Arch. Math. 97 (2011), 135–140

0003-889X/11/020135-6
published online July 14, 2011
DOI 10.1007/s00013-011-0288-4

Archiv der Mathematik

New examples of curves with a one-dimensional family of pencils of minimal degree

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Abstract. We give a geometric construction of sub-linear systems on a K3 surface consisting of smooth curves C with infinitely many $g^1_{\mathrm{gon}(C)}$'s.

Mathematics Subject Classification (2010). Primary 14H51; Secondary 14J28.

Keywords. Curves on K3 surfaces, gonality, Brill–Noether theory.

1. Introduction. Let C be a smooth curve over $\mathbb{C}$ of genus $g \geq 2$. One defines $W^r_d(C)$ to be the variety parametrising complete $g^{r'}_d$'s on C such that $r' \geq r$. The *expected dimension* is $\rho(g, r, d) := g-(r+1)(g-d+r)$, and is the dimension of $W^r_d(C)$ when C is general in $\mathscr{M}_g$, provided that $\rho(g, r, d) \geq 0$. We will be interested in finding new examples of smooth curves C with an infinite number of pencils of minimal degree on certain K3 surfaces.

A number of results have been proved for curves on K3 surfaces throughout the last 25 years. In 1986, it was proved by Lazarsfeld [10] that if the linear system $|L|$ on a K3 surface S doesn't contain non-reduced or reducible curves, then $\dim W^r_d(C) = \rho(g, r, d)$ for general $C \in |L|$. Knutsen [9] proved that the only cases of exceptional curves (i.e., curves C satisfying $\mathrm{Cliff}(C) < \mathrm{gon}(C)-2$) on K3 surfaces are the Donagi–Morrison example [3, (2.2)] and the generalised ELMS example (a generalisation of [4, Theorem 4.3] presented in Knutsen's article, see "Generalised ELMS examples"). He furthermore proved that the gonality is constant for all curves in a linear system on any K3 surface as long as it is not as in the Donagi–Morrison example. A weaker result of Knutsen's gonality theorem was proved in [1], for the case of ample linear systems on K3 surfaces.

Among the most recent results on curves on K3 surfaces, are one by Knutsen in [8, Theorem 1.1 (b)], proving that there exist curves C of any possible gonality $\mathrm{gon}(C) \leq \lfloor (g+3)/2 \rfloor$ on K3 surfaces; and one in [9, Theorem 3.1], proving that whenever $|L|$ is a base-point free linear system on a K3 surface not as in the Donagi–Morrison example and the smooth $C \in |L|$ satisfy $\rho(g, 1, \mathrm{gon}(C)) < 0$, then $\dim W^1_{\mathrm{gon}(C)}(C) = 0$ for the general smooth $C \in |L|$.

For the curves C in the Donagi–Morrison and generalised ELMS examples, we have $\dim W^1_{\mathrm{gon}(C)}(C) = 1$ in both cases, as is the case in general for all exceptional curves C, by [2, Corollary 2.3.1], and also all curves of odd genus and maximal gonality. All finite covers C' of curves C of degree z with $\dim W^1_{\mathrm{gon}(C)}(C) = 1$ and $g(C') \geq \mathrm{gon}(C') \cdot (z-1) + z \cdot g(C)$ will also give us $\dim W^1_{\mathrm{gon}(C')}(C') = 1$, by the Castelnuovo–Severi inequality [6]. By [4, Remark 3.8], there are no other known examples of curves with 1 dimension of $g^1_{\mathrm{gon}(C)}$'s. Moreover, as far as we know, the curves in the generalised ELMS example and the plane and bi-elliptic curves in the Donagi–Morrison example are the only cases known so far of curves on K3 surfaces with $\dim W^1_{\mathrm{gon}(C)}(C) = 1$.

In this paper, we give the following further examples of smooth curves on K3 surfaces with 1 dimension of $g^1_{\mathrm{gon}(C)}$'s. The curves given here are non-exceptional.

Theorem 1.1. *Let S be a smooth projective K3 surface with* $\mathrm{Pic}(S) \cong \mathbb{Z}H$*, where H is a smooth curve on S with $H^2 \geq 4$. Then for any integer $m \geq 3$, $|mH|$ contains a sub-linear system where the smooth curves C satisfy* $\dim W^1_{\mathrm{gon}(C)}(C) = 1$.

2. The result. Let C be a smooth curve over $\mathbb{C}$. One defines the *gonality* of C, denoted $\mathrm{gon}(C)$, to be the smallest k such that there exists a g^1_k on C. For $g(C) \geq 4$, the *Clifford index* of C is defined to be

$$\mathrm{Cliff}(C) := \min\{\deg(A) - 2(h^0(C,A) - 1) \mid h^0(C,A) \geq 2 \text{ and } h^1(C,A) \geq 2\}.$$

Hyperelliptic curves C of genus $g = 2$ are defined to have $\mathrm{Cliff}(C) = 0$, and trigonal curves C' of genus $g = 3$ are defined to have $\mathrm{Cliff}(C') = 1$.

By [2, Theorem 2.3], we have $\mathrm{Cliff}(C) \in \{\mathrm{gon}(C) - 2, \mathrm{gon}(C) - 3\}$. Curves C satisfying $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$ are called *exceptional*.

A *K3 surface* is defined to be a smooth, projective surface S over $\mathbb{C}$ satisfying $h^1(S, \mathscr{O}_S) = 0$ and $\mathscr{O}_S(K_S) \cong \mathscr{O}_S$. By the adjunction formula, it follows that $L^2 = 2p_a - 2$, where p_a is the arithmetic genus of the curves in $|L|$. By [9, Theorem 2], the only exceptional curves on K3 surfaces are the ones found in [3, (2.2)] and [9, "Generalized ELMS Examples"] (the latter a generalisation of [4]). By [5, Theorem], the Clifford index of the smooth curves in a linear system on a K3 surface is constant.

Lemma 2.1. *Suppose $C \in |L|$ is a smooth curve of gonality k such that $C \sim D_1 + D_2$, where D_1, D_2 are two divisors on S satisfying $h^0(S, \mathscr{O}_S(D_i)) \geq 2$ for $i = 1, 2$. Then* $\mathrm{Cliff}(C) \leq D_1.D_2 - 2$. *It follows that if C is non-exceptional, then $D_1.D_2 \geq k$.*

Proof. We show that $D_1|_C$ contributes to the Clifford index of C. Consider the exact sequence $0 \to \mathscr{O}_S(-C) \to \mathscr{O}_S \to \mathscr{O}_C \to 0$ tensored with $\mathscr{O}_S(D_1)$:

$$0 \to \mathscr{O}_S(-D_2) \to \mathscr{O}_S(D_1) \to \mathscr{O}_C(D_1|_C) \to 0.$$

Taking global sections, we see that $h^0(C, \mathscr{O}_C(D_1|_C)) \geq h^0(S, \mathscr{O}_S(D_1))$, which is ≥ 2. Tensoring with $\mathscr{O}_S(D_2)$ instead of $\mathscr{O}_S(D_1)$ gives us $h^0(C, \mathscr{O}_C(D_2|_C)) \geq h^0(S, \mathscr{O}_S(D_2))$, which is also ≥ 2. Since $K_C \sim (D_1+D_2)|_C$, then $K_C - D_1|_C \sim D_2|_C$, and so we conclude that $h^1(C, \mathscr{O}_C(D_1|_C)) \geq 2$, so that $D_1|_C$ contributes to Cliff(C). It follows that

$$\begin{aligned}\mathrm{Cliff}(C) &\leq \mathrm{Cliff}(D_1|_C) = \deg(D_1|_C) - 2(h^0(C, \mathscr{O}_C(D_1|_C)) - 1) \\ &\leq \deg(D_1|_C) - 2(h^0(S, \mathscr{O}_S(D_1)) - 1) \leq D_1.C - 2 \cdot \left(2 + \frac{1}{2}D_1^2 - 1\right) \\ &= D_1.D_2 - 2,\end{aligned}$$

as desired. □

Proof of Theorem 1.1. Let S' be a smooth projective K3 surface satisfying $\mathrm{Pic}(S') \cong \mathbb{Z}H'$, where H' is a smooth curve on S' with $(H')^2 = 2g - 2 \geq 4$. Then by a classical result in [12], H' is very ample (this result is stated more clearly in [7, Theorem 1.1] with $k = 1$, where it says in particular that if H' is not very ample, then there exists an effective divisor D on S' satisfying $D^2 \leq 0$). It follows that $\mathscr{O}_{S'}(H')$ defines an embedding from S' into $\mathbb{P}^g$, which restricted to any smooth curve $C' \in |H'|$ is the canonical embedding of C' into a hyperplane of $\mathbb{P}^g$. Note that the smooth curves in $|H'|$ are non-hyperelliptic. Let S be the image of S', and let H be the image of H'.

Now consider a smooth plane curve X of degree $m \geq 3$ lying inside a plane $F \subseteq \mathbb{P}^g$, let $T \subseteq \mathbb{P}^g$ be a codimension 3 linear space that neither intersects S nor F, and consider the cone over X consisting of all codimension 2 spaces spanned by T and points in X. This cone is a hypersurface of degree m in $\mathbb{P}^g$, and if we consider the cones over all curves of degree m in F, including the singular and reducible ones, we get a linear system $\mathfrak{d}$ inside $|\mathscr{O}_{\mathbb{P}^g}(m)|$ (making a coordinate change and letting $x_0, \ldots, x_g$ be the coordinates of $\mathbb{P}^g$, we can assume that F is given by $x_3 = \cdots = x_g = 0$ and T given by $x_0 = x_1 = x_2 = 0$, in which case the cones are precisely the zero-sets of the homogeneous polynomials of degree m in x_0, x_1, x_2, which is clearly a linear system). Since the base locus of $\mathfrak{d}$ is T and $T \cap S = \emptyset$, then $\mathfrak{d}$ doesn't have base-points in S. We can hence apply Bertini's theorem and conclude that the general $C \in \mathfrak{d}|_S$ is nonsingular. Each curve is connected because the number of connectedness components for each $C \in |mH|$ is constant and $\mathfrak{d}|_S \subseteq |mH|$.

We now prove that $\mathrm{gon}(C) = (m-1)H^2 = (m-1)(2g-2)$. By Lemma 2.1, we see that $\mathrm{gon}(C) \leq (m-1)H.H = (m-1)(2g-2)$. Now suppose that $|A|$ is a g^1_d of minimal degree on C, and note that $|A|$ must be base-point free. Then $d \leq (m-1)(2g-2)$. Let g' be the genus of C. We have $2g'-2 = (mH)^2 = m^2(2g-2)$, so that $g' = m^2(g-1)+1$, and so we get $\rho(g, 1, d) = -m^2(g-1)+2d-3$. Using that $d \leq (m-1)(2g-2)$, we get $\rho(g, 1, d) \leq -(g-1)m^2+(4g-4)m-4g+1$, and by solving this as a second-degree equation in m, we see that this expression is always < 0. (The expression $\rho(g, r, d)$ was defined in the introduction.)

Following the work of Lazarsfeld and Tyurin [10,13], for any base-point free line-bundle A on $C \in |mH|$, there is defined a rank-2 vector bundle $\mathscr{E}_{C,A}$ by

$$0 \to \mathscr{E}_{C,A}^{\vee} \to H^0(C, A) \otimes \mathscr{O}_S \to A \to 0,$$

where A is considered as a sheaf on S by expansion by 0. We have that $c_1(\mathscr{E}_{C,A}) \cong \mathscr{O}_S(C)$ and $c_2(\mathscr{E}_{C,A}) = d$. Furthermore, $\chi(\mathscr{E}_{C,A} \otimes \mathscr{E}_{C,A}^\vee) = 2 - 2\rho(g,1,d)$, and since $h^2(S, \mathscr{E}_{C,A} \otimes \mathscr{E}_{C,A}^\vee) = h^0(S, \mathscr{E}_{C,A} \otimes \mathscr{E}_{C,A}^\vee)$, it follows that $h^0(S, \mathscr{E}_{C,A} \otimes \mathscr{E}_{C,A}^\vee) \geq 1 - \rho(g,1,d)$. Since $\rho(g,1,d) < 0$ in our case, then we get that $\mathscr{E}_{C,A}$ is non-simple. By Green and Lazarsfeld [5, Lemma 3.1], and independently by Donagi and Morrison [3], any such non-simple vector bundle $\mathscr{E}_{C,A}$ lies inside an extension

$$0 \to M \to \mathscr{E}_{C,A} \to N \otimes \mathscr{I}_\xi \to 0,$$

where M, N are two line bundles on S satisfying $h^0(S,M) \geq 2$, $h^0(S,N) \geq 2$, $M \otimes N \cong \mathscr{O}_S(C)$, $|N|$ is base-point free, and where ξ is a 0-dimensional subscheme of finite length. We have $c_2(\mathscr{E}_{C,A}) = d = M.N + \mathrm{length}(\xi)$.

Now, since $\mathrm{Pic}(S)$ is generated by H, we have $M \cong \mathscr{O}_S(aH)$ and $N \cong \mathscr{O}_S(bH)$, where $a \geq 1$ and $b \geq 1$ since $h^0(S,M) \geq 2$ and $h^0(S,N) \geq 2$. It follows that $M.N + \mathrm{length}(\xi) = abH^2 + \mathrm{length}(\xi) \geq ab(2g-2)$. Since $M \otimes N \cong \mathscr{O}_S(mH)$, we have $a + b = m$, and so $ab \geq m - 1$. We conclude that $d \geq (m-1)(2g-2)$, as desired.

So $\mathrm{gon}(C) = (m-1)(2g-2)$ for all $C \in |mH|$. And so in particular, this applies when C is the intersection of S with a cone Q over a general smooth curve X of degree m in the plane F. Now let Z be one of the infinitely many codimension 2 subspaces of $\mathbb{P}^g$ that exist in Q, and let $\mathscr{H}_Z = |\mathscr{O}_{\mathbb{P}^g}(1) \otimes \mathscr{I}_Z|$, i.e., the linear system of all hyperplanes J in $\mathbb{P}^g$ that contain Z. We show that the set

$$|A_Z| = \{(J \cap C) \setminus (C \cap Z) \mid J \in \mathscr{H}_Z\}$$

is a linear system of degree $(m-1)(2g-2)$ on C.

It is clear that all the elements in $|A_Z|$ are linearly equivalent, since all elements in $\mathscr{H}_Z$ cut out linearly equivalent divisors on C. To find the degree of $|A_Z|$, it suffices to show that $\#(Z \cap C) = 2g-2$. To do this, consider a hyperplane $J \in \mathscr{H}_Z$. The points where this intersects C are precisely the intersection of the varieties $J \cap Q$ and $J \cap S$. Now $J \cap Q = Z + V_{m-1}$ where V_{m-1} is a hypersurface in J of degree $m-1$. And $J \cap S = C_{2g-2}$, a curve in J of degree $2g-2$. Since Z and C_{2g-2} are both contained in J, then these intersect in $2g-2$ points, as desired.

We now show that two distinct codimension 2 subspaces Z and $Z' \neq Z$ give divisors A_Z and $A_{Z'}$ that are not linearly equivalent. Suppose they were. Then $A_Z < J_1|_C$ for some $J_1 \in \mathscr{H}_Z$ and $A_{Z'} < J_2|_C$ for some $J_2 \in \mathscr{H}_{Z'}$. But since $J_1|_C \sim J_2|_C$ and we are assuming $A_Z \sim A_{Z'}$, then $J_1|_C - A_Z \sim J_2|_C - A_{Z'}$, and so $Z|_C \sim Z'|_C$. But since $Z \neq Z'$ and have T as the only space of intersection, then $Z|_C \neq Z'|_C$, and since we showed that each such codimension 2 subspace intersects C in $2g-2$ points and $m \geq 3$, then this contradicts the gonality of C.

It follows that each linear system $|A|$, $A \in \{A_Z\}_Z$, arises from a unique codimension 2 subspace Z in Q, and so it follows that $\dim W^1_{\mathrm{gon}(C)}(C) = 1$. In fact, we see from our construction that $W^1_{\mathrm{gon}(C)}(C)$ has an irreducible component isomorphic to the plane curve X. □

Note that the curves in this example are finite covers of plane curves.

Remark 2.2. In the case S is a smooth quartic in $\mathbb{P}^3$ containing a line ℓ, and H is a plane section on S, then the curves $C \sim mH$ have gonality $< 4(m-1)$ for $m \geq 5$. Indeed, by Proposition 2.1, putting $D_1 = (m-1)H + \ell$, $D_2 = H - \ell$ and using that $H^2 = 4$, we get $\text{gon}(C) \leq D_1.D_2 = 3m$, which is $< 4(m-1)$ whenever $m \geq 5$.

A more geometrical way to see this is that if S contains a line ℓ and C is a smooth curve in the linear system $|mH|$, then ℓ intersects C in m points, and so all planes going through ℓ define a g^1_d on C where $d = 4m - m = 3m$. (We actually have $\text{gon}(C) = 3m$. See Remark 2.4.).

We see that in these cases, the infinitely many codimension 2 subspaces Z in the cone Q of the proof of Theorem 1.1 no longer define pencils of minimal degree on C.

When $m \leq 4$ and S is a smooth quartic in $\mathbb{P}^3$, we don't need any conditions on $\text{Pic}(S)$, as we see in the next example.

Example 2.3. Let S be any smooth quartic in $\mathbb{P}^3$, let H be a plane in $\mathbb{P}^3$, and let C be a smooth curve in $|mH|$, where $m \in \{3, 4\}$. By Proposition 2.1, we see that $\text{gon}(C) \leq (m-1)H.H = 4(m-1)$. However, by [11, Theorem 4.12], since C is a complete intersection, we have $\text{gon}(C) \geq 4(m-1)$, and so equality follows.

We can now construct infinitely many $g^1_{4(m-1)}$'s on C precisely as we did in the proof of Theorem 1.1.

Remark 2.4. The lower bound for $\text{gon}(C)$ found in [11] is as follows: If $C \in \mathbb{P}^n$ is a complete intersection of hypersurfaces of degree $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1}$ and $|A|$ is a g^1_d on C, then $d \geq (a_1 - 1) \cdot a_2 \cdots a_{n-1}$. So for $m \geq 5$ in Remark 2.2, we get $d \geq 3m$, so that the curves in those cases have gonality precisely equal to $3m$.

Acknowledgements. The proof of this theorem is due to some interesting conversations with Gian Pietro Pirola for the case $H^2 = 4$, i.e., when S is a quartic surface in $\mathbb{P}^3$. Thanks also to Andreas Leopold Knutsen for valuable comments.

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Received: 9 November 2010