Appendix A

Derivation of linear stability equations

This appendix provides a detailed derivation of the linear stability equations for miscible displacement with a transient base state. The non-dimensional model problem is:

$$\mathbf{u} = -(\nabla P - c\mathbf{e_z}), \tag{A.1}$$

$$\partial_t c = -\mathbf{u} \cdot \nabla c + \nabla^2 c, \tag{A.2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{A.3}$$

where $\mathbf{u} = (u, w)$. The derivation does not depend on the initial condition or the boundary conditions which are therefore omitted here.

Linearization and base state

Darcy's law may be rewritten as:

$$\frac{\partial u}{\partial z} = -\frac{\partial^2 P}{\partial x \partial z}, \tag{A.4}$$

$$\frac{\partial w}{\partial x} = -\frac{\partial^2 P}{\partial x \partial z} + \frac{\partial c}{\partial x} \quad \Rightarrow \tag{A.5}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{\partial c}{\partial x} \quad \Rightarrow \tag{A.6}$$

$$\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 c}{\partial x^2}. \tag{A.7}$$

From equation (A.3) we have:

$$\frac{\partial^2 u}{\partial x \partial z} = -\frac{\partial^2 w}{\partial z^2}. \tag{A.8}$$

Inserting this above leads to:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 c}{\partial x^2}. \tag{A.9}$$

Now, the concentration and the vertical velocity component are written as:

$$c(x,z,t,k) = c_0(z,t) + \hat{c}(z,t,k)\, e^{ikx}, \tag{A.10}$$
$$w(x,z,t,k) = w_0(z,t) + \hat{w}(z,t,k)\, e^{ikx}, \tag{A.11}$$

where $\hat{c}(z,t,0) = \hat{w}(z,t,0) = 0$ (base state; no introduced perturbations). For the base state we further have that:

$$c_0(z,t) = 1 - \operatorname{erf}\left(z/(2\sqrt{t})\right), \quad (z > 0) \tag{A.12}$$
$$w_0(z,t) = 0, \tag{A.13}$$

where the first condition follows from mass balance and the latter is chosen. This gives us the first perturbation equation:

$$-k^2 \hat{w} e^{ikx} + \frac{\partial^2 \hat{w}}{\partial z^2} e^{ikx} = -k^2 \hat{c} e^{ikx} \Rightarrow \tag{A.14}$$
$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) \hat{w} = -k^2 \hat{c}, \tag{A.15}$$

Equation (A.3) with the base state and perturbation component is:

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z} = -\frac{\partial \hat{w}}{\partial z} e^{ikx}. \tag{A.16}$$

Integrating with respect to x gives:

$$u = -\frac{\partial \hat{w}}{\partial z} \frac{1}{ik} e^{ikx} + K(z,t). \tag{A.17}$$

Here $K = 0$ because there is no base velocity. Now we start from the mass balance equation (A.2) and rewrite it in terms of the base state and perturbation components:

$$\partial_t c = -\mathbf{u} \cdot \nabla c + \nabla^2 c.$$

The spatial derivatives are:

$$\frac{\partial c}{\partial x} = \hat{c} ik e^{ikx}, \qquad \frac{\partial^2 c}{\partial x^2} = -k^2 \hat{c} e^{ikx}, \tag{A.18}$$
$$\frac{\partial c}{\partial z} = \frac{\partial c_0}{\partial z} + \frac{\partial \hat{c}}{\partial z} e^{ikx}, \qquad \frac{\partial^2 c}{\partial z^2} = \frac{\partial^2 c_0}{\partial z^2} + \frac{\partial^2 \hat{c}}{\partial z^2} e^{ikx}, \tag{A.19}$$

and therefore:

$$\nabla^2 c = \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{c}e^{ikx} + \frac{\partial^2 c_0}{\partial z^2}. \tag{A.20}$$

The term $\mathbf{u}\cdot\nabla c$ is rewritten using (A.17):

$$\begin{aligned}
\mathbf{u}\cdot\nabla c &= u\frac{\partial c}{\partial x} + w\frac{\partial c}{\partial z} && \text{(A.21)}\\
&= u\hat{c}ike^{ikx} + w\left(\frac{\partial c_0}{\partial z} + \frac{\partial \hat{c}}{\partial z}e^{ikx}\right) && \text{(A.22)}\\
&= -\frac{\partial \hat{w}}{\partial z}\frac{1}{ik}e^{ikx}\hat{c}ike^{ikx} + \hat{w}e^{ikx}\left(\frac{\partial c_0}{\partial z} + \frac{\partial \hat{c}}{\partial z}e^{ikx}\right) && \text{(A.23)}\\
&= -\frac{\partial \hat{w}}{\partial z}\hat{c}e^{2ikx} + \hat{w}e^{ikx}\left(\frac{\partial c_0}{\partial z} + \frac{\partial \hat{c}}{\partial z}e^{ikx}\right). && \text{(A.24)}
\end{aligned}$$

Now, dropping nonlinear terms, we get the approximate:

$$\mathbf{u}\cdot\nabla c = \hat{w}e^{ikx}\frac{\partial c_0}{\partial z}. \tag{A.25}$$

Applying the mass balance equation for the base state (no perturbations) leads to:

$$\frac{\partial c_0}{\partial t} = \frac{\partial^2 c_0}{\partial z^2}. \tag{A.26}$$

The assembled linearized mass balance equation is:

$$\begin{aligned}
\frac{\partial c_0}{\partial t} + \frac{\partial \hat{c}}{\partial t}e^{ikx} &= -\hat{w}e^{ikx}\frac{\partial c_0}{\partial z} + \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{c}e^{ikx} + \frac{\partial^2 c_0}{\partial z^2} && \text{(A.27)}\\
\frac{\partial \hat{c}}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{c} &= -\frac{\partial c_0}{\partial z}\hat{w}. && \text{(A.28)}
\end{aligned}$$

In summary, the model problem in terms of perturbation and base state components is:

$$\begin{aligned}
\left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{w} &= -k^2\hat{c}, && \text{(A.29)}\\
\frac{\partial \hat{c}}{\partial t} - \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{c} &= -\frac{\partial c_0}{\partial z}\hat{w}. && \text{(A.30)}
\end{aligned}$$

Self similar coordinates

Now the linearized model problem is written in terms of the similarity variable of the base state, $\xi = z/(2\sqrt{t})$. First note that:

$$
\begin{aligned}
c_0 &= 1 - \mathrm{erf}(\xi), && \text{(A.31)}\\
\frac{\partial c_0}{\partial \xi} &= -\frac{2}{\sqrt{\pi}} e^{-\xi^2}, && \text{(A.32)}\\
\frac{\partial \xi}{\partial z} &= \frac{1}{2\sqrt{t}}, && \text{(A.33)}\\
\frac{\partial \xi}{\partial t} &= -\frac{z}{4t\sqrt{t}}, && \text{(A.34)}\\
\frac{\partial c_0}{\partial t} &= \frac{\partial c_0}{\partial \xi}\frac{\partial \xi}{\partial t}, && \text{(A.35)}\\
\frac{\partial \hat{c}(\xi, t)}{\partial t} &= \frac{\partial \hat{c}(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \hat{c}(t)}{\partial t}, && \text{(A.36)}\\
\frac{\partial \hat{c}}{\partial z} &= \frac{\partial \hat{c}}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{1}{2\sqrt{t}}\frac{\partial \hat{c}}{\partial \xi}, && \text{(A.37)}\\
\frac{\partial^2 \hat{c}}{\partial z^2} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z}\left(\frac{1}{2\sqrt{t}}\frac{\partial \hat{c}}{\partial \xi}\right) = \frac{1}{4t}\frac{\partial^2 \hat{c}}{\partial \xi^2}. && \text{(A.38)}
\end{aligned}
$$

Then, the first perturbation equation is:

$$
\left(\frac{1}{4t}\frac{\partial^2}{\partial \xi^2} - k^2\right)\hat{w} = -k^2 \hat{c}. \qquad \text{(A.39)}
$$

The second perturbation equation is obtained as:

$$
\begin{aligned}
\frac{\partial \hat{c}(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \hat{c}(t)}{\partial t} - \left(\frac{1}{4t}\frac{\partial^2}{\partial \xi^2} - k^2\right)\hat{c} &= -\frac{\partial c_0}{\partial \xi}\frac{\partial \xi}{\partial z}\hat{w}, && \text{(A.40)}\\
\frac{\partial \hat{c}(\xi)}{\partial \xi}\left(-\frac{z}{4t\sqrt{t}}\right) + \frac{\partial \hat{c}(t)}{\partial t} - \frac{1}{t}\left(\frac{1}{4}\frac{\partial^2}{\partial \xi^2} - k^2 t\right)\hat{c} &= \frac{2}{\sqrt{\pi}} e^{-\xi^2}\frac{\hat{w}}{2\sqrt{t}}, && \text{(A.41)}\\
-\frac{\partial \hat{c}(\xi)}{\partial \xi}\frac{\xi}{2t} + \frac{\partial \hat{c}(t)}{\partial t} - \frac{1}{t}\left(\frac{1}{4}\frac{\partial^2}{\partial \xi^2} - k^2 t\right)\hat{c} &= \sqrt{\frac{1}{\pi t}} e^{-\xi^2}\hat{w}, && \text{(A.42)}\\
\frac{\partial \hat{c}(t)}{\partial t} - \frac{1}{t}\left(\frac{1}{4}\frac{\partial^2}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial}{\partial \xi} - k^2 t\right)\hat{c} &= \sqrt{\frac{1}{\pi t}} e^{-\xi^2}\hat{w}. && \text{(A.43)}
\end{aligned}
$$