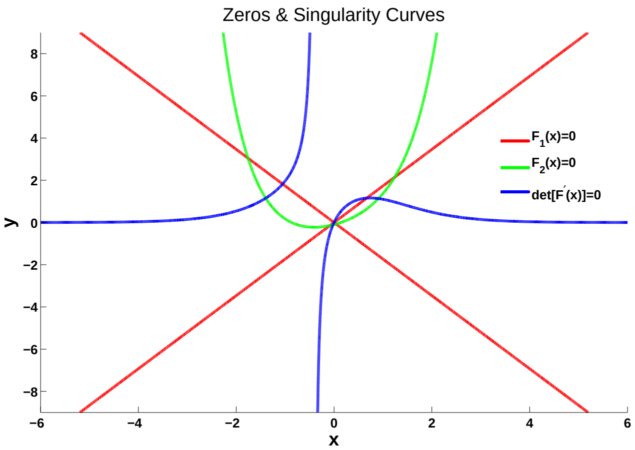


FIGURE 3.2: The figure shows the curves for the system $F(x) = 0$ in (3.23) and the singularity property of the system. The solutions of this system are in the intersections of the two curves (marked by red and green).

In Figure 3.2 the equations $F_i(x_1, x_2) = 0$ are plotted $i = 1, 2$. And the intersections points of $F_1(x_1, x_2) = 0$ and $F_2(x_1, x_2) = 0$ are the solutions of $F(x) = 0$. These solutions are shown clearly in Figure 3.1. In addition the points (x_1, x_2) so that $det[F'(x_1, x_2)] = 0$ are plotted.

The starting points for methods are chosen to be a grid of $N_1.N_2$ points in the rectangle $[a_1, b_1] \times [a_2, b_2]$. The first interval divided in N_1 pieces and the second interval in N_2 pieces. Let $h_1 = \frac{b_1 - a_1}{N_1 - 1}$ and $h_2 = \frac{b_2 - a_2}{N_2 - 1}$. The starting points are then given by:

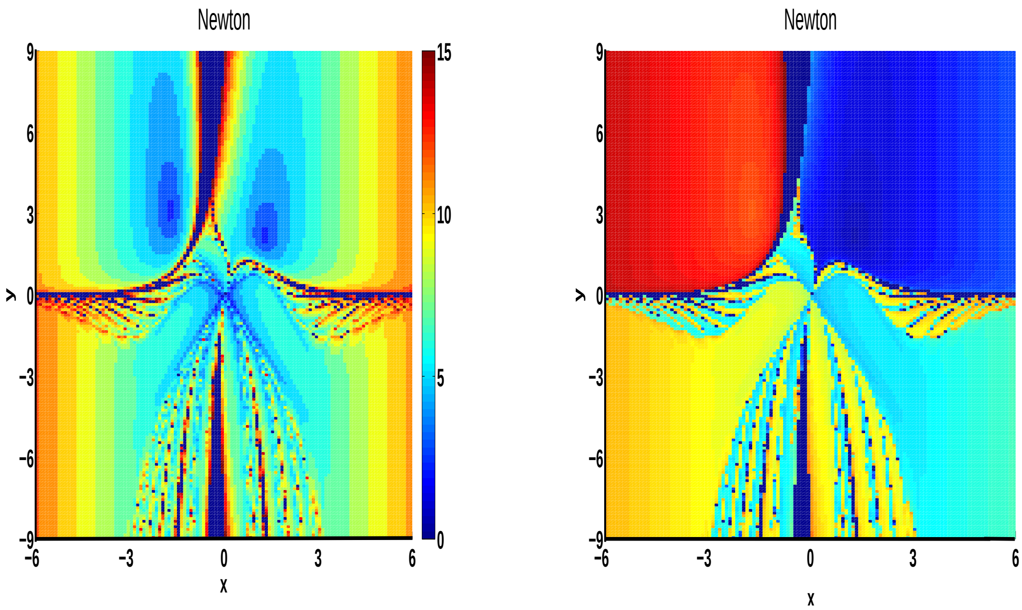
$$x_0 = (a_1 + (i-1)h_1, a_2 + (j-1)h_2), \tag{3.24}$$

where $i = 1, 2, \ldots, N_1$ and $j = 1, 2, \ldots, N_2$.

The grid and the stop criteria which we will use is taken from Cira [5]. The rectangle is $[-6, 6] \times [-9, 9]$ and a grid $121.181 = 21901$ points. The iteration is terminated when $\|F(x)\| \leq Tolerance$, where Tolerance is chosen to be 10^{-3}.

The following experiments applies Newton's method, Chebyshev's method, Halley's method and Super-Halley's method for all the starting points in the rectangle. In Figure 3.3, 3.4, 3.5 and 3.6 the starting points plotted versus the number of iterations.

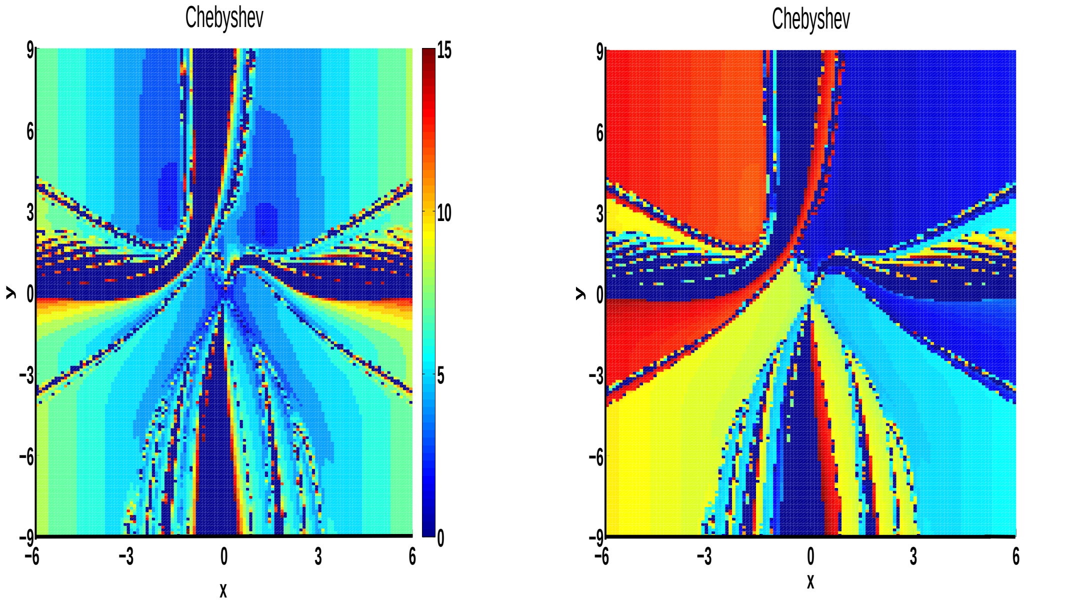
In all Figures in the left plots we give a different color for each starting point in the rectangle according to number of iterations needed to converge to one of the roots. In the right plots we give a different color for each starting points according to the solution that the method converge to, the color degree represents the number of iterations. When the method does not converge for a specific starting point, we color it with dark blue which has number zero in the colorbar.



(*a*) The colors indicate to the number of iterations. (*b*) The colors indicate to which solution the method converges.

Figure 3.3: Newton method

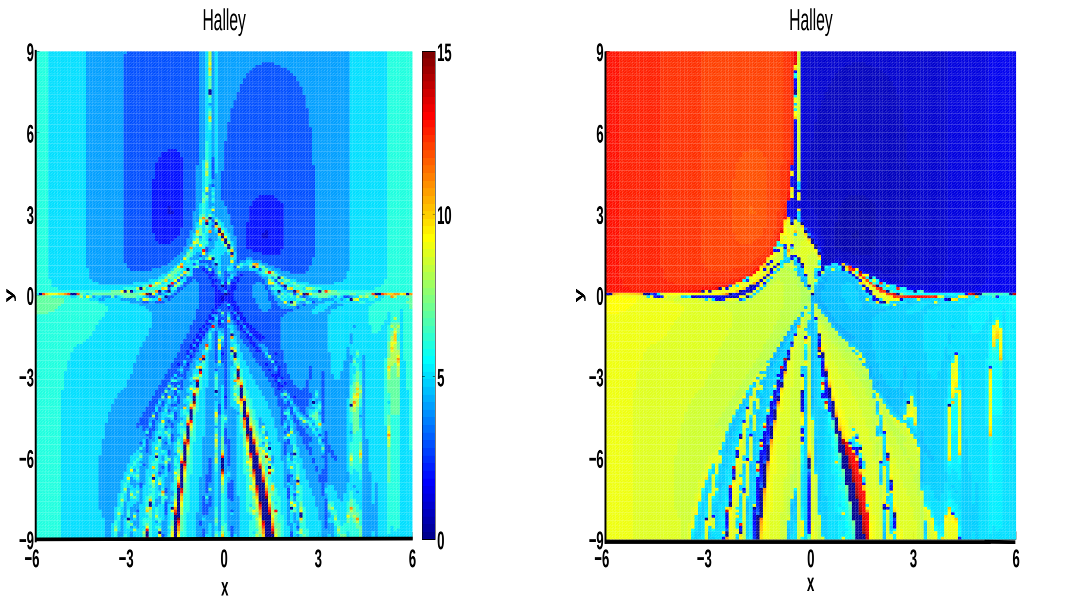
We see in all figures that the region of convergence of the methods ordered from smaller to bigger as the following: Super-Halley, Chebyshev, Newton, and Halley (look at the dark blue areas). Some of the non-convergent points for all methods have the singular property of the first derivative of the function (compared to Figure 3.2). The property of cubic convergence appears clearly in the lighter areas.



(a) The colors indicate to the number of iterations.

(b) The colors indicate to which solution the method converges.

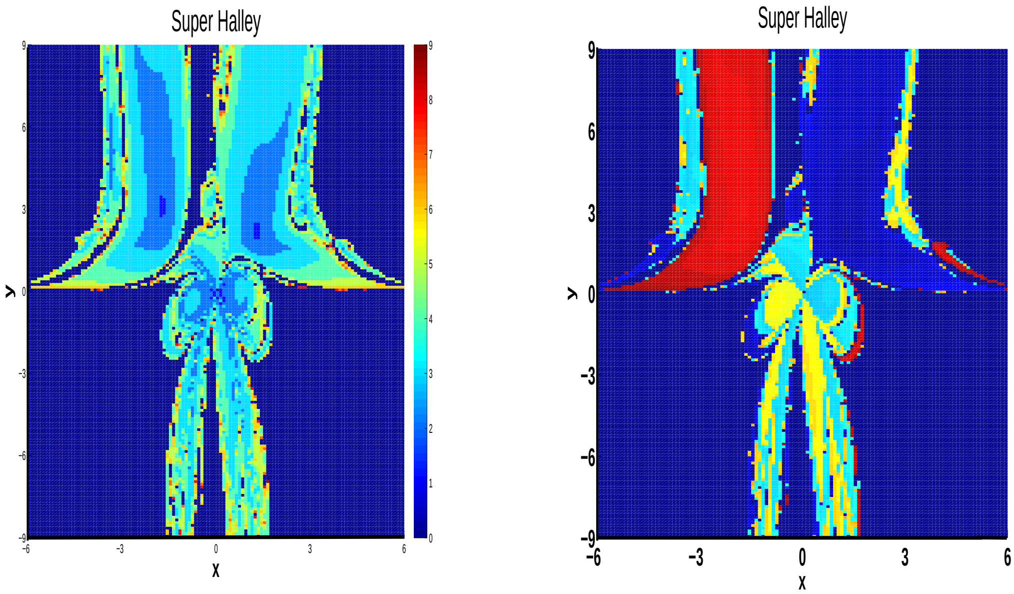
FIGURE 3.4: Chebyshev method



(a) The colors indicate to the number of iterations.

(b) The colors indicate to which solution the method converges.

FIGURE 3.5: Halley method



(a) The colors indicate to the number of iterations. (b) The colors indicate to which solution the method converges.

FIGURE 3.6: Super-Halley method

Chapter 4

Methods Based on Quadratic Model

This chapter introduces a new iterative framework to solve the nonlinear system of equations $F(x) = 0$. The framework solves approximately the quadratic model of the system around the current iterate. We will study the convergence property of this framework.

Section 4.1 gives a general description of the iterative framework, and some observations. Section 4.2 deals with solving the quadratic model. Finally, Section 4.3 discuss the convergence properties of the iterative framework.

4.1 Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \tag{4.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the basic assumptions stated in Assumption 3.1.

For a given point $x_k \in \mathbb{R}^n, k = 0, 1, 2, \ldots$, the model $T_k(s)$ is obtained from Taylor expansion of $F(x)$ at the point x_k,

$$F(x_k + s) \approx T_k(s) \equiv F(x_k) + F'(x_k)s + \frac{1}{2}F''(x_k)ss. \tag{4.2}$$

In other words, $T_k(s)$ is the quadratic approximation of $F(x)$ at the point $x_k + s$ and k is the current iteration. In this strategy, the next iterate would be

$$x_{k+1} = x_k + \tilde{s}, \tag{4.3}$$

where $\tilde{s}$ is the approximate solution of the following subproblem

$$T_k(s) = 0. \tag{4.4}$$

Therefore, we generate the sequence $\{x_k\}$ which converges to the solution x^* of (4.1). Algorithm 4.1 gives the general framework of the method described above.

Algorithm 4.1: Framework

```
1 Given x_0
2 for k = 0, 1, 2, ... until converge do
3     Find approximate solution s̃ for T_k(s) = 0
4     Update x_{k+1} = x_k + s̃
5 end
```

Note that step 3 in Algorithm 4.1 solves the subproblem (4.4) at each iteration. The method to solve (4.4) is discussed in more detail in Section 4.2.

For the rest of the thesis we use the following observations. Let x^* be the solution of the problem (4.1), the ball $\mathcal{N}(\epsilon, x^*) = \{x : \|x - x^*\|_2 \leq \epsilon\}$.

Observation 4.1.

1. *For x close to x^*, the norm of the function F at x will be small. This follows from the continuity of F. For any $0 < \delta < 1$ there exists $\epsilon > 0$ so that*

$$\|F(x)\| \leq \delta, \tag{4.5}$$

   for $x \in \mathcal{N}(\epsilon, x^*)$.

2. *There exists a constant $B_1 > 0$ such that*

$$\|F''(x)\| \leq B_1, \tag{4.6}$$

for $x \in \mathcal{N}(\epsilon, x^*)$.

3. *There exist constant* $c_6 > 0$ *and* $c_7 > 0$ *such that*

$$\left\| \left(F'(x_k + s)\right)^{-1} \right\| \le c_6, \quad \left\| \left(T_k'(s)\right)^{-1} \right\| \le c_7, \tag{4.7}$$

for $x_k \in \mathcal{N}(\epsilon, x^*)$ *and* $x_k + s \in \mathcal{N}(\epsilon, x^*)$.

We assume that these observations are true.

4.2 Solving the quadratic subproblem

This section describes the solution methods to solve the subproblem (4.4) approximately using the two steps of Newton's method, which is equivalent to one step of Super-Halley (Shown in chapter 3), and then generalize to the rest of Halley class methods.

4.2.1 Two steps Newton method

Let $T_k(s)$ be the quadratic function (4.2). Consider two steps of Newton method approximate solving the subproblem $T_k(s) = 0$

$$s^{(0)} = 0,$$

$$T_k'(s^{(0)})s^{(1)} = -T_k(s^{(0)}), \tag{4.8}$$

$$T_k'\big(s^{(0)} + s^{(1)}\big)s^{(2)} = -T_k\big(s^{(0)} + s^{(1)}\big), \tag{4.9}$$

$$\tilde{s} = s^{(0)} + s^{(1)} + s^{(2)}. \tag{4.10}$$

Then Algorithm 4.1 to solve $F(x) = 0$ will be Super Halley method. Assume that equations (4.8) and (4.9) are solved by a direct method such as LU factorization.

Eventually, we will have a residual t, $T_k(\tilde{s}) = t$, where t is the error obtained from approximate solution of the subproblem $T_k(s) = 0$. The following Lemma 4.1 calculate the error t when the two steps of Newton method is used to solve the subproblem (4.4).

Lemma 4.1. *Assume that (4.4) is solved approximately by two steps of Newton method for the starting point* $s^{(0)} = 0$*. And assume that* $\|\big(T_k'(s^{(1)})\big)^{-1}\|$ *is bounded by a constant* $c_1 > 0$*. There exists* $\mu > 0$ *so that the function* $T_k(s)$ *at* $s = s^{(1)} + s^{(2)}$ *satisfies*

$$\big\|T_k\big(s^{(1)} + s^{(2)}\big)\big\| \leq \mu \|F(x_k)\|^4. \tag{4.11}$$

Proof: From equation (4.8), the first step $s^{(1)}$ is given by

$$s^{(1)} = -T_k'(s^{(0)})^{-1} T_k(s^{(0)}),$$

by taking the norm for both sides of the above equation, we get

$$\|s^{(1)}\| \leq \|T_k'(s^{(0)})^{-1}\| \|T_k(s^{(0)})\|,$$

since $s^{(0)} = 0$, and by using the definition of $T_k(s)$ in (4.2), we observe that

$$T_k(s^{(0)}) = F(x_k), \;\; T_k'(s^{(0)}) = F'(x_k) \tag{4.12}$$

and using (4.7) in Observation 4.1 for a constant $c_6 > 0$, we have

$$\|s^{(1)}\| \leq c_6 \|F(x_k)\|. \tag{4.13}$$

Consider the function $T_k(s)$ (4.2) at the point $s^{(1)}$

$$T_k(s^{(1)}) = F(x_k) + F'(x_k)s^{(1)} + \frac{1}{2} F''(x_k) s^{(1)} s^{(1)}.$$

From equation (4.8), the term $F(x_k) + F'(x_k)s^{(1)}$ will vanish and we get

$$T_k(s^{(1)}) = \frac{1}{2} F''(x_k) s^{(1)} s^{(1)},$$

by taking the norm for both sides, we get

$$\|T_k(s^{(1)})\| \leq \frac{1}{2} \|F''(x_k)\| \|s^{(1)}\|^2. \tag{4.14}$$

Using (4.6) in Observation 4.1, therefore, (4.14) becomes

$$\|T_k(s^{(1)})\| \leq \frac{1}{2}B_1\|s^{(1)}\|^2,$$

by substitute the inequality (4.13) into the above equation, we obtain

$$\|T_k(s^{(1)})\| \leq C_1\|F(x_k)\|^2, \tag{4.15}$$

where $C_1 = \frac{1}{2}B_1c_6^2$. From equation (4.9), the second step $s^{(2)}$ is given by

$$s^{(2)} = -T_k'(s^{(1)})^{-1}T_k(s^{(1)}),$$

by taking the norm, we get

$$\|s^{(2)}\| \leq \|T_k'(s^{(1)})^{-1}\| \; \|T_k(s^{(1)})\|,$$

using the assumption that $\|T_k'(s^{(1)})^{-1}\|$ is bounded by $c_1 > 0$ in the above equation, we have

$$\|s^{(2)}\| \leq c_1 \; \|T_k(s^{(1)})\|,$$

by substituting the inequality (4.15) into the above equation, we obtain

$$\|s^{(2)}\| \leq \beta\|F(x_k)\|^2, \tag{4.16}$$

where $\beta = c_1C_1$. To prove (4.11), using Taylor expansion of $T_k(s)$ (4.2) at the point $s^{(1)}$

$$T_k\big(s^{(1)} + s^{(2)}\big) = T_k(s^{(1)}) + T_k'(s^{(1)})s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

from equation (4.9), the term $T_k(s^{(1)}) + T_k'(s^{(1)})s^{(2)}$ vanished. From the definition of $T_k(s)$, we have $T_k''(s^{(1)}) = F''(x_k)$. Therefore, the above equation becomes

$$T_k\big(s^{(1)} + s^{(2)}\big) = \frac{1}{2}F''(x_k)s^{(2)}s^{(2)},$$

by taking the norm, we get

$$\|T_k\big(s^{(1)} + s^{(2)}\big)\| \leq \frac{1}{2}\|F''(x_k)\|\|s^{(2)}\|^2,$$

using (4.6) in Observation 4.1 and (4.16) , we obtain

$$\|T_k\big(s^{(1)} + s^{(2)}\big)\| \leq \mu\|F(x_k)\|^4,$$

where $\mu = \frac{B_1}{2}\beta^2$. □

Lemma 4.1 shows that the error $t = T_k(\tilde{s})$ is $\mathcal{O}(\|F(x_k)\|^4)$ when two steps of Newton is used to solve the subproblem.

4.2.2 One step of Halley class methods

The subproblem (4.4) will be approximately solved using one step of Halley class method (3.2) with some value of α. In this Case, our method stated in Algorithm 4.1 is a Halley class method with some value of α.

We have shown that one step of Halley class can be written as two steps in chapter 3. Then the one step of Halley class method approximate solve $T_k(s) = 0$ is

$$s^{(0)} = 0, \tag{4.17}$$
$$T'_k(s^{(0)})s^{(1)} = -T_k(s^{(0)}), \tag{4.18}$$
$$\Big(T'_k(s^{(0)}) + \alpha T''_k(s^{(0)})s^{(1)}\Big)s^{(2)} = -\frac{1}{2}T''_k(s^{(0)})s^{(1)}s^{(1)}, \tag{4.19}$$
$$\tilde{s} = s^{(0)} + s^{(1)} + s^{(2)}. \tag{4.20}$$

We want to rewrite the second equation in terms of $s^{(1)}$. We have from definition of $T_k(s)$

$$T'_k(s) = F'(x_k) + F''(x_k)s, \tag{4.21}$$

and

$$T''_k(s) = F''(x_k), \tag{4.22}$$

therefore, since $s^{(0)} = 0$, the left hand side of equation (4.19) can be written as

$$T_k'(s^{(0)}) + \alpha T_k''(s^{(0)})s^{(1)} = F'(x_k) + \alpha F''(x_k)s^{(1)}, \tag{4.23}$$

by adding and subtracting the term $F''(x_k)s^{(1)}$ to the right hand side of the above equation, we get

$$T_k'(s^{(0)}) + \alpha T_k''(s^{(0)})s^{(1)} = F'(x_k) + F''(x_k)s^{(1)} - (1-\alpha)F''(x_k)s^{(1)},$$

thus by substituting again equations (4.21) and (4.22) into the above equation, we obtain

$$T_k'(s^{(0)}) + \alpha T_k''(s^{(0)})s^{(1)} = T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}. \tag{4.24}$$

Therefore, one step of Halley class can be written as

$$\begin{aligned} s^{(0)} &= 0, \\ T_k'(s^{(0)})s^{(1)} &= -T_k(s^{(0)}), && (4.25) \\ \big(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\big)s^{(2)} &= -T_k(s^{(1)}), && (4.26) \\ \tilde{s} &= s^{(0)} + s^{(1)} + s^{(2)}. && (4.27) \end{aligned}$$

Assume that the linear systems (4.25) and (4.26) solved by a direct method such as LU factorization.

The following theorem calculate the error $t = T_k(\tilde{s})$ when one step of the Halley class method used for solving the subproblem (4.4).

Theorem 4.2. *Assume that (4.4) is solved by one step of Halley class method with some value of α. Let $s^{(1)}$ be the solution of the equation (4.25). Assume that $\left\| \left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)} \right)^{-1} \right\|$ is bounded by a constant $c_3 > 0$. Then*

$$\|T_k(s^{(1)} + s^{(2)})\| \le |1-\alpha|\mu_1 \|F(x_k)\|^3 + \mu_2 \|F(x_k)\|^4. \tag{4.28}$$

Proof: The first equation (4.25) in one step of Halley class method is equal to the first one (4.8) in two steps Newton method. Therefore, the inequality (4.13) with a constant $c_6 > 0$ is valid for $s^{(1)}$ and also the inequality (4.15) with a constant $C_1 = \frac{1}{2}B_1 c_6^2$ is verified for $T_k(s^{(1)})$.

From equation (4.19) $s^{(2)}$ is given by

$$s^{(2)} = \left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\right)^{-1} T_k(s^{(1)}),$$

by taking the norm and using assumption $\left\|\left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\right)^{-1}\right\|$ is bounded by c_3, we get

$$\|s^{(2)}\| \leq c_3 \|T_k(s^{(1)})\|,$$

by substituting the inequality (4.15) into the above equation, we obtain

$$\|s^{(2)}\| \leq C_2 \|F(x_k)\|^2, \tag{4.29}$$

where $C_2 = c_3 C_1$.

Consider the function $T_k(s)$ at $s = s^{(1)} + s^{(2)}$, using the Taylor expansion of $T_k(s)$ at $s^{(1)}$ we have

$$T_k(s^{(1)} + s^{(2)}) = T_k(s^{(1)}) + T_k'(s^{(1)})s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

using equation (4.26), we get

$$T_k(s^{(1)} + s^{(2)}) = (1-\alpha)T_k''(s^{(1)})s^{(1)}s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

by taking the norm

$$\|T_k(s^{(1)} + s^{(2)})\| \leq |1-\alpha| \; \|T_k''(s^{(1)})\|\|s^{(1)}\|\|s^{(2)}\| + \frac{1}{2}\|T_k''(s^{(1)})\|\|s^{(2)}\|^2$$

since $T_k''(s^{(1)}) = F''(x_k)$, then by using (4.6) in Observation 4.1 we obtain

$$\|T_k(s^{(1)} + s^{(2)})\| \leq |1-\alpha| \; B_1\|s^{(1)}\|\|s^{(2)}\| + \frac{1}{2}B_1\|s^{(2)}\|^2,$$

by substituting the inequalities (4.13) and (4.29) into the above equation, we get

$$\|T_k(s^{(1)} + s^{(2)})\| \leq |1-\alpha| \; \mu_1 \|F(x_k)\|^3 + \mu_2 \|F(x_k)\|^4 \tag{4.30}$$

where $\mu_1 = c_6 B_1 C_2$ and $\mu_2 = B_1 C_2^2/2$. □

We observe that from Theorem 4.2 the error t is depend on α. And for the whole Halley class, there are two cases: when $\alpha = 1$ the error is $\mathcal{O}(\|F(x_k)\|^4)$. And for $\alpha \neq 1$, using (4.6) in Observation 4.1, therefore, the error t is $\mathcal{O}(\|F(x_k)\|^3)$. Thus Super-Halley method $\alpha = 1$ give us more accurate solution of the quadratic equation $T_k(s) = 0$ than the other method in the Halley class.

4.3 Convergence properties

Assume that the problem $F(x_k) = 0$ solved by Algorithm 4.1. Also assume that the subproblem $T_k(s) = 0$, where $T_k(s)$ is the quadratic model (4.2) is solved by an iterative method. The iterative method generates the iteration $\{s_j\}$, $j = 0, 1, 2, \ldots$ which converges to $s_{\hat{j}}$, where $s_{\hat{j}}$ is the solution of $T_k(s) = 0$. Thus the sequence x_k updates by $x_{k+1} = x_k + s_{\hat{j}}$.

Assume that in each iteration s_j and for given point x_k, there exist constants $0 \leq \eta_k \leq \eta_0 < 1$ and $\theta > 0$ so that

$$\|s_j\| \leq \theta \|F(x_k)\|, \;\; \|T_k(s_j)\| \leq \eta_k \|F(x_k)\|. \tag{4.31}$$

We will show that the function $F(x)$ at the sequence points $\{x_k\}$ is decreasing

$$\|F(x_{k+1})\| \leq \lambda \|F(x_k)\|, \tag{4.32}$$

where $0 < \lambda < 1$. Let $s_{\hat{j}}$ be the solution of (4.4), since $x_{k+1} = x_k + s_{\hat{j}}$, we have

$$F(x_{k+1}) = F(x_k + s_{\hat{j}}).$$

By using Taylor expansion of $F(x)$ at the point x_k, we have

$$F(x_{k+1}) = F(x_k) + F'(x_k)s_{\hat{j}} + \frac{1}{2}F''(x_k)s_{\hat{j}}s_{\hat{j}} + \mathcal{O}(\|s_{\hat{j}}\|^3),$$

using the definition of $T_k(s)$ in (4.2), we have

$$F(x_{k+1}) = T_k(s_{\hat{j}}) + \mathcal{O}(\|s_{\hat{j}}\|^3),$$

taking the norm, we get

$$\|F(x_{k+1})\| \leq \|T_k(s_{\hat{j}})\| + \mathcal{O}(\|s_{\hat{j}}\|^3),$$

using the assumption (4.31), we obtain

$$\|F(x_{k+1})\| \leq \eta_k\|F(x_k)\| + \mathcal{O}(\|s_{\hat{j}}\|^3), \tag{4.33}$$

using the definition of big $\mathcal{O}$ in [17], then there exists a constant $C > 0$ so that for all k , we have

$$\|F(x_{k+1})\| \leq \eta_k\|F(x_k)\| + C\|s_{\hat{j}}\|^3,$$

using (4.31), we have

$$\|F(x_{k+1})\| \leq (\eta_k + C\theta^3\|F(x_k)\|^2)\|F(x_k)\|$$

using (4.5) in Observation 4.1 for $0 < \delta < 1$, we get

$$\begin{aligned}\|F(x_{k+1})\| &\leq (\eta_k + C\theta^3\delta^2)\|F(x_k)\| \\ &= \lambda\|F(x_k)\|\end{aligned}$$

where $\lambda = \eta_k + C\theta^3\delta^2$. By choosing δ to be too small so that $\lambda < 1$, then we obtain (4.32). And then $F(x)$ called a contractive function (see A.2 in Appendix).

Since

$$\frac{1}{\gamma}\|x_k - x^*\| \leq \|F(x_k)\| \leq \gamma\|x_k - x^*\| \tag{4.34}$$

which proved in lemma(3.1) in [7], we have the following lemma.

Lemma 4.3. *Assume that the problem (4.4) is solved using an iterative method which satisfies (4.31). Let x_0 be the starting point, so that, $x_0 \in \mathcal{N}(x^*, \epsilon)$. Assume that (4.32) is verified. Then $x_k \to x^*$.*

Proof. From (4.34), we have

$$\frac{1}{\gamma}\|x_k - x^*\| \leq \|F(x_k)\|.$$

Since the function $F(x)$ is a contraction function, then $\|F(x_k)\| \to 0$ as $k \to \infty$. Then x_k is convergent to x^* when $k \to \infty$. □

Theorem 4.4. *Consider the problem (4.1), Suppose that the Jacobian matrix at the solution x^* is nonsingular. Assume that (4.1) solved by an iterative method, which generates a sequence $\{x_k\}$ which converges to the solution x^*. Then the framework converges quadratically when $\eta_k = \mathcal{O}(\|F(x_k)\|)$. The framework converges cubically if $\eta_k = O(\|F(x_k)\|^2)$.*

Proof. By using (4.34), we get

$$\|x_{k+1} - x^*\| \leq \gamma\|F(x_{k+1})\|,$$

by substituting the inequality (4.33)

$$\|x_{k+1} - x^*\| \leq \gamma(\|T_k(s_k^j)\| + \mathcal{O}(\|s_j\|^3)$$

using the assumption (4.31) and definition of $\mathcal{O}$ for constant $C > 0$, we obtain

$$\|x_{k+1} - x^*\| \leq \gamma\eta_k\|F(x_k)\| + \gamma C\theta^3\|F(x_k)\|^3. \tag{4.35}$$

Let $\eta_k = \mathcal{O}(\|F(x_k)\|$, using the definition of $\mathcal{O}$ for constant $L_2 > 0$, then (4.35) becomes

$$\|x_{k+1} - x^*\| \leq \gamma L_2\|F(x_k)\|^2 + \gamma C\theta^3\|F(x_k)\|^3,$$

using (4.5) in Observation 4.1, we have

$$\|x_{k+1} - x^*\| \leq (\gamma L_2 + \gamma C\theta^3\delta)\|F(x_k)\|^2$$

using again the inequality (4.34), we get

$$\|x_{k+1} - x^*\| \leq \gamma^3(L_2 + C\theta^3\delta)\|x_k - x^*\|^2,$$

then the method converges quadratically to the solution.

Now by putting $\eta_k = \mathcal{O}(\|F(x_k)\|^2$ in (4.35), using the definition of $\mathcal{O}$ for constant $L_4 > 0$, we get

$$\|x_{k+1} - x^*\| \leq \gamma L_4\|F(x_k)\|^3 + \gamma C\theta^3\|F(x_k)\|^3,$$

collecting the terms with $\|F(x_k)\|^3$

$$\|x_{k+1} - x^*\| \leq \gamma\big(L_4 + C\theta^3\big)\|F(x_k)\|^3$$

using again the inequality (4.34), we get

$$\|x_{k+1} - x^*\| \leq \gamma^4(L_4 + C\theta^3)\|x_k - x^*\|^3,$$

then the method has cubic convergence to the solution. □

Chapter 5

Practical Methods

Consider the system of quadratic equations

$$T_k(s) = 0, \tag{5.1}$$

where the function $T_k(s)$ defined in (4.2). We have shown that one step of Super-Halley is two steps of Newton's method with starting point $s^{(0)} = 0$ on a quadratic function.

$$T_k'(s^{(0)})s^{(1)} = -T_k(s^{(0)}), \tag{5.2}$$
$$T_k'(s^{(1)})s^{(2)} = -T_k(s^{(1)}), \tag{5.3}$$

where we solve for the unknown variables $s^{(1)}$ and $s^{(2)}$. $\tilde{s} = s^{(0)} + s^{(1)} + s^{(2)}$ will then be an approximate solution of (5.1).

In the previous chapter, the linear system (5.2) and (5.3) were solved using a direct method such as Gaussian elimination or LU Factorization. But in this chapter, (5.2) and (5.3) are solved approximately using an iterative method such as Conjugate gradient (CG) method. Also in this chapter the Observation 4.1 is used in the analysis.

5.1 Inexact Super-Halley

Suppose that the subproblem (5.1) approximately solved by two steps Newton method (4.8, 4.9 and 4.10) for starting point $s^{(0)} = 0$. And assume that the linear systems (4.8) and (4.9) solved by an iterative method. Then we have

$$T'_k(s^{(0)})s^{(1)} = -T_k(s^{(0)}) + r_k^{(1)}, \tag{5.4}$$
$$T'_k(s^{(1)})s^{(2)} = -T_k(s^{(1)}) + r_k^{(2)}, \tag{5.5}$$

and $\tilde{s} = s^{(0)} + s^{(1)} + s^{(2)}$ is an approximate solution of (5.1). It is called two steps of inexact Newton method [7]. We assume that the residual $r_k^{(1)}$ and $r_k^{(2)}$ for $0 < p, q \leq 1$ satisfy the following

$$\|r_k^{(1)}\| \leq \|T_k(s^{(0)})\|^{1+p}, \tag{5.6}$$
$$\|r_k^{(2)}\| \leq \|T_k(s^{(1)})\|^{1+q}. \tag{5.7}$$

In terms of the function F, the two iterations of inexact Newton method can be written as:

$$F'(x_k)s^{(1)} = -F(x_k) + r_k^{(1)}, \tag{5.8}$$
$$(F'(x_k) + F''(x_k)s^{(1)})s^{(2)} = -r_k^{(1)} - \frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + r_k^{(2)}, \tag{5.9}$$
$$\tilde{s} = s^{(0)} + s^{(1)} + s^{(2)}.$$

Lemma 5.1. *Let $T_k(s)$ be a function of the form (4.2). Let $s^{(1)}$ and $s^{(2)}$ as given by (5.4) and (5.5), respectively. Assume that $r_k^{(1)}$ and $r_k^{(2)}$ satisfy (5.6) and (5.7), respectively. Then there exists a constant $L > 0$ such that*

$$\|s^{(1)}\| \leq L\|T_k(s^{(0)})\|, \tag{5.10}$$

and

$$\|s^{(2)}\| \leq L\|T_k(s^{(1)})\|, \tag{5.11}$$

provided that

$$\max\{\|T_k(s^{(0)})\|,\ \|T_k(s^{(1)})\|\} \leq 1,$$

where $L = 2\max\left\{\left\|\left(T'_k(s^{(0)})\right)^{-1}\right\|,\ \left\|\left(T'_k(s^{(1)})\right)^{-1}\right\|\right\} < \infty.$

Proof : To prove (5.10), start from equation (5.4) and so $s^{(1)}$ defined as

$$s^{(1)} = -\big(T_k'(s^{(0)})\big)^{-1}\big(T_k(s^{(0)}) + r_k^{(1)}\big),$$

by taking the norm, we get

$$\|s^{(1)}\| \le \big\|\big(T_k'(s^{(0)})\big)^{-1}\big\|\big(\|T_k(s^{(0)})\| + \|r_k^{(1)}\|\big),$$

using equation (5.6) for $p \in (0, 1]$, we get

$$\|s^{(1)}\| \le \big\|\big(T_k'(s^{(0)})\big)^{-1}\big\|\big(\|T_k(s^{(0)})\| + \|T_k(s^{(0)})\|^{1+p}\big),$$

since $\|T_k(s^{(0)})\| \le 1$, then $\|T_k(s^{(0)})\|^{1+p} < \|T_k(s^{(0)})\|$ and so the above equation becomes

$$\|s^{(1)}\| \le 2\big\|\big(T_k'(s^{(0)})\big)^{-1}\big\|\big\|T_k(s^{(0)})\big\|,$$

The above equation can be rewritten as

$$\|s^{(1)}\| \le L\big\|T_k(s^{(0)})\big\|,$$

where $L = 2\max\big\{\big\|\big(T_k'(s^{(0)})\big)^{-1}\big\|,\ \big\|\big(T_k'(s^{(1)})\big)^{-1}\big\|\big\}$.

Prove (5.11), from equation (5.5) $s^{(2)}$ is given by

$$s^{(2)} = -\big(T_k'(s^{(1)})\big)^{-1}\big(T_k(s^{(1)}) + r_k^{(2)}\big),$$

taking the norm for both sides

$$\|s^{(2)}\| \le \big\|\big(T_k'(s^{(1)})\big)^{-1}\big\|\big(\|T_k(s^{(1)})\| + \|r_k^{(2)}\|\big),$$

using the assumption on the residual $r_k^{(2)}$ (5.7) for $q \in (0, 1]$, we get

$$\|s^{(2)}\| \le \big\|\big(T_k'(s^{(1)})\big)^{-1}\big\|(\|T_k(s^{(1)})\| + \|T_k(s^{(1)})\|^{1+q}).$$

Since $\|T_k(s^{(1)})\| \le 1$, and so $\|T_k(s^{(1)})\|^{1+q} < \|T_k(s^{(1)})\|$. Then we get

$$\|s^{(2)}\| \le 2\big\|\big(T_k'(s^{(1)})\big)^{-1}\big\|\|T_k(s^{(1)})\|.$$

The above equation can be rewritten as

$$\|s^{(2)}\| \le L\|T_k(s^{(1)})\|,$$

where $L = 2\max\big\{\big\|\big(T_k'(s^{(0)})\big)^{-1}\big\|,\ \big\|\big(T_k'(s^{(1)})\big)^{-1}\big\|\big\}$. □

Theorem 5.2. *Let $T_k(s)$ be a function of the form (4.2). And assume that $s^{(1)}$ and $s^{(2)}$ are the solutions of the equations of the two steps inexact Newton method in (5.4) and (5.5) respectively, $0 < p, q \le 1$. Then there exists a constant $M > 0$, such that*

$$\|T_k(s^{(1)})\| \le \|T_k(0)\|^{1+p}\big(1 + M\|T_k(0)\|^{1-p}\big), \tag{5.12}$$

and

$$\|T_k(s^{(1)} + s^{(2)})\| \le \|T_k(s^{(1)})\|^{1+q}\big(1 + M\|T_k(s^{(1)})\|^{1-q}\big), \tag{5.13}$$

where $M = \frac{1}{2}B_1L^2$. Moreover, for x sufficiently close to x^* where $F(x^*) = 0$. Then there exists a constant $M_1 > 0$ so that

$$\|T\big(s^{(1)} + s^{(2)}\big)\| \le M_1\|T_k(0)\|^{(1+p)(1+q)} \tag{5.14}$$

Proof : We start by proving (5.12). Using the definition of $T_k(s)$ in (4.2), we get

$$T_k(s^{(1)}) = F(x_k) + F'(x_k)s^{(1)} + \frac{1}{2}F''(x_k)s^{(1)}s^{(1)},$$

using the definition of $s^{(1)}$ in (5.8), we get

$$T_k(s^{(1)}) = r_k^{(1)} + \frac{1}{2}F''(x_k)s^{(1)}s^{(1)},$$

taking the norm for the both sides, we get

$$\|T_k(s^{(1)})\| \leq \|r_k^{(1)}\| + \frac{1}{2}\|F''(x_k)\|\|s^{(1)}\|^2,$$

using the assumption on $r_k^{(1)}$ (5.6), Lemma 5.1 and (4.6), we get

$$\begin{aligned}\|T_k(s^{(1)})\| &\leq \|T_k(s^{(0)})\|^{1+p} + \frac{1}{2}B_1L^2\|T_k(s^{(0)})\|^2,\\ &= \|T_k(0)\|^{1+p}\big(1 + M\|T_k(0)\|^{1-p}\big),\end{aligned}$$

where $M = \frac{1}{2}B_1L^2$. Then we obtain (5.12).

The inequality (5.13) will be shown as the following. Let $T_k(s)$ defined in (4.2). Using Taylor expansion of $T_k(s)$ at $s^{(1)}$, we have

$$T_k(s^{(1)} + s^{(2)}) = T_k(s^{(1)}) + T_k'(s^{(1)})s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)}.$$

Combining the equation (5.5) and the above equation, we get

$$T_k(s^{(1)} + s^{(2)}) = r_k^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

since $T_k''(s^{(1)}) = F''(x_k)$, we have

$$T_k(s^{(1)} + s^{(2)}) = r_k^{(2)} + \frac{1}{2}F''(x_k)s^{(2)}s^{(2)},$$

taking the norm to above equation, we get

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \|r_k^{(2)}\| + \frac{1}{2}\|F''(x_k)\|\|s^{(2)}\|^2,$$

using Lemma 5.1, the assumption on $r_k^{(2)}$ (5.7) and (4.6), we obtain

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \|T_k(s^{(1)})\|^{1+q} + \frac{1}{2}B_1L^2\|T_k(s^{(1)})\|^2,$$

collecting the terms with $\|T_k(s^{(1)})\|^{1+q}$, we get (5.13)

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \|T_k(s^{(1)})\|^{1+q}\big(1 + M\|T_k(s^{(1)})\|^{1-q}\big),$$

where $M = \frac{1}{2}L^2B_1$.

To prove (5.14), equation (5.13) used with q in the interval $(0, 1]$

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \|T_k(s^{(1)})\|^{1+q}\big(1 + M\|T_k(s^{(1)})\|^{1-q}\big),$$

substituting the inequality (5.12) with p in $(0, 1]$ into the above equation, we get

$$\begin{aligned}\|T_k(s^{(1)} + s^{(2)})\| \leq & \|T_k(0)\|^{(1+p)(1+q)}\big(1 + M\|T_k(0)\|^{1-p}\big)^{1+q} \\ & \big(1 + M\|T_k(0)\|^{(1+p)(1-q)}\big(1 + M\|T_k(0)\|^{1-p}\big)^{1-q}\big)\end{aligned}$$

Since x is sufficiently close to the solution x^* and $T_k(0) = F(x_k)$. Then by using (4.5) in Observation (4.1), we get (5.14)

$$\|T_k(s^{(1)} + s^{(2)})\| \leq M_1\|T_k(0)\|^{(1+p)(1+q)}$$

where $M_1 = \big(1 + M\delta^{1-p}\big)^{1+q}\big(1 + M\delta^{(1+p)(1-q)}\big(1 + M\delta^{1-p}\big)^{1-q}\big)$. □

Now look at equation (5.14) and equation (4.31), then η_k is this case defined as the following,

$$\eta_k \leq O\big(\|F(x_k)\|^{(1+p)(1+q)-1}\big) \tag{5.15}$$

from Theorem 4.4, the method is Q-third order if $(1 + p)(1 + q) \geq 3$.

5.2 The inexact Halley class

In this section we consider the case when $\alpha \neq 1$.

Consider one step in a method of the Halley class (4.25, 4.26 and 4.27) for approximately solve the subproblem (5.1). Assume that the linear systems (4.25) and (4.26) are solved by an iterative method. Then we obtain the one step of Inexact

Halley class

$$
\begin{aligned}
s^{(0)} &= 0 \\
T_k'(s^{(0)})s^{(1)} &= -T_k(s^{(0)}) + r_k^{(1)} && (5.16) \\
\left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\right)s^{(2)} &= -T_k(s^{(1)}) + r_k^{(2)} && (5.17) \\
\tilde{s} &= s^{(0)} + s^{(1)} + s^{(2)}. && (5.18)
\end{aligned}
$$

Since $s^{(1)}$ has the same definition as in (5.8), then equation (5.10) is verified and also equation (5.12) will be satisfied. Then to find the bounded for $s^{(2)}$ and $T_k(s^{(1)} + s^{(2)})$, we assume that there exists $c_3 > 0$ such that

$$
\left\| \left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)} \right)^{-1} \right\| \leq c_3. \tag{5.19}
$$

Define $G = T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}$. From equation (5.17), $s^{(2)}$ is given by

$$
\begin{aligned}
s^{(2)} = &- \left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\right)^{-1} T_k(s^{(1)}) \\
&+ \left(T_k'(s^{(1)}) - (1-\alpha)T_k''(s^{(1)})s^{(1)}\right)^{-1} r_k^{(2)},
\end{aligned}
$$

taking the norm for both sides of the above equation

$$
\|s^{(2)}\| \leq \|(G))^{-1}\| \|T_k(s^{(1)})\| + \|(G)^{-1}\| \|r_k^{(2)}\|,
$$

using the assumption on the residual $r_k^{(2)}$ (5.7) and (5.19)

$$
\|s^{(2)}\| \leq c_3(\|T_k(s^{(1)})\| + \|T_k(s^{(1)})\|^{1+q})
$$

Since $\|T_k(s^{(1)})\| \leq 1$, and so $\|T_k(s^{(1)})\|^{1+q} < \|T_k(s^{(1)})\|$. Then we get

$$
\|s^{(2)}\| \leq L_3 \|T_k(s^{(1)})\| \tag{5.20}
$$

where $L_3 = 2c_3$.

Now look at $T_k(s)$ at $s = s^{(1)} + s^{(2)}$, using Taylor expansion of $T_k(s)$ at $s^{(1)}$

$$T_k(s^{(1)} + s^{(2)}) = T_k(s^{(1)}) + T_k'(s^{(1)})s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

using the definition of $s^{(2)}$ (5.17), we get

$$T_k(s^{(1)} + s^{(2)}) = r_k^{(2)} + (1-\alpha)T_k''(s^{(1)})s^{(1)}s^{(2)} + \frac{1}{2}T_k''(s^{(1)})s^{(2)}s^{(2)},$$

since $T_k''(s^{(1)}) = F''(x_k)$

$$T_k(s^{(1)} + s^{(2)}) = r_k^{(2)} + (1-\alpha)F''(x_k)s^{(1)}s^{(2)} + \frac{1}{2}F''(x_k)s^{(2)}s^{(2)},$$

taking the norm for the above equation, we get

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \|r_k^{(2)}\| + |1-\alpha|\|F''(x_k)\|\|s^{(1)}\|\|s^{(2)}\| + \frac{1}{2}\|F''(x_k)\|\|s^{(2)}\|^2,$$

using the assumption on $r_k^{(2)}$ (5.7), (5.20), (5.10) and (4.6)

$$\begin{aligned}\|T_k(s^{(1)} + s^{(2)})\| \leq \|T_k(s^{(1)})\|^{1+q} &+ |1-\alpha|D_1\|F(x_k)\|\|T_k(s^{(1)})\| \\ &+ M\|T_k(s^{(1)})\|^2,\end{aligned} \tag{5.21}$$

where $D_1 = B_1L_3L$ and $M = \frac{1}{2}B_1L^2$.

from equation (5.12), equation (5.21) becomes

$$\begin{aligned}\|T_k(s^{(1)} + s^{(2)})\| \leq &\|T_k(0)\|^{(1+q)(1+p)}\big(1 + M\|T_k(0)\|^{1-p}\big)^{1+q} \\ &+ |1-\alpha|D_1\|T_k(0)\|^{2+p}\big(1 + M\|T_k(0)\|^{1-p}\big) \\ &+ M\|T_k(0)\|^{2(1+p)}\big(1 + M\|T_k(0)\|^{1-p}\big)^2,\end{aligned}$$

since $T_k(0) = F(x_k)$, so (4.5) used for small $\delta > 0$. Then we obtain

$$\|T_k(s^{(1)} + s^{(2)})\| \leq \sigma_1\|T_k(0)\|^{(1+q)(1+p)} + \sigma_2|1-\alpha|\|T_k(0)\|^{2+p} + \sigma_3\|T_k(0)\|^{2(1+p)}, \tag{5.22}$$

where $\sigma_1 = \big(1 + M\delta^{1-p}\big)^{1+q}, \sigma_2 = D_1(1 + M\delta^{1-p})$ and $\sigma_3 = M\big(1 + M\delta^{1-p}\big)^2$. Since $q \in (0, 1]$, we observe that $(1 + p)(1 + q) \leq 2(1 + p)$. And by using equation (4.5), equation (5.22) becomes

$$\|T_k(s^{(1)} + s^{(2)})\| \le \sigma_4 \|T_k(0)\|^{(1+q)(1+p)} + \sigma_2 |1-\alpha| \|T_k(0)\|^{2+p}, \tag{5.23}$$

where $\sigma_4 = \sigma_1 + \sigma_3$.

Remark 5.3.

1. Equation (5.23) is valid for all methods which $\alpha \neq 1$ in the Inexact Halley class (5.18).

2. The case of the inexact Halley class when $\alpha = 1$ (i.e. the inexact Super-Halley), the inequality (5.23) becomes

$$\|T_k(s^{(1)} + s^{(2)})\| \le \sigma_4 \|T_k(0)\|^{(1+q)(1+p)},$$

where $\sigma_4 = \left(1 + M\delta^{1-p}\right)^{1+q} + M\left(1 + M\delta^{1-p}\right)^2$. This case was discussed in Section 5.1. And from Theorem 5.2, we have

$$\|T_k(s^{(1)} + s^{(2)})\| \le M_1 \|T_k(0)\|^{(1+q)(1+p)},$$

where $M_1 = \left(1 + M\delta^{1-p}\right)^{1+q} + M\delta^{(1+p)(1+q)}\left(1 + M\delta^{1-p}\right)^2$. Since $\delta < 1$, then we have $M_1 < \sigma_4$ and

$$\|T_k(s^{(1)} + s^{(2)})\| < \sigma_4 \|T_k(0)\|^{(1+q)(1+p)}.$$

Then the inequality (5.23) is also valid for $\alpha = 1$.

3. Puting $\sigma_5 = \sigma_4 + \sigma_2|1-\alpha|$ and $\alpha \neq 1$, equation (5.23) can be rewritten as

$$\|T_k(s^{(1)} + s^{(2)})\| \le \sigma_5 \|T_k(0)\|^{\min\{(1+q)(1+p), 2+p\}}. \tag{5.24}$$

In the above equation, $\min\{(1+q)(1+p), 2+p\}$ means that we have two possibilities. The first case : $(1+q)(1+p) \le 2+p$, and then

$$q \le \frac{1}{1+p}. \tag{5.25}$$

The second one : $(1+q)(1+p) \geq 2+p$, and then

$$q \geq \frac{1}{1+p}. \tag{5.26}$$

Thus for both possibilities, $p \in (0,1]$ and q depend on p by (5.25) or (5.26).

4. From equations (5.24) and (4.31), the sequence η_k is satisfied

$$\eta_k \leq O\big(\|F(x_k)\|^{\min\{(1+q)(1+p),2+p\}-1}\big). \tag{5.27}$$

Therefore, by using Theorem 4.4, we conclude that the method is Q-third order when $\min\{(1+q)(1+p), 2+p\} \geq 3$ which means $p = 1$ and $\frac{1}{2} \leq q \leq 1$. For special case $\alpha = 1$, also we have Q-third order of convergence when $p = 1$ and $\frac{1}{2} \leq q \leq 1$.

5.3 Comparison

This section compares between the method used in the thesis and the method used in the literature.

Assume that the nonlinear system of equations $F(x) = 0$ solved by the Chebyshev method ($\alpha = 0$) (3.13)

$$F'(x_k)s^{(1)} = -F(x_k), \tag{5.28}$$

$$F'(x_k)s^{(2)} = -\frac{1}{2}F''(x_k)s^{(1)}s^{(1)}, \tag{5.29}$$

$$x_{k+1} = x_k + s^{(1)} + s^{(2)},\ k = 0, 1, 2, \ldots, \tag{5.30}$$

and the linear systems solved iteratively. Then we have the Inexact Chebyshev method (Inexact Halley class for $\alpha = 0$)

$$F'(x_k)s^{(1)} = -F(x_k) + r_k^{(1)}, \tag{5.31}$$

$$F'(x_k)s^{(2)} = -r_k^{(1)} - \frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + r_k^{(2)}, \tag{5.32}$$

$$x_{k+1} = x_k + s^{(1)} + s^{(2)},\ k = 0, 1, 2, \ldots, \tag{5.33}$$

where $r_k^{(1)}$ and $r_k^{(2)}$ obtained by solving iteratively the linear systems (5.28) and (5.29), respectively. We assumed that $r_k^{(1)}$ and $r_k^{(2)}$ are satisfied

$$\|r_k^{(1)}\| \leq \|F(x_k)\|^{1+p}, \tag{5.34}$$

$$\|r_k^{(2)}\| \leq \|T_k(s^{(1)})\|^{1+q}, \tag{5.35}$$

where $p, q \in (0, 1]$, the function $T_k(s)$ defined in (4.2) and $s^{(1)}$ is given by equation (5.31). Since the first step $s^{(1)}$ is defined by one way regardless α. We use Theorem 5.2 in equation (5.35) to get

$$\|r_k^{(2)}\| \leq \|T_k(0)\|^{(1+p)(1+q)}\big(1 + M\|T_k(0)\|^{1-p}\big)^{1+q},$$

since $T_k(0) = F(x_k)$ and using equation (4.5)

$$\|r_k^{(2)}\| \leq \sigma_1\|F(x_k)\|^{(1+p)(1+q)}, \tag{5.36}$$

where $\sigma_1 = \big(1 + M\delta^{1-p}\big)^{1+q}$.

Deng and Zhang [8], Gui-feng and Xiang [10] and Zhang, Cheng, Xue and Deng [27] solved the system of nonlinear equations $F(x) = 0$ using the Inexact Chebyshev method in the form

$$F'(x_k)s^{(1)} = -F(x_k) + \tilde{r}_k^{(1)}, \tag{5.37}$$

$$F'(x_k)s^{(2)} = -\frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + \tilde{r}_k^{(2)}, \tag{5.38}$$

$$x_{k+1} = x_k + s^{(1)} + s^{(2)}. \tag{5.39}$$

This method is the Inexact Halley class for $\alpha = 0$, except there is some change in the right hand side of the second equation. By adding and subtracting the term $\tilde{r}_k^{(1)}$ in the right hand side of the equation (5.38), we get

$$-\frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + \tilde{r}_k^{(2)} = -\tilde{r}_k^{(1)} - \frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + \tilde{r}_k^{(2)} + \tilde{r}_k^{(1)}.$$

Thus the method (5.37, 5.38, 5.39) can be written as

$$F'(x_k)s^{(1)} = -F(x_k) + \tilde{r}_k^{(1)}, \tag{5.40}$$

$$F'(x_k)s^{(2)} = -\tilde{r}_k^{(1)} - \frac{1}{2}F''(x_k)s^{(1)}s^{(1)} + \tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}, \tag{5.41}$$

$$x_{k+1} = x_k + s^{(1)} + s^{(2)}. \tag{5.42}$$

This method is the Inexact Halley class with $\alpha = 0$ (5.31, 5.32 and 5.33). Where the first residual $r_k^{(1)}$ is corresponding to $\tilde{r}_k^{(1)}$ and the second one $r_k^{(2)}$ is corresponding to $\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}$.

Deng and Zhang [8] proved that the residual -the stop condition- is satisfying the following conditions:

$$\|\tilde{r}_k^{(1)}\| \leq \omega' \|F(x_k)\|^{1+\min\{2,\, l_m^N/v\}}, \tag{5.43}$$

$$\|\tilde{r}_k^{(2)}\| \leq \omega'' \|F(x_k)\|^{2+\min\{1,\, l_m^H/v\}}, \tag{5.44}$$

where $\omega' > 0$ and $\omega'' > 0$ are constants and l_m^N and l_m^H are the maximum number of subiterations can be reached when we solved the linear system (5.28) and (5.29) respectively, and the progress index $v \geq 1$ (see Lemma 3.2 in Deng and Zhang [8]). We assume that we never reach the maximum number of iterations when the linear system was solved, otherwise, may be the exact solution is obtained. We set the values $\min\{2,\ l_m^N/v\} = 2$, $\min\{1,\ l_m^H/v\} = 1$, $\omega' = 1$ and $\omega'' = 1$. So equations (5.43) and (5.44) becomes

$$\|\tilde{r}_k^{(1)}\| \leq \|F(x_k)\|^3, \tag{5.45}$$

$$\|\tilde{r}_k^{(2)}\| \leq \|F(x_k)\|^3, \tag{5.46}$$

which are the error chosen in Algorithm PCG in Deng and Zhang [8].

Recall the first residual $r_k^{(1)}$ of the Inexact Halley class method is defined in equation (5.31). And the corresponding residual $\tilde{r}_k^{(1)}$ of Deng and Zhang method is defined in equation (5.40). By considering the conditions (5.45) and (5.34). Since the method is Q-third order of convergence then from Remark 5.3 p must be equal to one and q in the interval $[1/2, 1]$. And since $\|F(x_k)\| < 1$ from (4.5), we have

$$\|F(x_k)\|^3 \leq \|F(x_k)\|^2,$$

and then we get

$$\|\tilde{r}_k^{(1)}\| \leq \|r_k^{(1)}\|, \tag{5.47}$$

provided x_k is close to x^*.

The term $\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}$ in (5.41) correspond to $r_k^{(2)}$ in (5.32). Using triangular inequality, we have

$$\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| \leq \|\tilde{r}_k^{(1)}\| + \|\tilde{r}_k^{(2)}\|,$$

using the equations (5.45) and (5.46), we get

$$\begin{aligned}\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| &\leq \|F(x_k)\|^3 + \|F(x_k)\|^3 \\ &= 2\|F(x_k)\|^3.\end{aligned}$$

Now we can compare (5.36) with the above equation by looking at the right hand side of each equation. We know $p = 1$ and $1/2 \leq q \leq 1$ from previous stage, and we need to choose q and σ_1 such that the following is fulfilled

$$2\|F(x_k)\|^3 \leq \sigma_1 \|F(x_k)\|^{2(1+q)},$$

from above, we have $q \leq 1/2$ then $q = 1/2$. Clearly, to have $\sigma_1 \geq 2$ the value of $M \geq 0.5874$ (since ω depend on M see (5.36)).

$$\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| \leq \|r_k^{(2)}\|. \tag{5.48}$$

Gui-feng and Xiang [10] used the termination criteria

$$\|\tilde{r}_k^{(1)}\| \leq \|F(x_k)\|^{3+\epsilon}, \tag{5.49}$$

$$\|\tilde{r}_k^{(2)}\| \leq \|F(x_k)\|^{3+\epsilon}, \tag{5.50}$$

where $\epsilon > 0$ is small number. Put $p = 1$ to have Q-third order of convergence (see Remark 5.3). By using the same way that used in Deng, we get

$$\|\tilde{r}_k^{(1)}\| \leq \|r_k^{(1)}\|. \tag{5.51}$$

Now we look at $\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}$, by using triangular inequality

$$\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| \leq \|\tilde{r}_k^{(1)}\| + \|\tilde{r}_k^{(2)}\|,$$

using the equations (5.49) and (5.50), we get

$$\begin{aligned}\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| &\leq \|F(x_k)\|^{3+\epsilon} + \|F(x_k)\|^{3+\epsilon} \\ &= 2\|F(x_k)\|^{3+\epsilon}.\end{aligned}$$

To have Q-third order of convergence from Remark 5.3 q must be in the interval $[1/2, 1]$. Choose $q \in [1/2, 1]$ such that the following is satisfied

$$2\|F(x_k)\|^{3+\epsilon} \leq \sigma_1 \|F(x_k)\|^{2(1+q)},$$

then $q = 1/2$, and as shown above $M \geq 0.5874$. And therefore, we get

$$\|\tilde{r}_k^{(1)} + \tilde{r}_k^{(2)}\| \leq \|r_k^{(2)}\|. \tag{5.52}$$

Remark 5.4.

1. The result obtained from the analysis in Section 5.2 used to investigate the local convergence and cubic rate of convergence.

2. The condition $\|r_k^{(1)}\| \leq \|F(x_k)\|^2$ is sufficient to have Q-third order of convergence. The conditions (5.49) and (5.45) are so rigid .

Chapter 6

Numerical Experiments of a Problem with Singular $F'(x^*)$

This chapter illustrates in detail the derivation of Schröder's method. Analyzing the Chebyshev method, Halley method, Super-Halley method, Newton method and Schröder's method in the case where the first derivative of the function at the solution is singular.

6.1 Schröder's method

Schröder's method [22] is derived using Newton's method on the system of equations

$$F'(x)^{-1}F(x) = 0.$$

Let $G(x) = F'(x)^{-1}F(x)$. Newton's method is then

$$x_{k+1} = x_k - G'(x_k)^{-1}G(x_k),\ k = 0, 1, 2, \ldots.$$

G' computed by using the definition of $G(x)$, we have

$$F'(x)G(x) = F(x),$$

by taking the derivative on both sides, we have

$$F''(x)G(x) + F'(x)G'(x) = F'(x),$$

post-multiplication with $F'(x)^{-1}$, then we have

$$G'(x) + F'(x)^{-1}F''(x)G(x) = I,$$

rearranging the above equation, we get

$$G'(x) = I - F'(x)^{-1}(x)F''(x)G(x).$$

Using the definition of $G(x)$, we have

$$G'(x) = I - F'(x)^{-1}(x)F''(x)F'(x)^{-1}F(x),$$

define $L(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x)$, the above equation becomes

$$G'(x) = I - L(x). \tag{6.1}$$

Using the definition of $G(x)$ and substituting the value of $G'(x)$ in equation (6.1) into the Newton method iteration, we have

$$x_{k+1} = x_k - (I - L(x_k))^{-1}F'(x_k)^{-1}F(x_k),\ k = 0, 1, 2, 3, \dots. \tag{6.2}$$

This is called Schröder's method.

6.2 Numerical experiments

The numerical example to solve the nonlinear system of equations (4.1) in two dimension, is taken from Rall [21]. The function is

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_1x_2 + x_2^2 + x_1 - 2 \\ 3x_1^2 + 2x_1x_2 + 2x_2 - 7 \end{bmatrix} \tag{6.3}$$

In Figure (6.1) the intersection of the plotted $F_1(x) = 0$ and $F_2(x) = 0$ (marked by red and green) is defined the solutions of the problem (6.3). Also the points where have singular property to the first derivative are plotted. We observe that the solution $x = (1, 1)$ lies on the singularity curve. This can be shown clearly using the definition of the first derivative (see A.3 in the Appendix) at that point.

$$F'(1,1) = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}, \ \det[F'(1,1)] = 0. \tag{6.4}$$

Applying Chebyshev's method, Halley's method, Super-Halley's method, New-

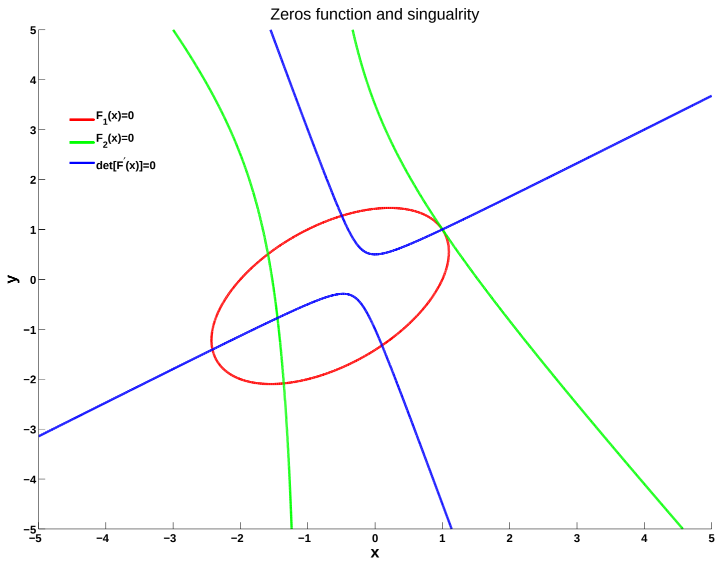


FIGURE 6.1: The curves for the system $F_1(x) = 0$ and $F_2(x) = 0$ and the singularity property of the first derivative of example (6.3).

ton's method and Schröder's method for the starting point $(0, 0)$. All these methods implemented in Matlab for double precision and stop criteria $\|F(x)\| \leq 1.410^{-10}$ and the maximum number of iterations is 25.

The following Tables show the number of iterations, the iterates and the function norm at these iterates.

In particular, Table 6.3 and 6.4 depict that one iteration from Super-Halley's method equivalent to two iterations of Newton's method. Table 6.1 shows that the Chebyshev's method did not converge to the solution $(1, 1)$ from standard starting point. However, when we choose the starting point to be the first iterate in Newton method $x_0 = (2, 3.5)$, it will converge to the desired solution. Figure 6.2

Table 6.1: Chebyshev's Method

k	x_1	x_2	$\|F(x_k)\|$
0	0	0	7.280109889280500
1	-7.250000000000000	-9.500000000000000	277.0939174043703
2	-2.968996813954100	-4.114769660009700	36.66220242433760
3	-1.583699601962200	-2.371038965535100	3.386002884748000
4	-1.358793773909200	-2.087570589247100	0.038032185681500
5	-1.355996429862800	-2.084039365035900	0.000000103579600
6	-1.355996422236900	-2.084039355394600	0.000000000000000

Table 6.2: Halley's Method

k	x_1	x_2	$\|F(x_k)\|$
0	0	0	7.280109889280518
1	0.506666666666667	0.546666666666667	4.740849095798747
2	1.026144685213466	0.898943403525560	0.201407441244827
3	1.014789524024398	0.970365418262718	0.001544436414059
4	1.004936854453253	0.990126269702986	0.000172331325504
5	1.001645620854597	0.996708758288070	0.000019148931309
6	1.000548540285213	0.998902919429573	0.000002127659163
7	1.000182846761747	0.999634306476506	0.000000236406574
8	1.000060948920403	0.999878102159194	0.000000026267397
9	1.000020316306819	0.999959367386362	0.000000002918600
10	1.000006772102609	0.999986455794781	0.000000000324289
11	1.000002257372266	0.999995485255467	0.000000000036032

Table 6.3: Super-Halley's Method

k	x_1	x_2	$\|F(x_k)\|$
0	0	0	7.280109889280518
1	1.133720930232558	1.909883720930232	5.355214977605924
2	0.927122115124097	1.145199552758963	0.037174889898064
3	0.981833265991789	1.036333431153082	0.002333646958440
4	0.995458319993551	1.009083360012750	0.000145853906513
5	0.998864579998406	1.002270840003187	0.000009115869161
6	0.999716144999560	1.000567710000881	0.000000569741823
7	0.999929036249757	1.000141927500485	0.000000035608864
8	0.999982259062446	1.000035481875109	0.000000002225554
9	0.999995564764280	1.000008870471441	0.000000000139097

Table 6.4: Newton's Method

k	x_1	x_2	$\|F(x_k)\|$
0	0	0	7.280109889280518
1	2.000000000000000	3.500000000000000	27.596421869510547
2	1.133720930232558	1.909883720930232	5.355214977605924
3	0.899185401750811	1.320530880585975	0.514445410962792
4	0.927122115124097	1.145199552758963	0.037174889898064
5	0.963669012504749	1.072668440837878	0.009337399130329
6	0.981833265991790	1.036333431153081	0.002333646958440
7	0.990916640026858	1.018166720050791	0.000583415670517
8	0.995458319993556	1.009083360012741	0.000145853906513
9	0.997729159996805	1.004541680006390	0.000036463476644
10	0.998864579998402	1.002270840003197	0.000009115869161
11	0.999432289999206	1.001135420001588	0.000002278967290
12	0.999716144999638	1.000567710000724	0.000000569741822
13	0.999858072499790	1.000283855000420	0.000000142435456
14	0.999929036249960	1.000141927500080	0.000000035608864
15	0.999964518125401	1.000070963749198	0.000000008902216
16	0.999982259063008	1.000035481873983	0.000000002225554
17	0.999991129530671	1.000017740938658	0.000000000556389
18	0.999995564767107	1.000008870465786	0.000000000139097

Table 6.5: Schroder's Method

k	x_1	x_2	$\|F(x_k)\|$
0	0	0	7.280109889280518
1	0.267441860465117	0.319767441860465	6.196980990894691
2	0.664649873269063	0.731178040241718	3.348760512962943
3	0.955058828497382	0.969868224931951	0.485943964480452
4	1.000215909885402	1.000137328916793	0.002346862051056
5	0.999997519478809	0.999998421468273	0.000026963328146
6	1.000000014062650	1.000000008948954	0.000000152861064
7	1.000000000016669	1.000000000010592	0.000000000181131
8	1.000000000000038	0.999999999999994	0.000000000000291

TABLE 6.6: Chebyshev's Method

k	x_1	x_2	$\|F(x_k)\|$
0	2.000000000000000	3.500000000000000	27.596421869510547
1	0.975315263121487	1.577120469738513	2.260813650918351
2	0.917090720554475	1.164191135947230	0.047657957656361
3	0.969142465137966	1.061720107585319	0.006735964013134
4	0.988427705570531	1.023144582887343	0.000946940352832
5	0.995660390439370	1.008679219123930	0.000133163843203
6	0.998372646414380	1.003254707171239	0.000018726165291
7	0.999389742405399	1.001220515189201	0.000002633366994
8	0.999771153402029	1.000457693195942	0.000000370317234
9	0.999914182525885	1.000171634948229	0.000000052075861
10	0.999967818446913	1.000064363106174	0.000000007323168
11	0.999987931916919	1.000024136166162	0.000000001029821
12	0.999995474470645	1.000009051058710	0.000000000144818
13	0.999998302916068	1.000003394167864	0.000000000020366

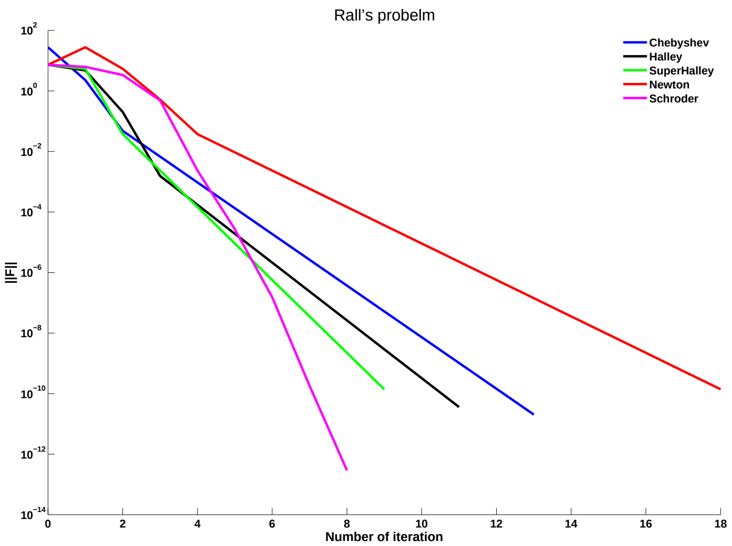


FIGURE 6.2: Comparing the number of iterations of Schröder's method, Chebyshev's method, Halley's method, Super-Halley's method, and Newton's method to solve example (6.3).

These experiments show that in the case where the first derivative of the function at the solution is singular, the Chebyshev's method, Halley's method, Super-Halley's and Newton's method are linearly convergent. However, Schröder's method converges super-linearly, and then reaches the solution faster than the above methods.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this thesis, we introduced a new iterative framework to solve the system of nonlinear equations in n-dimensional real space. The framework is based on approximate solving the quadratic equations which is an approximation of the function at the current iterate. We showed that the Halley class methods is based on the quadratic equation, which is approximately solved by two linear systems. We considered the inexact Halley class methods by solving iteratively the two linear systems. We have shown that the convergence rate for this class is cubic.

We assume that the existence and uniqueness of the solution obtained by the framework. There are some experiments on the rate of convergence of Chebyshev's method, Halley's method, Super-Halley's method and Newton's method that did not include in the thesis regard to the limited time.

7.2 Future work

The suggestions for future work we state as the following:

1. Study the region of convergence for Chebyshev's method, Halley's method, Super-Halley's method, Newton's method and Schröder method in higher dimension ($n \geq 2$).

2. Study (theoretically and experimentally) the radius of the attraction basin for the above methods in Halley class compared to Newton's method using the both weighted norm and regular norm.

3. Implementing the Halley class methods for large scale problems using sparsity structure.

4. Study and analyze the convergence properties of Halley class methods and Schröder's method in the case when the first derivative is singular at the solution.

5. Using the stationary refinement iteration to solve the linear system.

Appendix A

Basic Definitions

We consider the nonlinear equations

$$F(x) = 0 \tag{A.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, which can written as

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{bmatrix}. \tag{A.2}$$

The first derivative of the function F at the point x is $F'(x)$. $F'(x)$ is a matrix in $\mathbb{R}^{n\times n}$, and

$$F'(x) = \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \frac{\partial F_2(x)}{\partial x_1} & \frac{\partial F_2(x)}{\partial x_2} & \cdots & \frac{\partial F_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n(x)}{\partial x_1} & \frac{\partial F_n(x)}{\partial x_2} & \cdots & \frac{\partial F_n(x)}{\partial x_n} \end{bmatrix}. \tag{A.3}$$

Then the component ij of $F'(x)$ is defined by

$$[F'(x)]_{ij} = \frac{\partial F_i(x)}{\partial x_j}.$$

The second derivative of the function F at the point x is $F''(x)$. $F''(x)$ is a tensor in $\mathbb{R}^{n\times n\times n}$, where

$$[F''(x)]_{ijk} = \frac{\partial^2 F_i(x)}{\partial x_k \partial x_j}. \tag{A.4}$$

Definition A.1. A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz-continuous on $D_0 \subset D$, if there exist constant $L > 0$ so that, for all $x, y \in D$,

$$\|F(y) - F(x)\| \leq L\|x - y\|. \tag{A.5}$$

The tensor $F''(x)$ is Lipschitz-continuous on $D_0 \subset D$, if there exist constant $L > 0$ so that, for all $x, y \in D$,

$$\|\!|F''(y) - F''(x)\|\!| \leq L\|x - y\|. \tag{A.6}$$

The computation of the tensor norm $\|\!|.\|\!|$ is mentioned in Appendix B.

Definition A.2. A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is contractive on $D_0 \subset D$, if there exist a constant $\beta < 1$ such that, for all $x, y \in D_0$,

$$\|F(y) - F(x)\| \leq \beta\|x - y\|. \tag{A.7}$$

Definition A.3. A sequence $\{x_k\},\ k \geq 0$ converges to x^* with Q-order (at least) $q \geq 1$, if there exist two constants $\beta_q \geq 0$ and $k_q \geq 0$ such that for all $k \geq k_q$, we have

$$\|x_{k+1} - x^*\| \leq \beta_q \|x_k - x^*\|^q. \tag{A.8}$$

For $q = 2, 3$ the convergence is said to be (at least) Q-quadratic, Q-cubic respectively. For $q = 1$ the expression Q-linear convergence if in the above $0 \leq \beta_1 < 1$.

Definition A.4. If A is a square matrix, $\|A\| < 1$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A + A^2 + \cdots = \sum_{k=0}^{\infty} A^k$. This is the Neumann series.

Theorem A.5. *Let L be a matrix in $\mathbb{R}^{n\times n}$, α be a real number and I be the identity matrix. Assume that $I - \alpha L$ is invertible, then*

$$L(I - \alpha L)^{-1} = (I - \alpha L)^{-1} L$$

Proof. Consider

$$L(I - \alpha L) = L - \alpha L^2 = (I - \alpha L)L,$$

So we have

$$L(I - \alpha L) = (I - \alpha L)L, \tag{A.9}$$

By pre-multiplying and post-multiplying the equation (A.9) with $(I - \alpha L)^{-1}$, we get

$$(I - \alpha L)^{-1}L(I - \alpha L)(I - \alpha L)^{-1} = (I - \alpha L)^{-1}(I - \alpha L)L(I - \alpha L)^{-1}$$

The multiplication $(I - \alpha L)(I - \alpha L)^{-1}$ and the swap gives the identity matrix, we obtain that

$$(I - \alpha L)^{-1}L = L(I - \alpha L)^{-1}.$$

□

Lemma A.6. *Let $L \in \mathbb{R}^{n \times n}$, I be the identity matrix and α be a real number. Assuming that $I - \alpha L$ is invertible, So we have*

$$(I - \alpha L)^{-1}(I + (\frac{1}{2} - \alpha)L) = I + \frac{1}{2}L(I - \alpha L)^{-1} \tag{A.10}$$

Proof. We starting with the left hand side of (A.10) and rearrangement the interior of the right multiplication

$$\begin{aligned}(I - \alpha L)^{-1}(I + (\frac{1}{2} - \alpha)L) &= (I - \alpha L)^{-1}(I - \alpha L + \frac{1}{2}L) \\ &= I + \frac{1}{2}(I - \alpha L)^{-1}L\end{aligned}$$

By using the theorem A.5, so then we get the right hand side of (A.10). □

Appendix B

Norm

we will review the definitions of the norm for vector and matrix. we will define the norm of the tensor and make sure it satisfy the norm conditions.

Vector's norms

Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, recall the vector's norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \;\; \|x\|_\infty = \max_{1\le i\le n} |x_i|, \;\; \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{B.1}$$

where x_i is the component number i in x.

The matrix norms defined by

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |A_{ij}|, \;\; \|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |A_{ij}|, \tag{B.2}$$

where A_{ij} is the element (i,j) in the matrix. However, the induced norm of the matrix A is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2. \tag{B.3}$$

Moreover, there exists $z \in \mathbb{R}^n$ so that $\|z\|_2 = 1$ and

$$\|A\|_2 = \|Az\|_2. \tag{B.4}$$

Let $\mathcal{T} \in \mathbb{R}^{m\times n\times n}, p \in \mathbb{R}^n$, we define tensor multiplication with a vector $\mathcal{T}p \in \mathbb{R}^{m\times n}$ as

$$[\mathcal{T}p]_{ij} = \sum_{k=1}^{n} \mathcal{T}_{ijk} p_k. \tag{B.5}$$

Then

$$\begin{aligned}
\|\mathcal{T}p\|_\infty &= \max_{1\le i\le m} \sum_{j=1}^{n} |\sum_{k=1}^{n} \mathcal{T}_{ijk} p_k| \\
&= \sum_{j=1}^{n} |\sum_{k=1}^{n} \mathcal{T}_{\hat{\imath}jk} p_k| \\
&\le \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{T}_{\hat{\imath}jk}| \, |p_k| \\
&\le \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{T}_{\hat{\imath}jk}| \, \|p\|_\infty \\
&\le \max_{1\le i\le m} \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{T}_{ijk}| \, \|p\|_\infty
\end{aligned}$$

using that there exists an index $\hat{\imath}$ so that the maximum occurs, and $|p_k| \le \max_{1\le k\le n} |p_k| = \|p\|_\infty$.

Tensor's norms

The tensor norm (B.9) is a norm iff the following conditions is satisfied:
For all $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)} \in \mathbb{R}^{m\times n\times n}$ and for all $\alpha \in \mathbb{R}^n$

$$\|\mathcal{T}^{(1)}\| = 0 \Leftrightarrow \mathcal{T}^{(1)} = 0 \tag{B.6}$$

$$\|\alpha\mathcal{T}^{(1)}\| = |\alpha| \, \|\mathcal{T}^{(1)}\| \tag{B.7}$$

$$\|\mathcal{T}^{(1)} + \mathcal{T}^{(2)}\| \le \|\mathcal{T}^{(1)}\| + \|\mathcal{T}^{(2)}\| \tag{B.8}$$

Definition B.1. The infinity tensor norm $\mathcal{T} \in \mathbb{R}^{m\times n\times n}$ can be defined by the following

$$\|\mathcal{T}\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\mathcal{T}_{ijk}| \tag{B.9}$$

The norm defined in (Referencesnorminfty) is a norm.

Proof. Using the definition (B.9), So it is clear to see that the condition (B.6) is satisfied. Since α is a sclar then we have that

$$\begin{aligned}\|\alpha\mathcal{T}^{(1)}\|_\infty &= \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\alpha\mathcal{T}^{(1)}_{ijk}| \\ &= |\alpha|\, \|\mathcal{T}^{(1)}\|_\infty\end{aligned}$$

Then the condition (B.7) is satisfied. To prove the triangular inequality, we define a tensor $\mathcal{T} = \mathcal{T}^{(1)} + \mathcal{T}^{(2)}$ then ,

$$\begin{aligned}\|\mathcal{T}\|_\infty &= \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\mathcal{T}_{ijk}| \\ &= \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\mathcal{T}^{(1)}_{ijk} + \mathcal{T}^{(2)}_{ijk}| \\ &\le \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\mathcal{T}^{(1)}_{ijk}| + \max_{1\le i\le m} \sum_{j=1}^{n}\sum_{k=1}^{n} |\mathcal{T}^{(2)}_{ijk}| \\ &= \|\mathcal{T}^{(1)}\|_\infty + \|\mathcal{T}^{(2)}\|_\infty\end{aligned}$$

Therefore the infinty tensor norm is a norm. □

To derive an easy computable $\|\ .\ \|_1$ consider

$$\begin{aligned}\mathcal{T}p &= \sum_{k=1}^{n} \mathcal{T}_{.,.,k}\, p_k \\ \|\mathcal{T}p\|_1 &\le \sum_{k=1}^{n} \|\mathcal{T}_{.,.,k}\|_1\, |p_k| \le \max_{1\le k\le n} \|\mathcal{T}_{.,.,k}\|_1 \sum_{k=1}^{n} |p_k| \\ &= \max_{1\le k\le n} \|\mathcal{T}_{.,.,k}\|_1\, \|p_k\|_1 \|\mathcal{T}p\|_1 \le \|\mathcal{T}\|_1 \|p\|_1.\end{aligned}$$

Definition B.2. The tensor 1-norm of $\mathcal{T} \in \mathbb{R}^{m\times n\times n}$ is defined by

$$\|\mathcal{T}\|_1 = \max_{1\leq k\leq n} \max_{1\leq j\leq n} \sum_{i=1}^{m} |\mathcal{T}_{ijk}| \tag{B.10}$$

Using the same argument used in the above proof we can verify that the norm defined in Definition B.2 is a norm.

Definition B.3. The tensor Forbinous norm of $\mathcal{T} \in \mathbb{R}^{m\times n\times n}$ is defined as the following

$$\|\mathcal{T}\|_F = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{m} |\mathcal{T}_{ijk}|^2 \tag{B.11}$$

Let $p \in \mathbb{R}^n$, then tensor Forbinous norm satisfy the following induced property

$$\|\mathcal{T}p\|_2 \leq \|\mathcal{T}\|_F \|p\|_2 \tag{B.12}$$

Proof. We start by using the vector-tensor product as follows

$$\begin{aligned}\|\mathcal{T}p\|_2^2 &= \left(\| \sum_{k=1}^{n} \mathcal{T}_{.,.,k} |p| \|_2 \right)^2 \\ &\leq \left(\sum_{k=1}^{n} \|\mathcal{T}_{.,.,k}\|_2 |p| \right)^2,\end{aligned}$$

using the Cuachy Schwartz inequality we get

$$\begin{aligned}\|\mathcal{T}p\|_2^2 &\leq \left(\sum_{k=1}^{n} \|\mathcal{T}_{.,.,k}\|_2^2 \right) \left(\sum_{k=1}^{n} |p_k|^2 \right) \\ &= \|p_k\|_2^2 \sum_{k=1}^{n} \|\mathcal{T}_{.,.,k}\|_2^2,\end{aligned}$$

using the relation between the Forbinous norm and 2-norm

$$\|\mathcal{T}p|\|_2^2 \leq \|p_k\|_2^2 \sum_{k=1}^{n} \|\mathcal{T}_{.,.,k}\|_F^2 \leq \|\mathcal{T}\|_F^2 \|p\|_2^2,$$

this complete the proof. □

Bibliography

[1] S. Amat and S. Busquier. Third-Order Iterative Methods Under Kantorovich Conditions. *Journal of Mathematical Analysis and Applications*, 336(1):243–261, 2007.

[2] V. Candela and A. Marquina. Recurrence Relations for Rational Cubic Methods I: The Halley Method. *Computing*, 44(2):169–184, 1990.

[3] V. Candela and A. Marquina. Recurrence Relations for Rational Cubic Methods II: The Chebyshev Method. *Computing*, 45(4):355–367, 1990.

[4] D. Chen, I.K. Argyros, and Q. Qian. A Local Convergence Theorem for the Super-Halley Methodin a Banach Space. *Appl. Math. Lett*, 7(5):49–52, 1994.

[5] O. Cira. Numerical Experiments on Attraction Basin. *AMO–Advanced Modeling and Optimization*, 2(3):122–134, 2000.

[6] O. Cira and C. M. Cira. The Algorithm that Determines the Start Iteration of the Halley-Altman Method. *International Symposium on Symbolic and Numeric Algorithms for Scientific Computing*, pages 381–386, 2007.

[7] R. S. Dembo, S. C. Eisenstat, and T. Steihaug. Inexact Newton Methods. *SIAM Journal on Numerical Analysis*, 19(2):400–408, 1982.

[8] N. Deng and H. Zhang. Theoretical Efficiency of a New Inexact Method of Tangent Hyperbolas. *Optimization Methods and Software*, 19(3-4):247–265, 2004.

[9] J. A. Ezquerro and M. A. Hernández. New Class of Third-Order Methods in Banach Spaces. *Bulletin of the Institute of Mathematics Academia Sinica*, 31(1):33–58, 2003.

[10] Y. Gui-feng and T. Xiang. An Inexact Halley's Method. *Journal of Beijing Institute of Technology*, 14(3):340–343, 2005.

[11] G. Gundersen and T. Steihaug. On Large Scale Unconstrained Optimization Problems and Higher Order Methods. *Submitted to a special issue of Optimization Methods and Software for the Veszprm Optimization Conference: Advanced Algorithms (VOCAL) conference*, 2006.

[12] J. M. Gutiérrez and M. A. Hernández. Third-Order Iterative Methods for Operators with Bounded Second Derivative. *Journal of Computational and Applied Mathematics*, 82(1–2):171–183, 1997.

[13] J. M. Gutiérrez and M. A. Hernández. An acceleration of Newton's method: Super-Halley method. *Applied Mathematics and Computation*, 117(2):223–239, 2001.

[14] J. M. Gutiérrez, M. A. Hernández, and M. A. Salanova. Resolution of Quadratic Equations in Banach Spaces. *Numerical Functional Analysis and Optimization*, 17(1–2):113–121, 1996.

[15] M. A. Hernández and J. M. Gutiérrez. A Family of Chebyshev–Halley Type Methods in Banach Spaces. *Bull. Austral. Math. Soc.*, 55:113–130, 1997.

[16] M. A. Hernández and M. A. Salanova. A Family of Chebyshev–Halley Type Methods. *International Journal of Computer Mathematics*, 47:59–63, 1993.

[17] J. F. Bonnans, J. C. Gilbert, C. Lemaréchal, C. A. Sagastizábal. *Numerical Optimization: Theoretical and Practical Aspects*. Springer, 2003.

[18] C. T. Kelley. *Iterative methods for Optimization*. SIAM, 1999.

[19] J. M. Ortega and W. G. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York, 1970.

[20] B. T. Polyak. Newton's Method and Its Use in Optimization. *European Journal of Operational Research*, 127(3):1086–1096, 2007.

[21] L. B. Rall. Convergence of Newton Process to Multiple Solutions. *Numerische Mathematik*, 9(1):23–37, 1966.

[22] E. Schröder. Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen. *Mathematische Annalen*, 2(2):317–365, 1870.

[23] H. Schwetlick. *Numerische Lösung nichtlinearer Gleichungen*. R. Oldenburg Verlag, München-Wien, 1979.

[24] J.R. Sharma. A Family of Third-Order Methods to Solve Nonlinear Equations by Quadratic Curves Approximation. *Applied Mathematics and Computation*, 184:210–215, 2007.

[25] T. Yamamoto. On the Method of Tangent Hyperbolas in Banach Spaces. *Journal of Computational and Applied Mathematics*, 21(1):75–86, 1988.

[26] T. Yamamoto. Historical Developments in Convergence Analysis for Netwon's and Newton-like Methods. *Journal of Computational and Applied Mathematics*, 124(1–2):1–23, 2000.

[27] H. Zhang, Q. Cheng, Y. Xue, and N. Deng. An Efficient Version on a New Improved Method of Tangent Hyperbolas. *K. Li et al. (Eds): LSMS 2007, Lecture Notes in Computer Science*, 4688:205–214, 2007.