Boundary-value problems and shoaling analysis for the BBM equation

Master of Science Thesis in Applied and Computational Mathematics

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Abstract

In this thesis we study the BBM equation

$$u_t + u_x + \frac{3}{2}uu_x - \frac{1}{6}u_{xxt} = 0$$

which describes approximately the two-dimensional propagation of surface waves in a uniform horizontal channel containing an incompressible and inviscid fluid which in its undisturbed state has depth h. Here $u(x,t)$ represents the deviation of the water surface from its undisturbed position, and the flow is assumed to be irrotational.

The BBM equation features a bounded dispersion relation (Benjamin, Bona and Mahony [3]). We utilize this boundedness to prove existence, uniqueness and regularity results for solutions of the BBM equation supplemented with an initial condition and various types of boundary conditions. We also treat the water-wave problem over an uneven bottom. In particular, we consider two different models for the propagation of long waves in channels of decreasing depth and we provide both analytical and numerical results for these models. For the numerical simulation we use a spectral discretization coupled with a four-stage Runge-Kutta time integration scheme. After verifying numerically that the algorithm is fourth-order accurate in time, we run the solitary wave with uneven bottom and examine how solitary waves respond to this non-uniform depth. Our numerical simulations are compared with previous numerical and experimental results of Madsen and Mei [16] and Peregrine [19].

Contents

Outline and motivation

The prediction of the behaviour of surface water waves is of great practical as well as theoretical, interest because for instance, a large ship, travelling at the wrong speed, may generate waves of nearly permanent form, which are destructive when they come ashore. The BBM equation is one of the important nonlinear equation and also a regularized model for long-water waves. Moreover, the BBM equation admits soliton solutions. An important part of this study will be to examine the solitary wave solution for an uneven bottom.

In chapter 1 we start with historical remarks about solitary waves and KdV equation. Next we introduce BBM equation and a derivation of the BBM equation. We review dispersion relation for the BBM equation. Finally we move on to a models for variable depth long waves.

In chapter 2 we first analyse the solution of BBM equation with an initial and different types of boundary conditions theoretically. Here we consider all possible types of boundary conditions. For instance, both Dirichlet, both Neumann and mixed boundary conditions. Further we present theory for the solitary wave propagation with uneven bottom.

In chapter 3 we solve the BBM equation numerically using spectral methods with the help of the Runge-Kutte 4-stage time integration method. Next we run solitary wave solution in an uniform bottom profile and observe the results. We check the algorithm by halving the time steps and this results 16 times protection of error.

In chapter 4 we study the solitary wave solution of our two new models (1.66) and (1.67) numerically. Here we compare the amplitudes of the solitary wave after some time with the initial wave amplitudes. Further we compare the numerical results with theoretical results.

In chapter 5 we are interested in solitary wave transformation on a slope. Here we examine how solitary waves behave on the slope with different initial amplitudes. Further we compare our results with existing experiment results made by Madsen and Mei [16] and Peregrine [19].

Chapter 1

Introduction

1.1 Historical remarks about solitary waves

In this section we will see how the KdV equation and the solitary waves appeared in science. The work is based on the books of Drazin and Johnson [6], Ablowitz and Clarkson [1]. The Korteweg-de Vries Equation (KdV equation) describes the theory of water waves in shallow channels. It is a non-linear equation which exhibits special solutions, known as solitons, which are stable and do not disperse with time. Drazin and Johnson mentioned some of the analytical properties of this wave and the KdV is indeed the relevant one for the solitary wave. A nice story about the history and the underlying physical properties of the Korteweg-de Vries equation can be found from the experiments handled by John Scott Russell (1808-1882). He first observed the solitary waves on the Edinburgh-Glasgow canal in 1834. He discovered a phenomenon that he called as the 'great wave of translation'. He described the discovery in the following words:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

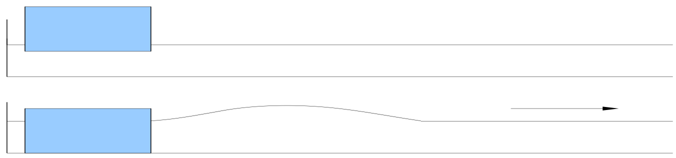


Figure 1.1: Scott Russell's experiment to generate a solitary wave.

Russell has done some laboratory experiments, generating waves by dropping a weight at one end of a water channel (Figure 1.1). He deduced that the volume of water in the wave is equal to the volume of water displaced and the wave speed, c, of the solitary wave is given by

$$c^2 = g(h + a), \tag{1.1}$$

where 'a' is the amplitude of the wave, 'h' the undisturbed depth of water and 'g' the acceleration of gravity. Because of the gravity term, solitary waves are gravity waves. From (1.1) we can find that higher waves travel faster. Boussinesq (1871) and Lord Rayleigh (1876) assumed that a solitary wave has a length scale much greater than the depth of the water. They found, from the equations of motion for an inviscid, incompressible fluid, Russell's formula for c. They showed that the wave form of a solitary wave is given as a function of distance x and time t by

$$\eta(x, t) = a \ \mathrm{sech}^2(k(x - ct)), \tag{1.2}$$

where a is the maximum wave height, c is the speed, although the $sech^2$ profile is strictly correct only if $a/h \ll 1$. The parameter k is defined by

$$k = \sqrt{\frac{3a}{4h^2(h + a)}},$$

'h' is the water depth for any $a > 0$.

Further investigations were made by Airy (1845), Stokes(1847), Boussinesq (1871, 1872) and Rayleigh (1876) in an attempt to understand this phenomenon. Boussinesq and Rayleigh separately obtained approximate descriptions of the solitary wave. Boussinesq derived one-dimensional non-linear evolution equation, which now called Boussinesq approximation. The investigations resulted much lively discussion and controversy as to whether the inviscid equations of water waves would posses such solitary wave solutions. Finally Korteweg and de Vries (1895) resolved this issue. They derived a non-linear evolution equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \chi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \chi^2}\right), \tag{1.3}$$

where $\sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}$, which governing small amplitude, long one dimensional, surface gravity waves propagating in a shallow channel of water. Here η is the surface elevation of the wave above equilibrium level 'h', α an small arbitrary constant related to the uniform motion of the liquid, g the gravitation constant, T the surface tension and ρ the density (here the terms "long" and "small" are meant in comparison to the depth of the channel). The equation (1.3) is called Korteweg-de Vries equation (KdV), has permanent wave solutions (see Drazin and Johnson [6] sec 1.3).

Finally we see a connection between the Korteweg-de Vries equation (1.3), the $sech^2$ profile and the Russell wave speed formula, under the assumption that $a/h \ll 1$, as follows. If the solution of equation (1.3) is stationary in the frame χ, then $\eta = \eta(\chi)$ and (1.3) becomes

$$\frac{2}{3}\alpha\eta' + \eta\eta' + \frac{1}{3}\sigma\eta''' = 0. \tag{1.4}$$

Here the prime denotes the derivative with respect to χ. If we consider $\eta \to 0$ as $|\chi| \to \infty$, then equation (1.4) can be integrated to give

$$\frac{2}{3}\alpha\eta + \frac{1}{2}\eta^2 + \frac{1}{3}\sigma\eta'' = 0. \tag{1.5}$$

The equation (1.5) can be integrated once again, we get that

$$2\alpha\eta^2 + \eta^3 + \sigma(\eta')^2 = 0. \tag{1.6}$$

It is easy to check that this equation admits A solitary-wave solution of the form

$$\eta(\chi) = a \text{ sech }^2(k\chi), \tag{1.7}$$

provided $a = 4\sigma k^2$ and $\alpha = -2\sigma k^2$. The coordinate χ is defined by Korteweg and de Vries in 1895 by

$$\chi = x - \sqrt{gh}\left(1 - \frac{\alpha}{h}\right)t, \tag{1.8}$$

and therefore the the solitary wave solution becomes

$$\eta(x,t) = a \text{ sech }^2\left[\frac{1}{2}\left(\frac{a}{\sigma}\right)^{1/2}\left\{x - \sqrt{gh}\left(1 + \frac{1}{2}\frac{a}{h}\right)t\right\}\right]. \tag{1.9}$$

If we neglect surface tension and assume $a/h \ll 1$, then above equation (1.9) agrees with c formula of Russell in (1.1) and (1.2), which also shows that the wave speed has a form

$$\begin{aligned} c &\sim \sqrt{gh}\left(1 + \frac{a}{2h}\right) \Rightarrow c^2 \sim g(h+a) + O(\frac{a}{h}), \\ k &\sim \frac{1}{2}\left(\frac{3a}{h^3}\right)^{1/2}, \end{aligned}$$

and this also agreed with the work of Boussinesq and Lord Rayleigh. That is

$$k^2 = \frac{3a}{4h^2(h+a)} = \frac{3a}{4h^3(1+a/h)} \sim \frac{3a}{4h^3} \quad [\because a/h \ll 1].$$

Therefore, Russell's solitary wave is a solution of the KdV equation.

1.2 BBM equation

The BBM equation is an alternative model for the Korteweg-de Vries (KdV) equation (Korteweg and de Vries (1895)) (in dimensionless form and in the absence of surface tension)

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.10}$$

Under the assumption of small wave-amplitude and large wave length, the KdV equation was originally derived for water waves and it is similarly justifiable as a model for long waves in many other physical systems (Benjamin, Bona, Mahony [3]).
The KdV equation is approximated by the propagation of uni-directional, two-dimensional, small amplitude long waves in non-linear dispersive media. Here the parameters are scaled into the definition of space x, time t and

$u(x,t)$ the deflection of the surface from its rest position at the point x at time t.
Joseph Boussinesq (1872) proposed a variety of possible models for describing the propagation of water waves in shallow channels and he included KdV equation. While it has some remarkable properties (Drazin and Johnson [6]), some other applications of this equation are less favourable. For e.g non-physical unbounded dispersion relation. There are several noticeable attempts to improve the KdV equation. Benjamin, Bona and Mahony (1972) proposed an another model which is an alternative model instead of the KdV equation (1.10). They used the following argument to obtain an alternative model.
Here the variables u, x and t are non-dimensional and when we scaled so that the dependent variable and its derivatives are of order one, (1.10) takes the form (see Kalisch [12])

$$u_t + u_x + \epsilon u u_x + \epsilon u_{xxx} = O(\epsilon^2), \tag{1.11}$$

where ϵ is of order $\frac{h^2}{\lambda^2} \cong \frac{a}{h}$. We can also observe from (1.11) that

$$u_t + u_x = O(\epsilon). \tag{1.12}$$

If we assume that the differentiation does not alter the ϵ- order of the dependent variable, (1.12) becomes

$$u_{xxt} + u_{xxx} = O(\epsilon), \tag{1.13}$$

so we can replace u_{xxx} by $-u_{xxt}$ in (1.11), we have

$$u_t + u_x + \epsilon u u_x - \epsilon u_{xxt} = O(\epsilon^2). \tag{1.14}$$

Again, eliminating terms of order ϵ^2 and then rescaling, we get

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1.15}$$

which is called BBM equation and is equally well justified as a model of the same phenomena which replaces the third- order derivative (u_{xxx}) in (1.10) by a mixed derivative, $-u_{xxt}$. This results a bounded dispersion (see fig. 1.4) relation which is useful to prove existence, uniqueness, and regularity results for solutions of the BBM equation. The KdV and the BBM equation is indeed the relevant one for the solitary wave.

1.3 Linear water wave theory

Waves are motions generated due to the existence of restoring force. The gravity waves are the waves with gravity playing the role of the restoring force. It occurs in stratified media, or on interfaces between two media with different properties. Surface gravity waves occurs at the free surface of a liquid. For example the ocean and the atmosphere. If the wave occur at the interface between two fluid of different density, then the waves are called internal gravity waves. Here we give attention to the surface gravity waves.

We assume an inviscid, incompressible fluid in a constant gravitational field including the assumption of small amplitude shallow water. The phenomena we study here have waves propagating only in one-horizontal direction. Let the space coordinates are (x,z) and x-axis be oriented in horizontal direction. Assume there is no motion takes place in the direction perpendicular to the xz-plane. The gravitational acceleration g is in the negative z direction. We assume here that the motion is irrotational (Kundu and Cohen [14] p. 219) and let $\phi(x, z, t)$ denote the velocity potential ($u = \Delta\phi$), where $u = (u, w)$ is the velocity vector and u,w are the horizontal and vertical components respectively. Then form the irrotational motion, the velocity potential ϕ satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{for} \quad -h_0 < z < \eta(x,t). \tag{1.16}$$

In order to solve this equation, we are in need of boundary conditions. We

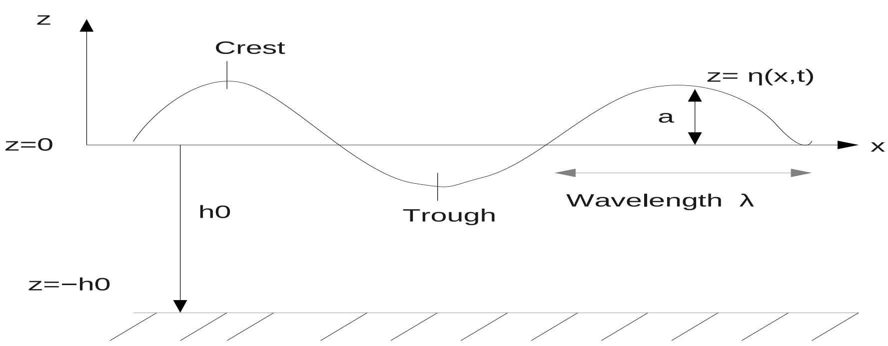


Figure 1.2: Geometrical configuration for water waves.

assume small perturbations of water surface that is initially at rest, i.e small-amplitude waves . In this case both velocities and surface excursion must be small. That is we can neglect quadratic terms and boundary conditions at the free surface can be evaluated at $z = 0$ rather than $z = \eta$. Therefore we have to solve

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1.17}$$

subject to the conditions

$$\frac{\partial \phi}{\partial z} = 0, \qquad \text{on } z = -h_0, \tag{1.18}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}, \quad \text{on } z = 0, \tag{1.19}$$

$$\frac{\partial \phi}{\partial t} + g\eta = 0, \qquad \text{on } z = 0. \tag{1.20}$$

Equation (1.19) is referred as a kinematic boundary condition and (1.20) represents the continuity of pressure at the free surface, as derived from Bernoulli's equation. We assume that the surface $\eta(x, t)$ in the form of a sinusoidal wave with wave number ξ, frequency ω and amplitude a.

$$\eta = a\cos(\xi x - \omega t). \tag{1.21}$$

Equations (1.19) and (1.20) show that ϕ must be a 'sine' function of $(\xi x - \omega t)$. Consequently, we assume that the solution of Laplace equation is separable,

$$\phi(x, z, t) = f(z)\sin(\xi x - \omega t). \tag{1.22}$$

Then (1.17) yields f on the form

$$f(z) = Ae^{\xi z} + Be^{-\xi z}, \tag{1.23}$$

where A and B are arbitrary constants. Equations (1.22) and (1.23) lead to

$$\phi = \frac{a\omega}{\xi}\frac{\cosh(\xi(z + h_0))}{\sinh(\xi h_0)}\sin(\xi x - \omega t), \tag{1.24}$$

and the velocity components are given by (Kundu and Cohen [14] p. 223)

$$u = a\omega\frac{\cosh(\xi(z + h_0))}{\sinh(\xi h_0)}\cos(\xi x - \omega t), \tag{1.25}$$

$$w = a\omega\frac{\sinh(\xi(z + h_0))}{\sinh(\xi h_0)}\sin(\xi x - \omega t). \tag{1.26}$$

Substitution of equations (1.21) and (1.24) into (1.20) gives the dispersion relation:

$$\omega^2 = g\xi \tanh(\xi h_0). \quad (1.27)$$

The propagation speed c of the waves are

$$c = \omega/\xi.$$

That is the phase velocity of the linearized water wave problem is given by

$$c = \omega/\xi = \sqrt{\frac{g}{\xi}\tanh(\xi h_0)} = \sqrt{gh_0}(1 - \frac{1}{6}(\xi h_0)^2 + ...). \quad (1.28)$$

In general, c depends on the wave length $\lambda = \frac{2\pi}{\xi}$. Further we say something about long waves. If $\frac{h_0}{\lambda} \ll 1$, then the waves are called long waves or shallow water waves and

$$c_0 = \sqrt{gh_0} \quad (1.29)$$

is called long wave speed, which is independent of wavelength (Kundu and Cohen [14] p. 229).
If we consider the Taylor's expansion of (1.28) approximated to second order, then we get the phase velocity in the KdV equation which is given by

$$c(\xi) = c_0 - \frac{1}{6}c_0 h_0^2 \xi^2. \quad (1.30)$$

1.4 Derivation of nonlinear equations

In this section we derive the BBM equation which represent a mathematical model of surface gravity waves on shallow water balancing nonlinear and dispersive effects. Here the derivation is based on Whitham ([24] pp.463-466). To derive BBM equation we assume an inviscid, incompressible fluid in a constant gravitational field including the assumption of small amplitude shallow water. We now introduce a vertical coordinate, $Z = z + h_0$, which represents the distance from the bottom. Then the velocity potential ϕ satisfies the Laplace equation

$$\phi_{xx} + \phi_{ZZ} = 0 \quad 0 < Z < \eta + h_0 \quad (1.31)$$

with

$$\phi_Z = 0 \text{ on } Z = 0. \quad (1.32)$$

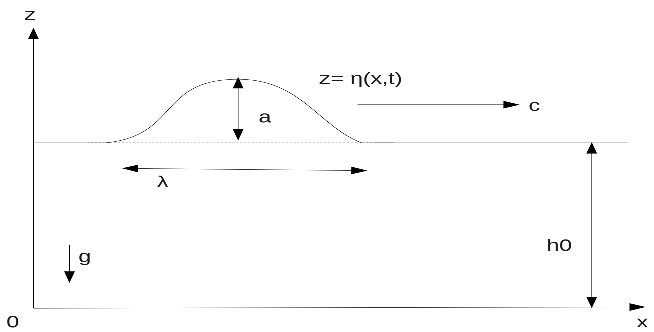


Figure 1.3: The terms used in the derivation of BBM equation.

The boundary conditions (1.18), (1.19) and (1.20) are changed correspondingly. We assume that the solution of equation (1.31) can be expressed by an expansion in Z by

$$\phi = \sum_{n=0}^{\infty} Z^n f_n(x,t). \tag{1.33}$$

Then use (1.32) and (1.31) to find

$$\phi = \sum_{m=0}^{\infty} (-1)^m \frac{Z^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial x^{2m}}, \tag{1.34}$$

here $f = f_0$. Next, we want to find f by using the boundary conditions on the free surface. Further for our convenience, we normalize the variables by

$$x = \lambda x', \quad Z = h_0 Z', \quad t = \frac{\lambda t'}{c_0}, \quad \eta = a\eta', \quad \phi = \frac{g\lambda a \phi'}{c_0}, \tag{1.35}$$

where the non-primed variables are the original and the primed variables are the normalised variables, c_0 is the phase velocity and a is the amplitude. We write (1.31) and the boundary conditions (1.18), (1.19) and (1.20) in normalised variables, we have

$$\beta\phi'_{x'x'} + \phi'_{Z'Z'} = 0, \quad \text{for } 0 < Z' < 1 + \alpha\eta', \tag{1.36}$$

$$\phi'_{Z'} = 0, \quad \text{at } Z' = 0, \tag{1.37}$$

$$\left.\begin{array}{l} \eta'_{t'} + \alpha\phi'_{x'}\eta'_{x'} - \frac{1}{\beta}\phi'_{Z'} = 0, \\ \eta' + \phi'_{t'} + \frac{1}{2}\alpha\phi'^2_{x'} + \frac{1}{2}\frac{\alpha}{\beta}\phi'^2_{Z'}, \end{array}\right\} \quad \text{at } Z' = 1 + \alpha\eta', \tag{1.38}$$

where $\alpha = a/h_0$ and $\beta = h_0^2/\lambda^2$, and (1.34) becomes

$$\phi = \sum_{m=0}^{\infty} (-1)^m \frac{Z'^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial x'^{2m}} \beta^m. \tag{1.39}$$

On substitution in the boundary conditions at the free surface we find that

$$\eta'_{t'} + \{(1+\alpha\eta')f_{x'}\}_{x'} - \left\{ \frac{1}{6}(1+\alpha\eta')^3 f_{x'x'x'x'} + \frac{1}{2}\alpha(1+\alpha\eta')^2 \eta'_{x'} f_{x'x'x'} \right\} \beta + O(\beta^2) = 0, \tag{1.40}$$

$$\eta' + f_{t'} + \frac{1}{2}\alpha f_{x'}^2 - \frac{1}{2}(1+\alpha\eta')^2 \left\{ f_{x'x't'} + \alpha f_{x'} f_{x'x'x'} - \alpha f_{x'x'}^2 \right\} \beta + O(\beta^2) = 0. \tag{1.41}$$

If we cancel all terms of order $\alpha\beta$ and differentiate (1.41) with respect to x', then we get the non-linear shallow water equations:

$$\eta'_{t'} + \{(1+\alpha\eta')\zeta\}_{x'} - \frac{1}{6}\beta\zeta_{x'x'x'} + O(\alpha\beta, \beta^2) = 0, \tag{1.42}$$

$$\zeta_{t'} + \alpha\zeta\zeta_{x'} + \eta'_{x'} - \frac{1}{2}\beta\zeta_{x'x't'} + O(\alpha\beta, \beta^2) = 0, \tag{1.43}$$

where $\zeta = f_{x'}$, these are a variant of Boussinesq's equations. Here ζ is the first term in the expansion of the velocity $\phi'_{x'}$, which is given by

$$\phi'_{x'} = \zeta - \beta \frac{Z'^2}{2} \zeta_{x'x'} + O(\beta^2). \tag{1.44}$$

If we integrate over the depth, then we get

$$\tilde{u} = \zeta - \frac{1}{6}\beta\zeta_{x'x'} + O(\alpha\beta, \beta^2). \tag{1.45}$$

The inverse is

$$\zeta = \tilde{u} + \frac{1}{6}\beta\tilde{u}_{x'x'} + O(\alpha\beta, \beta^2). \tag{1.46}$$

We first drive Korteweg-de Vries from any of these system by specializing to a wave moving to the right. If we neglect the terms of α and β from (1.42) and (1.43), we get (see Whitham [24] pp.466)

$$\zeta = \eta', \quad \eta'_{t'} + \eta_{x'} = 0. \tag{1.47}$$

This a plain linear transport equation. We however wish to keep the first order terms of α and β, in the form

$$\zeta = \eta' + \alpha A + \beta B + O(\alpha^2 + \beta^2), \tag{1.48}$$

where A and B are the functions of η' and derivatives with respect to x'. Then (1.42) and (1.43) becomes

$$\eta'_{t'} + \eta'_{x'} + \alpha\left\{A_{x'} + 2\eta'\eta'_{x'}\right\} + \beta\left(B_{x'} - \frac{1}{6}\eta'_{x'x'x'}\right) O(\alpha^2 + \beta^2) = 0, \quad (1.49)$$

$$\eta'_{t'} + \eta'_{x'} + \alpha\left\{A_{t'} + \eta'\eta'_{x'}\right\} + \beta\left(B_{t'} - \frac{1}{2}\eta'_{x'x't'}\right) + O(\alpha^2 + \beta^2) = 0. \quad (1.50)$$

And hence

$$\eta'_{t'} = \eta'_{x'} + O(\alpha, \beta). \quad (1.51)$$

The two equations (1.49) and (1.50) become identical only if

$$A = -\frac{1}{4}\eta'^2, \qquad B = \frac{1}{3}\eta'_{x'x'}. \quad (1.52)$$

If we substitute A and B into (1.49), (1.50), we obtain the KdV equation in normalised variables

$$\eta'_{t'} + \eta'_{x'} + \frac{3}{2}\alpha\eta'\eta'_{x'} + \frac{1}{6}\beta\eta'_{x'x'x'} + O(\alpha^2 + \beta^2) = 0, \quad (1.53)$$

and

$$\zeta = \eta' - \frac{1}{4}\alpha\eta'^2 + \frac{1}{3}\beta\eta'_{x'x'} + O(\alpha^2 + \beta^2). \quad (1.54)$$

Further we change the normalise variables into original variables using (1.35) and ignore α and β terms , we obtain the KdV equation in the form

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}c_0 h_0^2\eta_{xxx} = 0. \quad (1.55)$$

In the following we see how we derive the BBM equation using KdV equation. The argument is based on Benjamin, Bona, Mahony [3] and Kalisch [12]. It also from (1.51) that

$$\eta'_{t'} + \eta'_{x'} = O(\alpha, \beta), \quad (1.56)$$

that is in original variables

$$\frac{1}{c_0}\eta_t + \eta_x \approx O\left(\frac{a}{h_0}\right). \quad (1.57)$$

Under the assumptions that the differentiation does not alter the α, β order of the dependent variable, replace $-c_0\eta_x$ by η_t in the last term of (1.55) becomes

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{1}{6}h_0^2\eta_{xxt} = 0. \quad (1.58)$$

Next we wish to find the linear phase velocity of the BBM equation. Therefore we linearize the BBM equation and assume that the solution is of the form

$$\eta = e^{(ix\xi - i\omega t)}. \tag{1.59}$$

On substitution of (1.59) into (1.58) gives the dispersion relation

$$c(\xi) = \frac{\omega}{\xi} = \frac{c_0}{1 + \frac{1}{6}h_0^2\xi^2}. \tag{1.60}$$

Ehrnström and Kalisch [7] compared phase velocity $c(\xi) = \frac{\omega}{\xi}$ for the KdV, Euler, and BBM equations. They concluded that the linear phase velocity of BBM equation (1.60) is qualitatively closer to wave speed given by (1.28) than the phase velocity of KdV equation (1.30). We have the following figure (1.4) for the comparison of wave speed calculated by Ehrnström and Kalisch [7].

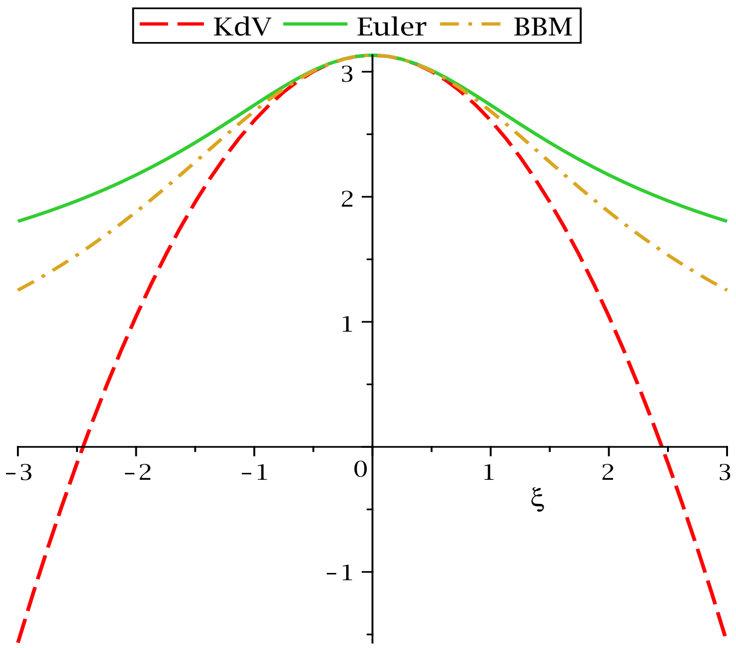


Figure 1.4: Comparision of phase velocity for the KdV, Euler and BBM equations made by Ehrnström and Kalisch [7] in the case $h_0 = 1$.

Therefore BBM equation has better dispersion relation than the KdV equation. Benjamin, Bona, Mahony [3] explained bit more about these issues of modeling long waves.

1.5 Models for variable depth

Our main work is focused on the solitary waves with long wave length, lower amplitude and in addition that the depth of the water is slowly varying. Many peoples have been interested in models which represent the changes that occur in a solitary wave as it travels over a slowly changing topography. The first study was conducted by Green in 1832. He considered the effects of a slowly changing depth on a linear surface wave and found that the amplitude of the linear wave changed inversely proportional to the fourth root of the depth (Synolakis [22]). Boussinesq found that the amplitude of a solitary wave changed inversely proportional to the depth. In recent studies of long waves in shallow water, the interplay between non-linearity and dispersiveness has received much attention (Whitham [24]).
When the bottom is not flat on water waves, which is obvious importance in engineering field and has been incorporated in the governing equations by Mei and LeMéhauté (1996) and by Peregrine (1967). Peregrine (1967) obtained quantitative results using a finite difference scheme to obtain the deformation of a solitary wave climbing on a beach. The governing equation in their works is the Korteweg-de Vries equation or a simple extension there of, thus corresponding to waters of constant depth. Ippen and Kulin (1995), Kishi and Saeki (1966), Camfield and Street (1969) obtained additional results which are in better conformation with the experiments. Madsen and Mei (1969) treated the related problem of a solitary wave propagating from a channel of constant depth, past a mild slope, onto a shelf of constant, smaller depth. They found that when a wave reaches the slope the amplitude has increased slightly. Schematically, the experiment setup is as illustrated in figure (1.5). In 1996 Mei and LeMéhauté obtained the set of approximate equations for the above physical system of solitary waves over uneven bottom ([Madsen, Mei [16]) are

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \frac{\partial[u(h+\eta)]}{\partial x} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} &= Au + B\frac{\partial u}{\partial x} + \frac{3}{2}h^2 h'\frac{\partial^2 u}{\partial x^2} + O(\epsilon^6),\\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\epsilon}\frac{\partial \eta}{\partial x} - \frac{1}{2}h^2\frac{\partial^3 u}{\partial t \partial x^2} &= [(h')^2 + hh'']\frac{\partial u}{\partial t} + 2hh'\frac{\partial^2 u}{\partial t \partial x} + O(\epsilon^5),
\end{aligned}$$

where

$$\begin{aligned}
u &= \text{horizontal velocity averaged over the depth},\\
A &= (h')^3 + 3hh'h'' + \frac{1}{2}h^2 h''',\\
B &= 3h(h')^2 + \frac{3}{2}h^2 h'',\\
\epsilon &= h_0/L \ll 1,
\end{aligned}$$

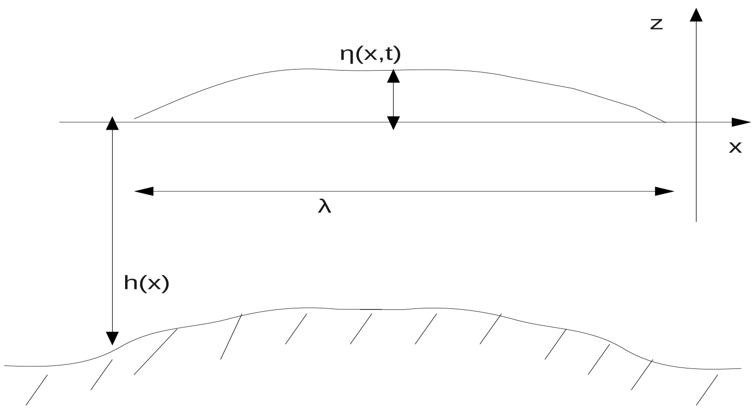


Figure 1.5: Geometry of the problem.

h_0 and L are typical vertical and horizontal length scales respectively. Peregrine (1996) obtained the following equation for two dimensional waves with beach of uniform slope α, water in the region $x > 0$ and crests parallel to the shore line

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} &= \frac{1}{3}\alpha^2 x^2 \frac{\partial^3 u}{\partial x^2 \partial t} + \alpha^2 x \frac{\partial^2 u}{\partial x \partial t}, \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\alpha x + \eta)u] &= 0. \end{aligned} \tag{1.61}$$

In our work, we consider the following two models:

$$\eta_t + c(x)\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{h_0^2}{6}\eta_{xxt} = 0, \tag{1.62}$$

$$\eta_t + (c(x)\eta)_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{h_0^2}{6}\eta_{xxt} = 0. \tag{1.63}$$

Here η is $(h_0+\eta(x,t))$ is the total depth at location x at time t, t elapsed time, x is the distance along the channel, h_0 undisturbed depth, $c(x) = \sqrt{gh(x)}$ where g is the acceleration gravity. The above equations (1.62) and (1.63) are not derived formally, but are reasonable when assuming a small gradient in the bottom profile.

If we use the standard non-dimensional variables $\hat{x} = \frac{x}{h_0}, \hat{\eta} = \frac{\eta}{h_0}, \hat{t} = \frac{t}{h_0/c_0}$. Then our model equations (1.62) and (1.63) becomes

$$\hat{\eta}_{\hat{t}} + \frac{c(\hat{x}h_0)}{c_0}\hat{\eta}_{\hat{x}} + \frac{3}{2}\hat{\eta}\hat{\eta}_{\hat{x}} - \frac{1}{6}\hat{\eta}_{\hat{x}\hat{x}\hat{t}} = 0, \tag{1.64}$$

$$\hat{\eta}_{\hat{t}} + (\frac{c(\hat{x}h_0)}{c_0}\hat{\eta})_{\hat{x}} + \frac{3}{2}\hat{\eta}\hat{\eta}_{\hat{x}} - \frac{1}{6}\hat{\eta}_{\hat{x}\hat{x}\hat{t}} = 0 \tag{1.65}$$

Further we rewrite the above equations in the form

$$\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0, \tag{1.66}$$

$$\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0. \tag{1.67}$$

In the next chapters we will discuss the existence and regularity results for the above equations (1.66) and (1.67).

Chapter 2

Mathematical theory

2.1 Definitions

Definition 2.1. *We define $C^k(a,b)$ as the Banach space of k-times continuously differentiable functions defined on [a,b], equipped with the norm*

$$\|f\|_{C^k} = \sup_{0\le j\le k}\sup_{a\le x\le b}\left|f^{(j)}(x)\right|.$$

We denote $\|f\|_{C^0}$ by $\|f\|$.

Definition 2.2. *The space $L^p = L^p(\mathbb{R})$, $1 \le p < \infty$ is the set of all measurable real-valued functions of a real variable f whose p^{th} powers are integrable over $\mathbb{R}$. That is*

$$\int_{\mathbb{R}} |f(x)|^p \, dx < \infty.$$

The norm is denoted by $\|f\|_{L^p}$ and defined by

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x))|^p \, dx\right)^{1/p}.$$

Similarly we can define

$$\|f\|_{L^p(0,L)} = \left(\int_0^L |f(x))|^p \, dx\right)^{1/p}.$$

Definition 2.3. *We define the Sobolev norm by*

$$\|f\|_{k,p} = \left(\sum_{0 \le n \le k,\ n \in \mathbb{N}} \|D^n u\|_p^p \right)^{1/p},$$

where k is a positive integer and $1 \le p < \infty$. In particular, we take p=2 and we define the space $H^k(\mathbb{R})$ which is the subspace of $L^2(\mathbb{R})$ by

$$\|f\|^2_{H^k(\mathbb{R})} = \sum_{0 \le n \le k,\ n \in \mathbb{N}} \|D^n u\|^2_{L^2(\mathbb{R})}$$

which is also a Banach space. Similarly we define $H^k(0,L)$ which is the subspace of $L^2(0,L)$ by

$$\|f\|^2_{H^k(0,L)} = \sum_{0 \le n \le k,\ n \in \mathbb{N}} \|D^n u\|^2_{L^2(0,L)}.$$

We can also define the Sobolev norm by using Fourier transforms which is given by

$$\|f\|^2_{H^k(\mathbb{R})} = \int_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^k \left|\hat{f}(\xi)\right|^2 d\xi < \infty,$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

Remark: There is no necessity for k to be an integer in this condition. If s is real we can consider those functions in H^s such that $\int_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi$ is finite to define the Sobolev space H^s.

Definition 2.4. *We define the space $C([0,T];X)$, for any Banach space X (for instance $X = C^k$ (or) H^k), is the Banach space of continuous maps $u(x,t) : [0,T] \to X$ with the norm*

$$\|u\|_{C([0,T]:X)} = \sup_{0 \le t \le T} \|u(.,t)\|_X .$$

In the same way, we define

$$\begin{aligned} C^n([0,T];C^k) &= \left\{ u(x,t) : \partial_t^k(u.,t) \in C([0,T];C^k) \text{ for } 0 \le k \le n \right\}, \\ C^n([0,T];H^k) &= \left\{ u(x,t) : \partial_t^k(u.,t) \in C([0,T];H^k) \text{ for } 0 \le k \le n \right\}. \end{aligned}$$

And the corresponding norms are defined by

$$\|u\|_{C^n([0,T]:X)} = \sum_{k\le n} \left\|\partial_t^k u\right\|_{C([0,T]:X)}.$$

Further, we define the space

$$C^\infty([0,T];X) = \cap_{n\ge 0} C^n([0,T];X).$$

Definition 2.5. *Given a measurable function $f : X \to R$, where X is a measure space with measure μ, the essential supremum is the smallest number α such that the set $\{x : f(x) > \alpha\}$ has measure zero. If no such number exists, then the essential supremum is ∞.*
The essential supremum of the absolute value of a function $|f|$ is usually denoted $\|f\|_\infty$, and this serves as the norm for L^∞-space.
Or in other words,

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{\alpha; |f(x)| \le \alpha \ a.e\ on\ X\}.$$

Definition 2.6. *We define the space $C_b([0,\infty];H^k) = C([0,\infty];H^k) \cap L^\infty$, which consists of all functions $u(x,t)$ such that $u(.,t)$ is a continuous function $t \to H^s$ for $t \in [0,\infty]$ and are bounded.*

Definition 2.7. *The L^2- inner product is defined as*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

Since all functions we consider are real-valued, we take the L^2- inner product.
The convolution of two functions is defined as

$$g * f(x) = \int_{-\infty}^{\infty} g(y) f(x-y) dy$$

2.2 Initial and boundary-value problems

In this section, we prove that the given initial and boundary data (2.2), there exists a unique solution defined at least in $[0,L]\times[0,T]$ for some $T > 0$ and also we examine regularity of this solution. We use contraction mapping principle to establish the existence theory. Here all my works are based on the works of Bona, Chen [4] and Benjamin, Bona, Mahony [3].
If we use the standard non-dimensional variables $\hat{x} = \frac{x}{h_0}$, $\hat{\eta} = \frac{\eta}{h_0}$ and $\hat{t} = \frac{t}{h_0/c_0}$, then our model equation (1.58) becomes

$$\hat{\eta}_{\hat{t}} + \hat{\eta}_{\hat{x}} + \frac{3}{2}\hat{\eta}\hat{\eta}_{\hat{x}} - \frac{1}{6}\hat{\eta}_{\hat{x}\hat{x}\hat{t}} = 0.$$

We rewrite the above system in the form

$$u_t + u_x + \frac{3}{2} u u_x - a^{-2} u_{xxt} = 0, \tag{2.1}$$

where $a^2 = 6$, the initial and boundary conditions for (2.1) is specified are

$$\begin{aligned} u(x,0) &= u_0(x), \\ u(0,t) &= h(t), \\ u(L,t) &= g(t). \end{aligned} \tag{2.2}$$

The function $u(x,t)$ represents the vertical deviation of the surface from its rest position at the point x at time t. The equation (2.1) can be written as

$$(1 - a^{-2}\partial_x^2) u_t = -u_x - \frac{3}{2} u u_x.$$

Now we use Green's function (Roach [20]) to find the solution of (2.1). Our first task is to determine the Green's function $G(x,s)$ for the operator

$$Q = (1 - a^{-2}\partial_x^2)$$

subject to the homogeneous boundary conditions $u(0) = 0, u(L) = 0$. Once this is obtained we extend the definition of Q to later for the actual boundary value (2.2).

To determine the Green's function $G(x,s)$, we use the following technique. Let $G(x,s)$ be a solution of

$$\begin{aligned} QG = (1 - a^{-2}\partial_x^2) G &= \delta(x-s), \\ G(0) &= 0, \\ G(L) &= 0. \end{aligned}$$

Since G(x,s) satisfies

$$(1 - a^{-2}\partial_x^2) G = 0 \tag{2.3}$$

everywhere except at $x = s$. For $x < s$ an arbitrary solution of (2.3) is

$$G(x,s) = A(s) e^{ax} + B(s) e^{-ax}, \tag{2.4}$$

where $A(s), B(s)$ are arbitrary functions. And the boundary condition at $x = 0$ implies $A(s) = -B(s)$.

For $x > s$ an arbitrary solution of (2.3) is

$$G(x,s) = C(s) e^{ax} + D(s) e^{-ax}, \tag{2.5}$$

where $C(s), D(s)$ are arbitrary functions. And the boundary condition at $x = L$ implies $C(s) = -D(s)e^{-2aL}$.
To summarize the result, we have

$$G(x,s) = \begin{cases} -B(s)e^{ax} + B(s)e^{-ax} & \text{if } x < s, \\ -D(s)e^{-2aL}e^{ax} + D(s)e^{-ax} & \text{if } x > s. \end{cases} \tag{2.6}$$

First we use continuity in the Green's function at $x = s$ to find that

$$B(s) = \frac{D(s)\left(e^{-as} - e^{-2aL}e^{as}\right)}{\left(e^{-as} - e^{as}\right)}. \tag{2.7}$$

Next we use derivative jump i,e

$$G'(s_{+0}, s) - G'(s_{-0}, s) = -a^2,$$

which implies that

$$D(s) = \frac{-a - B(s)\left(e^{as} + e^{-as}\right)}{-\left(e^{-2aL}e^{as} - e^{-as}\right)}. \tag{2.8}$$

When we solve the equations (2.7) and (2.8), we get

$$\begin{aligned} D(s) &= \frac{1}{2}a\left[\frac{e^{aL}e^{-as} - e^{aL}e^{as}}{e^{-aL} - e^{aL}}\right], \\ B(s) &= \frac{1}{2}a\left[\frac{e^{aL}e^{-as} - e^{-aL}e^{as}}{e^{-aL} - e^{aL}}\right]. \end{aligned}$$

We substitute D(s), B(s) into equation (2.6), we have

$$G(x,s) = \begin{cases} -a\frac{[\cosh(a(L-x-s)) - \cosh(a(L-(s-x)))]}{2\sinh(aL)} & \text{if } x < s, \\ -a\frac{[\cosh(a(L-x-s)) - \cosh(a(L-(x-s)))]}{2\sinh(aL)} & \text{if } x > s. \end{cases}$$

Returning to our original problem we first consider the extended definition of Q. Evidently, Q is formally self-adjoint (Roach [20] p.154), and so $Q = Q^*$. Let v_t be a testing function for Q (in this case any function in the domain of Q will do), then extending the definition of Q we obtain:

$$\begin{aligned}(Qu_t, v_t) &= (u_t, Qv_t) = \int_0^L u_t(1 - a^{-2}\partial_x^2)v_t dx \\ &= \int_0^L v_t(1 - a^{-2}\partial_x^2)u_t\, dx - a^{-2}[g'(t)v_t'(L) - h'(t)v_t'(0)] \\ &= \int_0^L v_t\left(-u_x - \frac{3}{2}uu_x\right) + a^{-2}g'(t)\int_0^L v_t\delta'(x-L)\, dx \\ &\quad - a^{-2}h'(t)\int_0^L v_t\delta'(x)\, dx,\end{aligned}$$

which implies that

$$Qu_t = \left(-u_x - \frac{3}{2}uu_x\right) + a^{-2}g'(t)\delta'(x-L) - a^{-2}h'(t)\delta'(x),$$

we first write

$$Qu_{1t} = \left(-u_x - \frac{3}{2}uu_x\right),$$

we now need to find a function u_2 such that

$$Qu_{2t} = a^{-2}g'(t)\delta'(x-L) - a^{-2}h'(t)\delta'(x)$$

since

$$QG(x,s) = \delta(x-s).$$

We see that

$$Q\left\{-a^{-2}g'(t)\frac{\partial}{\partial s}G(x,L) + a^{-2}h'(t)\frac{\partial}{\partial s}G(x,0)\right\} \\ = a^{-2}g'(t)\delta'(x-L) - a^{-2}h'(t)\delta'(x).$$

This shows that

$$u_{2t} = g'(t)\frac{\sinh(ax)}{\sinh(aL)} + h'(t)\frac{\sinh(a(L-x))}{\sinh(aL)}.$$

Therefore, we get

$$u_t = \int_0^L G(x,s)\left(-u_s - \frac{3}{2}uu_s\right) ds + S(L-x)h' + S(x)g' \tag{2.9}$$

where

$$G(x,s) = -a\frac{[\cosh(a(L-x-s)) - \cosh(a(L-|x-s|))]}{2\sinh(aL)},$$

and

$$S(x) = \frac{\sinh(ax)}{\sinh(aL)}.$$

Here $G(x,s)$ is Green's function, clearly G(x,s) is continuous at $s = x$, continuously differentiable except at $s = x$ and $G(x,L) = G(x,0) = 0$ for all $x \in [0,L]$, the integral on the right-hand sides may be integrated by parts, thereby leading to

$$u_t = S(L-x)h' + S(x)g' + \int_0^L K(x,s)\left(u + \frac{3}{4}u^2\right) ds, \tag{2.10}$$

where

$$\begin{aligned} K(x,s) &= \frac{\partial G}{\partial s} \\ &= \frac{a^2}{2}\left\{\frac{\sinh(a(L-x-s)) + sign(x-s)sinh(a(L-|x-s|))}{\sinh(aL)}\right\} \end{aligned}$$

Now integrate equation (2.10) with respect to t to obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^L \left(u + \frac{3}{4}u^2\right) K(x,s)\, ds\, d\tau + S(L-x)[h(t) - h(0)] \\ &\quad + S(x)[g(t) - g(0)] + c_3. \end{aligned}$$

Use the initial condition (2.2) to obtain

$$c_3 = u_0(x),$$

and hence

$$\begin{aligned} u(x,t) &= u_0(x) + \int_0^t \int_0^L \left(u + \frac{3}{4}u^2\right) K(x,s)\, ds\, d\tau \\ &\quad + S(L-x)[h(t) - h(0)] + S(x)[g(t) - g(0)]. \end{aligned} \tag{2.11}$$

Definition 2.8. *The mapping Ψ is defined by*

$$\Psi(v)(x) \;=\; \int_0^L K(x,s)v(s)\,ds$$

for any $v \in C(0,L)$.

We use some properties of the mapping Ψ defined by the following lemma to prove the local existence and uniqueness result.

Lemma 2.9. *For given L there exists a constant c_1 depending only on L and constants D_k, $k = 0,1,\ldots$ depending on k and L such that*
(a) if $A_j = \sup\limits_{0\le x\le L}\left\{\int_0^x \left|k_x^{(j)}(x,s)\right|\,ds + \int_x^L \left|k_x^{(j)}(x,s)\right|\,ds\right\}$, $j \ge 0$, then

$$A_j \le (a^j)c_1, \tag{2.12}$$

(b) if $v \in C^k(0,L)$ for some $k \ge 0$ then $\Psi(v) \in C^{k+1}(0,L)$ and

$$\|\Psi(v))\|_{C^{k+1}} \le D_k \|v\|_{c^k}\,. \tag{2.13}$$

Here $k_x^{(j)}$ denotes the j^{th} partial derivative of K with respect to x, computed classically on the intervals $[0,x]$ and $[x,L]$.

Proof:

$$\text{If}\,|x| \le L \text{ then } |S(x)| \;=\; \left|\frac{\sinh(ax)}{\sinh(aL)}\right| \le \left|\frac{\sinh(aL)}{\sinh(aL)}\right| = 1,$$

$$K(x,s) \;=\; \frac{a^2}{2}\left\{\frac{\sinh(a(L-x-s)) + sign(x-s)sinh(a(L-|x-s|))}{\sinh(aL)}\right\}$$

is continuous in s except at $s = x$, where there is a jump discontinuity with

$$K(x,x^+) - K(x,x^-) \;=\; \frac{a^2}{2}\left(1+1-1-(-1)\right) = a^2 = 6 \tag{2.14}$$

and

$$\begin{aligned}|k(x,s)| \;&\le\; \left|a^2\frac{\sinh(a(L-x-s))}{2\sinh(aL)}\right| \\ &\quad + |sign(x-s)|\left|a^2\frac{(sinh(a(L-(x-s)))}{2\sinh(aL)}\right| \\ &\le\; \frac{a^2}{2} + \frac{a^2}{2} \le 2a^2\end{aligned}$$

which shows that A_0 is bounded by constant $12L$. Since for $|x| \leq L$, $|S'(x)|$ is bounded by a constant depending only on L, and $K_x^{'}$ is continuous, it is seen that A_1 is bounded by a constant which also depends only on L. And for $x \neq s$

$$K_x^{(m+2)}(x,s) = a^2 K_x^{(m)}(x,s) \tag{2.15}$$

for any $m \geq 0$, this we can prove by induction, which gives (a) for $j \geq 0$

$$\begin{aligned} i.e, A_j &= \sup_{0\leq x\leq L} \left\{ \int_0^x |K_x^{(j)}(x,s)|\,ds + \int_x^L |K_x^{(j)}(x,s)|\,ds \right\} \\ &= a^2 \sup_{0\leq x\leq L} \left\{ \int_0^x |K_x^{(j-2)}(x,s)|\,ds + \int_x^L |K_x^{(j-2)}(x,s)|\,ds \right\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq (a^j)c_1, \end{aligned}$$

where c_1 being any constant which bounds A_0 and A_1.

Let $v \in C(0,L)$ and denote $\Psi(v)$ by ϕ. Using part (a), we can prove

$$\begin{aligned} \|\Psi(v)(x)\| = \|\phi\| &= \left\| \int_0^L K(x,s)v(s)\,ds \right\| \\ &\leq \sup_{0\leq x\leq L} \int_0^L |K(x,s)|\,\|v(s)\|\ ds \\ &\leq c_1 \|v\|. \end{aligned}$$

Using (2.14) and (2.15), we show that

$$\begin{aligned} \phi(x) &= \int_0^L K(x,s)v(s)\,ds = \int_0^x K(x,s)v(s)\,ds + \int_x^L K(x,s)v(s)\,ds, \\ \phi'(x) &= k(x,x^+)v(x) + \int_0^x K_x^{(1)}(x,s)v(s)\,ds - k(x,x^-)v(x) \\ &\quad + \int_x^L K_x^{(1)}(x,s)v(s)\,ds \\ &= v(x)(6) + \int_0^L K_x^{(1)}(x,s)v(s)\,ds, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \phi''(x) &= 6v'(x) + \int_0^L K_x^{(2)}(x,s)v(s)\,ds \\ &= 6v'(x) + 6\int_0^L K_x^{(0)}(x,s)v(s)\,ds \\ \phi''(x) &= 6v'(x) + 6\phi(x), \end{aligned}$$

and that

$$\phi^{(m+2)}(x) \;=\; 6v^{(m+1)}(x) + 6\phi^{(m)}(x), \tag{2.17}$$

is true for $m \geq 0$. The equation (2.16) gives that $\phi \in C^1$ in the case $v \in C^0$ with

$$\begin{aligned} \|\phi\|_{C^1} &\leq \sup_{0\leq j\leq 1} \sup_{0\leq x\leq L} |\phi^{(1)}(x)| \\ &\leq (c_1+6)\,\|v\|\,, \end{aligned}$$

use the equation (2.16), and by induction method we can prove that

$$\|\Psi(v))\|_{C^{k+1}} \leq D_k \|v\|_{c^k}\,,$$

and if $v \in C^k(0,L)$ for some $k \geq 0$ then $\Psi(v) \in C^{k+1}(0,L)$.

We can now prove the local existence and global uniqueness of solution of the integral equation (2.11) corresponding to the boundary conditions (2.2).

Theorem 2.10. *If $u_0(x) \in C(0,L)$, $h, g \in C(0,T)$ for some $T, L > 0$, and u_0,h,g satisfy (2.2), then there exists a $T_0 = T_0(L, T, \|h\|, \|g\|, \|u_0\|) \leq T$ and a unique solution u in $C([0,T_0]; C(0,L))$ that satisfies (2.11). Moreover for any $T_1 \leq T$, there is at most one solution of (2.11) in $C([0,T_1]; C(0,L))$.*

Proof: First we denote $\mathtt{C} = C([0,T_0]; C(0,L))$ and write the integral equation (2.11) in the compact form

$$\mathbf{U} = A\mathbf{U}.$$

We will prove that the operator A defined by the right-hand side of (2.11) has a fixed point in $\mathtt{C}$ for suitably chosen T_0 by using the contraction-mapping theorem. We use Lemma 2.9 and the initial and boundary conditions, we will obtain that if $\mathbf{U} \in \mathtt{C}$ then $A\mathbf{U} \in \mathtt{C}$. If for any U_1, U_2 lies in the closed ball $\mathbb{B}_R$ of radius R about 0 in $\mathtt{C}$, then we have

$$\begin{aligned} |AU_1 - AU_2| &= \left|\int_0^t \int_0^L K(x,s)\left[\left(U_1 + \frac{3}{4}U_1^2\right) - \left(U_2 + \frac{3}{4}U_2^2\right)\right] ds\, d\tau\right| \\ &\leq \|U_1 - U_2\|_{\mathtt{C}} \left\{1 + \frac{3}{4}\left(\|U_1\|_c + \|U_2\|_c\right)\right\} \\ &\quad \int_0^t \int_0^L |K(x,s)|\, ds\, d\tau \\ &\leq \|U_1 - U_2\|_{\mathtt{C}} \left\{1 + \frac{3}{4}\left(\|U_1\|_c + \|U_2\|_c\right)\right\} c_1 T_0 \\ \|AU_1 - AU_2\|_{\mathtt{C}} &\leq \|U_1 - U_2\|_c\,(1+2R)c_1T_0 \\ &\equiv \Theta\,\|U_1 - U_2\|_{\mathtt{C}}\,, \end{aligned}$$

where $\Theta = (1+2R)c_1T_0$, if we prove $\Theta < 1$, then we use contraction mapping theorem and we can say A is a contraction mapping and A has a fixed point in $\mathtt{C}$. For $U \in \mathbb{B}_R$ and let B denote the terms in (2.11) involving initial- and boundary-values, say

$$B = u_0(x) + S(L-x)(h(t)-h(0)) + S(x)(g(t)-g(0)).$$

Now consider

$$\begin{aligned} \|B\|_{C([0,T];C(0,L))} &= \|u_0(x) + S(L-x)(h(t)-h(0)) + S(x)(g(t)-g(0))\|_C \\ &\leq \|u_0(x)\| + \|h\| + \|g\| = b(say), \end{aligned}$$

and hence

$$\begin{aligned} \|AU\|_{\mathtt{C}} &= \|AU - A0 + A0\|_{\mathtt{C}} \leq \Theta \|U\|_{\mathtt{C}} + \|B\|_{\mathtt{C}} \\ &\leq \Theta R + b. \end{aligned}$$

If we choose $R = 2b$ and $T_0 = T_0(b) = \frac{1}{2(1+2R)c_1}$, we get

$$\Theta = \frac{1}{2},$$

and

$$\|AU\|_{\mathtt{C}} \leq R.$$

Now we apply the contraction-mapping theorem to establish the local existence of a solution (2.11).
For Uniqueness, let U_1 and U_2 are two solutions of (2.11) in $C = C([0,T];C(0,L))$ satisfies (2.2) and let $V = U_1 - U_2 = A(U_1 - U_2)$. Now consider

$$\begin{aligned} \|V\|_{\mathtt{C}} &\leq \left\{1 + \frac{3}{4}\|U_1\|_{\mathtt{C}} + \frac{3}{4}\|U_2\|_{\mathtt{C}}\right\} \int_0^t \int_0^L |K(x,s)| \, \|U_1 - U_2\|_C \, ds \, d\tau \\ &\leq c_1 \left\{1 + \frac{3}{4}\|U_1\|_{\mathtt{C}} + \frac{3}{4}\|U_2\|_{\mathtt{C}}\right\} \int_0^t \|V\|_{\mathtt{C}} \, d\tau \\ &\leq c \int_0^t \|V\|_{\mathtt{C}} \, d\tau, \end{aligned}$$

for $0 \leq t \leq T_1$, where c depends on both $\|U_1\|_C$ and $\|U_2\|_C$. Apply Gronwall's inequality which implies that

$$\|V\|_{\mathtt{C}} = 0$$

and hence $U_1 = U_2$, which finishes the proof of the theorem. □

Theorem 2.11. *If $u_0(x) \in C^2(0,L)$, $h(t), g(t) \in C^1(0,T)$ for some $T, L > 0$, satisfy the initial and boundary conditions (2.2), then any solution u in $C([0,T_0]; C(0,L))$ of (2.11) lies in $C^1([0,T_0]; C^2(0,L))$ and is a classical solution of the initial and boundary value problem (2.1) on the interval $[0,T_0]$.*

Proof: From Lemma 2.9(b) we obtain that if U has continuous functions, then AU is differentiable with respect to t. Therefore U_t exists and is given by (2.10). Since $h'(t)$ and $g'(t) \in C^0(0,T)$, it shows that $U_t \in C$. The equation (2.11) can be rewritten as

$$u = u_0(x) + \int_0^t \Psi\left(u + \frac{3}{4}u^2\right) d\tau + (h(t) - h(0))S(L-x) + (g(t) - g(0))S(x), \tag{2.18}$$

where Ψ is defied in definition (2.8). Lemma 2.9 gives that the terms on the right hand sides of the equations are in $C^1([0,T_0]; C^1(0,L))$ which is equivalent to saying that u in $C^1([0,T_0]; C^1(0,L))$. We use the same argument once again which gives that u in $C^1([0,T_0]; C^2(0,L))$.
We consider (2.11) again to show that (2.2) is valid because $S(0) = 0$, $S(L) = 1$ and $K(0,s) = K(L,s) = 0$. Thus the solution of (2.11) satisfies (2.1) can be established by observing that the derivation leading from (2.1) to (2.11) is reversible if u in $C^1([0,T_0]; C^2(0,L))$. □

Theorem 2.12. *Let $u_0(x) \in C^l(0,L)$, $h(t)$, $g(t) \in C^k(0,T)$ for some $T, L > 0$, $l \geq 2, k \geq 1$ satisfy the initial and boundary conditions (2.2). Then any solution u in $C([0,T_0]; C(0,L))$ lies in $C^k([0,T_0]; C^l(0,L))$ and is the classical solution of the initial and boundary value problem (2.1) on the interval $[0,T_0]$.*

Proof: This results from a straightforward extension of the argument in the proof of the theorem 2.11. □

2.2.1 Other Boundary conditions

Different variables in the environment may have different boundary conditions according to certain physical problems. We use different boundary conditions for the BBM equation and we find the following results.

- case (i)

  Here we consider mixed Dirichlet-Neumann boundary conditions instead of Dirichlet's boundary conditions, .

$$\begin{cases} u_t + u_x + \frac{3}{2}uu_x - a^{-2}u_{xxt} = 0 \quad \text{if } 0 < x < L \\ u(x,0) = u_0(x), \\ u(0,t) = g(t), \\ u_x(L,t) = h(t). \end{cases} \tag{2.19}$$

We use same Green's function technique to find the solution of (2.19), which is as follows:

The required Green's function $G(x,s)$ is given as the solution of the equation

$$QG = (1 - a^{-2}\partial_x^2)G \;\; = \;\; \delta(x-s)$$

with the conditions

$$\begin{aligned} G(0) &= 0, \\ G_x(L) &= 0. \end{aligned}$$

We have already seen that $G(x,s)$ satisfies

$$(1 - a^{-2}\partial_x^2)G \;\; = \;\; 0 \tag{2.20}$$

everywhere except at $x = s$. For $x < s$, an arbitrary solution of (2.20) is

$$G(x,s) = A(s)e^{ax} + B(s)e^{-ax},$$

where $A(s), B(s)$ are arbitrary functions. And the boundary condition at $x = 0$ implies $A(s) = -B(s)$. For $x > s$, an arbitrary solution of (2.20) is

$$G(x,s) = C(s)e^{ax} + D(s)e^{-ax}, \tag{2.21}$$

where $C(s), D(s)$ are arbitrary functions. And the boundary condition at $x = L$ implies $C(s) = D(s)e^{-2aL}$.

To summarize the result, we have

$$G(x,s) = \begin{cases} -B(s)e^{ax} + B(s)e^{-ax} & \text{if } x < s, \\ D(s)e^{-2aL}e^{ax} + D(s)e^{-ax} & \text{if } x > s. \end{cases} \tag{2.22}$$

We use continuity in the Green's function at $x = s$ to find

$$B(s) = \frac{D(s)\left(e^{-as} + e^{-2aL}e^{as}\right)}{\left(e^{-as} - e^{as}\right)}. \tag{2.23}$$

Further we use derivative jump i,e

$$G'(s_{+0}, s) - G'(s_{-0}, s) = -a^2,$$

which implies that

$$D(s) = \frac{-a - B(s)\left(e^{as} + e^{-as}\right)}{\left(e^{-2aL}e^{as} - e^{-as}\right)}. \tag{2.24}$$

We solve the equations (2.23) and (2.24) to get

$$\begin{aligned} D(s) &= \frac{1}{2}a\left[\frac{e^{aL}e^{as} - e^{aL}e^{-as}}{e^{aL} + e^{-aL}}\right], \\ B(s) &= \frac{-1}{2}a\left[\frac{e^{aL}e^{-as} + e^{-aL}e^{as}}{e^{-aL} + e^{aL}}\right]. \end{aligned}$$

We substitute D(s), B(s) into equation (2.22), we have

$$G(x,s) = \begin{cases} a\frac{[-\sinh(a(L-s-x)) + \sinh(a(L-(s-x)))]}{2\cosh(aL)} & \text{if } x < s, \\ a\frac{[-\sinh(a(L-s-x)) + \sinh(a(L-(x-s)))]}{2\cosh(aL)} & \text{if } x > s. \end{cases}$$

We have already seen that Q is formally self-adjoint. Let v be a testing function for Q, then

$$\begin{aligned} (Qu_t, v) &= (u_t, Qv) = \int_0^L u_t(1 - a^{-2}\partial_x^2)v dx \\ &= \int_0^L v(1 - a^{-2}\partial_x^2)u_t \, dx - a^{-2}[-h'(t)v(L) - g'(t)v'(0)] \\ &= \int_0^L v\left(-u_x - \frac{3}{2}uu_x\right) + a^{-2}h'(t)\int_0^L v(x)\delta(x-L)\, dx \\ &\quad - a^{-2}g'(t)\int_0^L v(x)\delta'(x)\, dx. \end{aligned}$$

Consequently, we see that

$$Qu_t = \left(-u_x - \frac{3}{2}uu_x\right) + a^{-2}h'(t)\delta(x-L) - a^{-2}g'(t)\delta'(x).$$

From the definition of G, we observe that

$$\begin{aligned} &Q\left\{a^{-2}h'(t)G(x,L) + a^{-2}g'(t)\frac{\partial}{\partial s}G(x,0)\right\} \\ &\qquad = a^{-2}h'(t)\delta(x-L) - a^{-2}g'(t)\delta'(x). \end{aligned}$$

This shows that

$$\begin{aligned} u_t &= \int_0^L G(x,s)\left(-u_s - \frac{3}{2}uu_s\right) ds + \frac{1}{a}\frac{\sinh(ax)}{\cosh(aL)}[h'(t)] \\ &+ \frac{\cosh(a(L-x))}{\cosh(aL)}[g'(t)], \end{aligned} \tag{2.25}$$

where

$$G(x,s) = a\frac{[-\sinh(a(L-x-s)) + \sinh(a(L-|x-s|))]}{2\cosh(aL)}.$$

Here $G(x,s)$ is Green's function, clearly G(x,s) is continuous at $s = x$, continuously differentiable except at $s = x$ and $G(x,L) = G(x,0) = 0$ for all $x \in [0,L]$, the integral on the right-hand sides may be integrated by parts, thereby leading to

$$\begin{aligned} u_t &= \int_0^L K(x,s)\left(u + \frac{3}{4}u^2\right) ds + \frac{1}{a}\frac{\sinh(ax)}{\cosh(aL)}[h'(t)] \\ &+ \frac{\cosh(a(L-x))}{\cosh(aL)}[g'(t)], \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} K(x,s) &= \frac{\partial G}{\partial s} \\ &= a^2 \left\{\frac{\cosh(a(L-x-s)) + sign(x-s)cosh(a(L-|x-s|))}{2\cosh(aL)}\right\}. \end{aligned}$$

Now integrate equation (2.26) with respect to t and use initial condition to obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^L \left(u + \frac{3}{4}u^2\right) K(x,s)\, ds\, d\tau + \frac{1}{a}\frac{\sinh(ax)}{\cosh(aL)}[h(t) - h(0)] \\ &+ \frac{\cosh(a(L-x))}{\cosh(aL)}[g(t) - g(0)] + u_0(x). \end{aligned} \tag{2.27}$$

It is easy to check that $K(x,s)$ satisfies the Lemma 2.9. Since $K(x,s)$ satisfies the Lemma 2.9, we use the same technique here as in section (2.2), we end up with the following theorem:

Theorem 2.13. *If $u_0(x) \in C(0,L)$, $h, g \in C(0,T)$ for some $T, L > 0$, and u_0,h,g satisfies (2.19), then there exists a $T_0 = T_0(L, T, \|h\|, \|g\|, \|u_0\|) \leq T$ and a unique solution u in $C([0,T_0]; C(0,L))$ that satisfies (2.27).*

- case (ii)

Here we consider the BBM equation with two Neumann boundary conditions which is as follows:

$$\begin{cases} u_t + u_x + \frac{3}{2}uu_x - a^{-2}u_{xxt} = 0 \quad \text{if } 0 < x < L \\ u(x,0) = u_0(x), \\ u_x(0,t) = h(t), \\ u_x(L,t) = g(t). \end{cases} \tag{2.28}$$

Theorem 2.14. *If $u_0(x) \in C(0,L)$, $h, g \in C(0,T)$ for some $T, L > 0$, and u_0,h,g satisfies (2.28), then there exists a $T_0 = T_0(L, T, \|h\|, \|g\|, \|u_0\|) \le T$ and a unique solution u in $C([0,T_0]; C(0,L))$.*

Proof: In this case

$$\begin{aligned} u_t &= \int_0^L G(x,s)\left(-u_s - \frac{3}{2}uu_s\right) ds \\ &\quad + \frac{1}{a}g'(t)\frac{\cosh(ax)}{\sinh(aL)} - \frac{1}{a}h'(t)\frac{\cosh(a(L-x))}{\sinh(aL)}], \end{aligned}$$

the Green's function

$$G(x,s) = a\frac{[\cosh(a(L-x-s)) + \cosh(a(L-|x-s|))]}{2\sinh(aL)}.$$

The integral on the right-hand sides may be integrated by parts, to get

$$\begin{aligned} u_t &= \int_0^L K(x,s)\left(u + \frac{3}{4}u^2\right) ds \\ &\quad + \frac{1}{a}g'(t)\frac{\cosh(ax)}{\sinh(aL)} - \frac{1}{a}h'(t)\frac{\cosh(a(L-x))}{\sinh(aL)}, \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} K(x,s) &= \frac{\partial G}{\partial s} \\ &= a^2 \left\{\frac{sign(x-s)sinh(a(L-|x-s|)) - \sinh(a(L-x-s))}{2\sinh(aL)}\right\}. \end{aligned}$$

Now integrate equation (2.29) with respect to t and use initial condition (2.28) to obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^L \left(u + \frac{3}{4}u^2\right) K(x,s)\, ds\, d\tau + u_0(x) \\ &\quad + \frac{1}{a}\frac{\cosh(ax)}{\sinh(aL)}[g(t) - g(0)] - \frac{1}{a}\frac{\cosh(a(L-x))}{\sinh(aL)}[h(t) - h(0)]. \end{aligned} \tag{2.30}$$

We also find that $K(x,s)$ satisfies the Lemma 2.9. Since $K(x,s)$ satisfies the Lemma 2.9, we use the same technique here as in section (2.2) to prove the remaining part of the proof. □

- case (iii)

  Again we consider here mixed Dirichlet-Neumann boundary conditions at different end points.

$$\begin{cases} u_t + u_x + \frac{3}{2}uu_x - a^{-2}u_{xxt} = 0 \quad \text{if } 0 < x < L \\ u(x,0) = u_0(x), \\ u_x(0,t) = g(t), \\ u(L,t) = h(t). \end{cases} \tag{2.31}$$

We use same procedure here as in previous cases we can find that

$$\begin{aligned} u_t &= \int_0^L G(x,s)\left(-u_s - \frac{3}{2}uu_s\right) ds + \frac{\cosh(ax)}{\cosh(aL)}[h'(t)] \\ &\quad - \frac{1}{a}\frac{\sinh(a(L-x))}{\cosh(aL)}[g'(t)], \end{aligned} \tag{2.32}$$

where the Green's function

$$G(x,s) = a\frac{[\sinh(a(L-x-s)) + \sinh(a(L-|x-s|))]}{2\cosh(aL)}.$$

After a formal integration by parts, becomes

$$\begin{aligned} u_t &= \int_0^L K(x,s)\left(u + \frac{3}{4}u^2\right) ds + \frac{\cosh(ax)}{\cosh(aL)}[h'(t)] \\ &\quad - \frac{1}{a}\frac{\sinh(a(L-x))}{\cosh(aL)}[g'(t)], \end{aligned} \tag{2.33}$$

where

$$K(x,s) = a^2\left\{\frac{sign(x-s)\cosh(a(L-|x-s|)) - \cosh(a(L-x-s))}{2\cosh(aL)}\right\}.$$

Now integrate equation (2.33) with respect to t and use initial condition to find that

$$\begin{aligned} u(x,t) &= \int_0^t\int_0^L \left(u + \frac{3}{4}u^2\right) G(x,s)\, ds\, d\tau + u_0(x) \qquad (2.34) \\ &\quad + [h(t)-h(0)]\frac{\cosh(ax)}{\cosh(aL)} - \frac{1}{a}[g(t)-g(0)]\frac{\sinh(a(L-x))}{\cosh(aL)}. \end{aligned}$$

Since $K(x,s)$ satisfies the Lemma 2.9, we use the same technique here as in section (2.2) to confirm the following theorem:

Theorem 2.15. *There exists a* $T_0 = T_0(L, T, \|h\|, \|g\|, \|u_0\|) \leq T$ *and the unique solution* $u(x,t)$ *in (2.34) satisfies (2.31) point-wise in* $C([0,T_0]; C(0,L))$ *and is a classical solution of the initial and boundary value problem (2.31) on the interval* $[0,T_0]$ *if* $u_0(x) \in C(0,L)$*,* $h, g \in C(0,T)$ *for some* $T, L > 0$*.*

2.2.2 Global existence

Further we consider the global existence of the boundary wave problem. For global existence, we prefer our works in sobolev space.
In order to simplify the analysis, we assume that all the coefficients in equation (2.1) must to be equal to one. We use the Sobolev space definitions and as usual the same technique here as in section (2.2), we have the following theorem:

Theorem 2.16. *If* $u_0(x) \in H^1(0,L)$*, then there exists a unique global solution u(x,t) in* $C^\infty([0,\infty]; H^1(0,L))$ *which satisfies (2.35)*

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0 \quad \text{if } 0 < x < L \\ u(x,0) = u_0(x), \\ u(0,t) = 0, \\ u(L,t) = 0. \end{cases} \tag{2.35}$$

Proof: Local existence: We use the same ideas as we have done in equation (2.2) to find the solution u of equation (2.35). And it is easy to check that

$$u_t = \int_0^L (-u_s - uu_s)\, G(x,s)\, ds, \tag{2.36}$$

where

$$G(x,s) = -\frac{[\cosh(L-x-s) - \cosh(L-|x-s|)]}{2\sinh(L)}.$$

We will prove that the solution u(x,t) of (2.35) for $u_0(x) \in H^1(0,L)$, is a global solution in $C^\infty([0,\infty]; H^1(0,L))$. We prove the local existence result by the following arguments:
We can rewrite $u(x,t)$ as

$$u(x,t) = \int_0^t \int_0^L \left(u + \frac{1}{2}u^2\right) K(x,s)\, ds\, d\tau + u_0(x), \tag{2.37}$$

where

$$K(x,s) = \frac{1}{2}\left\{\frac{\sinh(L-x-s) + sign(x-s)sinh(L-|x-s|)}{\sinh(L)}\right\}. \quad (2.38)$$

Write integral equation (2.37) in the compact form

$$U = AU. \quad (2.39)$$

We will show that the operator A defined by the right-hand side of (2.37) has a fixed point in $\mathcal{C} = C([0,t_0]; H^1(0,L))$ for suitably chosen t_0 by using the contraction mapping theorem. Suppose now that both U and W lie in the closed ball B_R of radius R about o in $\mathcal{C}$, the we obtain that

$$\|AW(x,t) - AU(x,t)\|_{H^1} \le \int_0^t \left\| \int_0^L K(x,s)\left[W + \frac{W^2}{2} - U - \frac{U^2}{2}\right] ds \right\|_{H^1} dt.$$

Next if we use the definition of $K(x,s)$, Minkowski's integral inequality and Cauchy-Schwarz inequality, then

$$\begin{aligned}\|AW(x,t) - AU(x,t)\|_{H^1} &\le t\int_0^L \|K(x,s)\|_{H^1}\left[W + \frac{W^2}{2} - U - \frac{U^2}{2}\right] ds \\ &\le t\,\|K(x,s)\|_{H^1}\,\|W-U\|_{L^2}\,(1+2R) \\ &\le tc_K\,\|W-U\|_{H^1}\,(1+2R),\end{aligned}$$

where $\|K(x,s)\|_{H^1} = c_K$. Let us take the supremum $t \in [0,t_0]$

$$\|AW(x,t) - AU(x,t)\|_{\mathcal{C}} \le t_0 c_K\,\|W-U\|_{\mathcal{C}}\,(1+2R),$$

it gives us if $U \in \mathcal{C}$ then $AU \in \mathcal{C}$. Now let $\|u_0\|_{\mathcal{C}} \le d$, then

$$\begin{aligned}\|AU\|_{\mathcal{C}} &\le \|AU - A0\|_{\mathcal{C}} + d \\ &\le t_0 c_K(1+2R)\,\|U\|_{\mathcal{C}} + d \\ &= \Theta R + d,\end{aligned}$$

where $\Theta = t_0 c_K(1+2R)$.
If we choose $R = 2d$ and $t_0 = \frac{0.5}{c_K(1+2R)}$, then $\Theta = 0.5 < 1$, and

$$\|AU\|_{\mathcal{C}} \le R.$$

The contraction-mapping theorem can be applied to establish the local existence result.

Global existence:
Further for the global existence we use the following argument:
If $u \in C([0,t_0];H^1(0,L))$, then we get that $u_t \in C([0,t_0];H^1(0,L))$. And

$$u_{xxt} = u_t + u_x + uu_x.$$

Since $u \in H^1(0,L) \Rightarrow u_x \in L^2(0,L)$ and $uu_x \in L^2(0,L)$, which shows that $u_{xxt} \in L^2(0,L) \Rightarrow u_{xt} \in H^1(0,L)$.
Multiply the BBM equation by u, we have

$$uu_t - uu_{xxt} + uu_x + u^2u_x = 0.$$

If we integrate the above equation between $x = 0$ to $x = L$, then we have

$$\begin{aligned}
\int_0^L uu_t\,dx - \int_0^L uu_{xxt}\,dx &= -\int_0^L uu_x\,dx - \frac{1}{3}\int_0^L \frac{\partial}{\partial x}(u^3)\,dx \\
\int_0^L \frac{1}{2}\frac{d}{dt}\left[u^2\right]\,dx - [uu_{xt}]_0^L + \int_0^L u_xu_{xt}\,dx &= \frac{-1}{2}\int_0^L \frac{\partial}{\partial x}(u^2)\,dx \\
&\quad - \frac{1}{3}\left[u^3\right]_0^L.
\end{aligned}$$

If we use Riemann-Lebesgue lemma and boundary conditions, we can prove that functions in $H^1(0,L)$ are vanish at boundary. And hence we have

$$\frac{1}{2}\int_0^L \frac{d}{dt}\left[u^2 + u_x^2\right]\,dx = 0.$$

Since each of the terms u, u_t, u_x, u_{xt} is in $\mathcal{C}([0,T_0];L^2(0,L))$, the dominated convergence theorem gives us

$$\frac{d}{dt}\int_0^L \left[u^2 + u_x^2\right]\,dx = 0.$$

It yields that the H^1-norm is constant on the time interval $[0,T_0]$. Consequently, the solution may be continued to any interval $[0,T]$ by repeating the contraction argument a sufficient number of times. That is we can conclude that $u \in C([0,\infty];H^1(0,L))$. Also that u can be differentiated any number of times with respect to t, and therefore $u \in C^\infty([0,\infty];H^1(0,L))$. $\square$

2.3 BBM equation with uneven bottom

When we consider the problem of a solitary wave propagating from a channel with uneven bottom, we have already seen that in section (1.5), there are two different models. First we interested in the following model $u_t + C(x)u_x + uu_x - u_{xxt} = 0$ and we have the following results.

2.3.1 Local existence of solutions in $\mathbb{R}$

In this subsection we consider the long wave equation with unbounded domain $\mathbb{R}$ of the form

$$u_t + C(x)u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{R} \tag{2.40}$$

with the initial condition

$$u(x,0) \;=\; u_0(x).$$

In the following we investigate well-posedness of (2.40).

Theorem 2.17. *If $u_0 = u(x,0) \in H^1(\mathbb{R})$, $C(x) \in H^1(\mathbb{R})$ then there exists a t_0 which depends on u_0 and $C(x)$ such that the equation $u_t + C(x)u_x + uu_x - u_{xxt} = 0$, $x \in \mathbb{R}$ has a solution $u \in C([0,t_0]; H^1(\mathbb{R}))$.*

Proof: We rewrite the BBM equation as

$$(1 - \partial_x^2)u_t = -[C(x)u_x + uu_x]. \tag{2.41}$$

The formal solution of (2.41) is

$$u_t = -\int_{-\infty}^{\infty} G(x-\xi)\left(C(\xi)u_\xi + uu_\xi\right)\, d\xi,$$

where $G(x) = \frac{1}{2}e^{-|x|}$ is Green's function.
Using integration by parts we rewrite the above equation as

$$u_t = \int_{-\infty}^{\infty} K(x-\xi)\left(C(\xi)u + \frac{1}{2}u^2\right)\, d\xi + \int_{-\infty}^{\infty} G(x-\xi)uC_\xi(\xi)\, d\xi,$$

where $K(x) = sign(x)\frac{e^{-|x|}}{2}$.
Using convolution we write u_t as

$$u_t \;=\; K * (Cu + \frac{1}{2}u^2) + G * (C_x u)\,. \tag{2.42}$$

If we integrate (2.42) with respect to t, then we get

$$u(x,t) = u_0(x) + \int_0^t \left[K * (Cu + \frac{1}{2}u^2) + G * (C_x u) \right] d\tau. \tag{2.43}$$

Next we use a fixed point theorem to prove the existence of solution in the sufficiently small interval. First we symbolize the equation (2.43) by

$$U = AU.$$

We have to prove that A is a contraction mapping in a closed ball $B_R \subset \mathcal{C} = C([0,t_0]; H^1(\mathbb{R}))$, where sufficiently small t_0.
Now consider

$$\begin{aligned} \|AU(t)\|_{H^1} &\leq \|u_0\|_{H^1} + \int_0^t \left\| K * (CU + \frac{1}{2}U^2) + G * (C_x U) \right\|_{H^1} d\tau \\ &\leq \|u_0\|_{H^1} + \int_0^t \left\| CU + \frac{1}{2}U^2) \right\|_{L^2} d\tau + \int_0^t \|C_x U\|_{H^{-1}} d\tau. \end{aligned}$$

And hence

$$\begin{aligned} \|AU_1(t) - AU_2(t)\|_{H^1} &\leq \int_0^t \left\| C(U_1 - U_2) + \frac{1}{2}(U_1^2 - U_2^2) \right\|_{L^2} d\tau \\ &\quad + \int_0^t \|C_x(U_1 - U_2)\|_{H^{-1}} d\tau \\ &\leq t\,\|U_1 - U_2\|_{L^2} \left[\left| C + \frac{1}{2}(U_1 + U_2) \right|_\infty + \|C_x\|_{L^2} \right] \\ &\leq t\,\|U_1 - U_2\|_{H^1} \left[|C|_\infty + R + \|C_x\|_{L^2} \right]. \end{aligned}$$

Now take the supremum $t \in [0,t_0]$ on both sides, we have

$$\|AU_1(t) - AU_2(t)\|_{\mathcal{C}} \leq t_0 \{ |C|_\infty + R + \|C_x\|_{L^2} \} \|U_1 - U_2\|_{\mathcal{C}}.$$

From the above equality it can be conformed that the operator A is a continuous mapping of the space $\mathcal{C}$ into itself. We can show that the mapping of the ball $\|U\|_{\mathcal{C}} \leq R$ satisfies Lipschitz condition with Lipschitz constant

$\Theta < 1$.
we let $\|u_0\|_{\mathcal{C}} \leq d$, then

$$\begin{aligned} \|AU\|_{\mathcal{C}} &\leq \|AU - A0\|_{\mathcal{C}} + d \\ &\leq t_0 \left\{ |C|_{\infty} + R + \|C_x\|_{L^2} \right\} \|U\|_{\mathcal{C}} + d \\ = \Theta R + d, \end{aligned}$$

where $\Theta = t_0 \left\{ |C|_{\infty} + R + \|C_x\|_{L^2} \right\}$.
If we choose $R = 2d$ and $t_0 = \frac{0.5}{\left\{ |C|_{\infty} + R + \|C_x\|_{L^2} \right\}}$, then $\Theta = 0.5 < 1$, and

$$\|AU\|_{\mathcal{C}} \leq \frac{1}{2}R + \frac{R}{2} = R.$$

Which shows that A is a contraction mapping on B. Therefore A has a fixed point u in the ball B_R. □

2.3.2 Extension to global existence

In 1971 Benjamin, Bona and Mahony [3] proved Global existence result for the case $C(x) =$ constant. They proved the following theorem;

Theorem 2.18. *Let $g(x)$ satisfy the conditions*

(i) $\int_{-\infty}^{\infty} (g^2 + g'^2)\, dx < \infty$,

(ii) $g \in C^2(\mathbb{R})$.

Then the partial differential equation $u_t + u_x + uu_x - u_{xxt} = 0$ has solution $C^{\infty}([0, \infty]; C^2(\mathbb{R}))$ (see Benjamin,Bona and Mahony [3]) which satisfies $u(x, 0) = g(x)$.

Now for the case $C(x) \neq$ constant, we examine the global existence solution of (2.40). If $u \in C([0, t_0]; H^1)$, then from (2.42) we observe that $u_t \in C([0, t_0]; H^1)$. And

$$u_{xxt} = u_t + C(x)u_x + uu_x.$$

Since $u \in H^1(\mathbb{R}) \Rightarrow u_x \in L^2(\mathbb{R})$ and $uu_x \in L^2(\mathbb{R})$ and if $C(x) \in H^1(\mathbb{R}) \Rightarrow Cu_x \in L^2(\mathbb{R})$. Which shows that $u_{xxt} \in L^2(\mathbb{R}) \Rightarrow u_{xt} \in H^1(\mathbb{R})$.
Multiply the BBM equation by u, we have

$$uu_t - uu_{xxt} + uC(x)u_x + u^2u_x = 0.$$

We integrate the above equation between $x = -R$ to $x = R$ to obtain

$$\int_{-R}^{R} uu_t\,dx - \int_{-R}^{R} uu_{xxt}\,dx = -\int_{-R}^{R} C(x)uu_x\,dx \quad - \quad \frac{1}{3}\int_{-R}^{R} \frac{\partial}{\partial x}(u^3)\,dx$$

$$\int_{-R}^{R} \frac{1}{2}\frac{d}{dt}\left[u^2\right]\,dx - \left[uu_{xt}\right]_{-R}^{R} + \int_{-R}^{R} u_x u_{xt}\,dx \quad = \quad \frac{-1}{2}\int_{-R}^{R} C(x)\frac{\partial}{\partial x}(u^2)\,dx - \frac{1}{3}\left[u^3\right]_{-R}^{R}.$$

We use the fact that functions in $H^1(\mathbb{R})$ are vanish at infinity. We can prove this fact by using Riemann-Lebesgue lemma. And we know that the terms u, u_t, u_x, u_{xt} is in $C([0,t_0];L^2)$, if we use dominated convergence theorem, we get the result for $R \to \infty$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(u^2 + u_x^2\right)\,dx \quad = \quad -\frac{1}{2}\int_{-\infty}^{\infty} C(x)\frac{\partial}{\partial x}(u^2)\,dx$$

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} \left(u^2 + u_x^2\right)\,dx \quad &= \quad \int_{-\infty}^{\infty} C'(x)u^2\,dx \\
&\le \quad |C'|_\infty \left\|u\right\|_{L^2}^2 \\
&\le \quad |C'|_\infty \left\|u\right\|_{H^1}^2 .
\end{aligned}$$

And hence

$$\frac{d}{dt}\left\|u(.,t)\right\|_{H^1}^2 \le |C'|_\infty \left\|u(.,t)\right\|_{H^1}^2 .$$

Further if we use Gronwall's inequality (differential form), then we can conclude that $u \in C([0,t_0];H^1(\mathbb{R}))$ for any $t_0 > 0$.

Special case:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(u^2 + u_x^2\right)\,dx \quad &= \quad \int_{-\infty}^{\infty} C'(x)\frac{u^2}{2}\,dx \\
&\le \quad 0 \text{ if } C'(x) \le 0.
\end{aligned}$$

It shows that

$$\frac{dE}{dt} \leq 0 \text{ if } h'(x) \leq 0 \tag{2.44}$$

$$\text{since } C(x) = \sqrt{gh(x)}, \text{ and } C'(x) = \frac{\sqrt{g}}{2\sqrt{h(x)}}h'(x) \leq 0.$$

And accordingly we conclude the following theorem from the above result:

Theorem 2.19. *If $u_0 = u(x,0) \in H^1(\mathbb{R})$, $C(x) \in H^1(\mathbb{R})$ and $C'(x) \leq 0$, then the equation $u_t + C(x)u_x + uu_x - u_{xxt} = 0$, $x \in \mathbb{R}$ has a solution $u \in C_b([0,\infty]; H^1(\mathbb{R}))$.*

2.3.3 Inclusion of a forcing term

We generalize the BBM equation as

$$u_t + C(x)u_x + uu_x - u_{xxt} = f(x,t), \ x \in \mathbb{R} \tag{2.45}$$

where the function $f(x,t)$ is a function may represent some kind of forcing action on the physical system, whose response evolving from a given initial state is described by the solution u. We assume that $f(x,t) \in C([0,t_0]; H^1(\mathbb{R}))$, for a given finite $t_0 > 0$.
The formal solution of (2.45) is

$$\begin{aligned} u(x,t) &= u_0(x) + \int_0^t \left[k * \left(Cu + \frac{1}{2}u^2 \right) + G * (C_x u) \right] d\tau \\ &+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-\xi|} f(\xi,t)\, d\xi\, d\tau. \end{aligned}$$

We rewrite $u(x,t)$ as

$$u = Au + Zf, \tag{2.46}$$

where

$$Zf = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-\xi|} f(\xi,t)\, d\xi\, d\tau.$$

Consider

$$|Zf| = \left| \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-\xi|} f(\xi,t)\, d\xi\, d\tau \right| \leq \|f\|_C \frac{1}{2} t.$$

Take supremum $x \in \mathbb{R}$, $t \in [0, t_0]$

$$\|Zf\|_{\mathcal{C}} \leq \|f\|_{\mathcal{C}}\, t_0.$$

Again making the assumption in theorem (2.17) and adapting the same argument in proof of the theorem (2.17), we can find sufficient conditions for the transformation $Zf + A$ to be contractive mapping of the ball B_R.

$$\|Au + Zf\|_{\mathcal{C}} \leq t_0 \left\{|C|_\infty + R + \|C_x\|_{L^2}\right\} \|u\|_{\mathcal{C}} + d + t_0 \|f\|_{\mathcal{C}}\,.$$

Hence with the suitable choice of t_0 and R we can prove $A + Zf$ is a contraction mapping. Hence it proves the local existence of the solution for the equation (2.45).

2.3.4 Equation with boundary conditions

In this subsection we consider the BBM equation with initial and boundary conditions.

$$u_t + C(x)u_x + uu_x - u_{xxt} = 0, \quad 0 < x < L. \tag{2.47}$$

And the initial and boundary conditions for (2.47) are

$$\begin{aligned} u(x,0) &= u_0(x), \\ u(0,t) &= f(t), \\ u(L,t) &= g(t). \end{aligned} \tag{2.48}$$

Again we will investigate here the regularity of the solution corresponding to the given initial and boundary value problem.
The formal solution of the above partial differential equations (2.47) is

$$u_t = S(L-x)f'(t) + S(x)g'(t) + \int_0^L G(x,s)[-C(s)u_s - uu_s]ds, \tag{2.49}$$

where

$$G(x,s) = -\frac{[\cosh(L-x-s) - \cosh(L-|x-s|)]}{2\sinh(L)},$$

and

$$S(x) = \frac{\sinh(x)}{\sinh(L)}.$$

Here $G(x,s)$ is green function, clearly G(x,s) is continuous at $s = x$ and continuously differentiable except at $s = x$ and $G(x,L) = G(x,0) = 0$ for all $x \in [0,L]$. After integration by parts, (2.49) becomes

$$\begin{aligned} u_t &= S(L-x)f'(t) + S(x)g'(t) + \int_0^L K(x,s)\left(Cu + \frac{u^2}{2}\right) ds \\ &+ \int_0^L C_s u G(x,s)\, ds, \end{aligned} \tag{2.50}$$

where

$$K(x,s) = \frac{1}{2}\left[S(L-x-s) + sign(x-s)S(L-|(x-s)|)\right].$$

Now integrate equation (2.50) with respect to t, we will obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^L \left(Cu + \frac{u^2}{2}\right) K(x,s)\, ds\, dr + \int_0^t \int_0^L C_s u G(x,s)\, ds\, dr \\ &+ S(x)[g(t) - g(0)] + S(L-x)[h(t) - h(0)] + u(x,o). \end{aligned}$$

Now use the initial condition (2.48), we will obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^L \left(Cu + \frac{1}{2}u^2\right) K(x,s)\, ds\, dr + S(x)[g(t) - g(0)] \\ &+ S(L-x)[h(t) - h(0)] + \int_0^t \int_0^L C_s u G(x,s)\, ds\, dr + u_0(x). \end{aligned} \tag{2.51}$$

We are nothing new to prove here. We use the same procedure as we done for the equation (2.1) and we summarize the following results.

Theorem 2.20. *If $u_0(x) \in C(0,L)$, $f, g \in C(0,T)$, $C(x) \in C^1(0,L)$, for some $T, L > 0$ and satisfies the initial and boundary conditions (2.48) then there exists a $T_0 = T_0(L,T,\|f\|,\|g\|,\|u_0\|) \leq T$ and a unique solution u of (2.51) in $C(0,T_0;C(0,L))$ that satisfies (2.47). Moreover for any $T_1 \leq T$, there is at most one solution of (2.47) in $C([0,T_1];C(0,L))$.*

Theorem 2.21. *Let $u_0(x) \in C^2(0,L)$, $f(t), g(t) \in C^1(0,T)$, $C(x) \in C^1(0,L)$ for some $T, L > 0$,satisfy the initial and boundary conditions (2.48).Then any solution u in $C(0,T_0;C(0,L))$ of (2.51) lies in $C^1([0,T_0];C^2(0,L))$ and is a classical solution of the initial and boundary value problem (2.47) on the interval $[0,T_0]$.*

Theorem 2.22. *Let $u_0(x) \in C^l(0,L)$, $h(t)$, $g(t) \in C^k(0,T)$, $C(x) \in C^{l-1}(0,L)$ for some $T, L > 0$, $l \geq 2, k \geq 1$ satisfy the conditions (2.48). Then any solution u in $C([0,T_0];C(0,L))$ lies in $C^k([0,T_0];C^l(0,L))$ and is the classical solution of the initial and boundary value problem (2.47) on the interval $[0,T_0]$.*

2.3.5 Local existence of solutions in $\mathbb{R}$ for another model

Now we interested in the following model $u_t+(C(x)u)_x+uu_x-u_{xxt}=0$ and we have the following results.

Theorem 2.23. *If $u_0=u(x,0)\in H^1(\mathbb{R})$ and $C(x)$ is a bounded continuous function then there exists a t_0 which depends on u_0 and $C(x)$ such that the equation $u_t+(C(x)u)_x+uu_x-u_{xxt}=0$, $x\in\mathbb{R}$ has a solution $u\in C([0,t_0],H^1)$.*

Proof: The given BBM equation can be written as

$$(1-\partial_x^2)u_t=-[(C(x)u)_x+uu_x]. \tag{2.52}$$

The formal solution of (2.52) is

$$u_t=-\frac{1}{2}\int_{-\infty}^{\infty} G(x-\xi)\left((C(\xi)u)_\xi+uu_\xi\right)\,d\xi,$$

where $G(x)=\frac{1}{2}e^{-|x|}$ is Green's function.
When we use integration by parts, we can rewrite the above equation as

$$u_t \;=\; \int_{-\infty}^{\infty} K(x-\xi)\left(C(\xi)u+\frac{1}{2}u^2\right)\,d\xi,$$

where $K(x)=sign(x)\frac{e^{-|x|}}{2}$.
Using convolution we rewrite u_t as

$$u_t \;=\; K*(Cu+\frac{1}{2}u^2), \tag{2.53}$$

If we integrate (2.53) with respect to t, we get

$$u(x,t)=u_0(x)+\int_0^t K*(Cu+\frac{1}{2}u^2)\,d\tau. \tag{2.54}$$

To prove the existence of solution in the sufficiently small interval, we use a fixed point theorem . First we notate the equation (2.54) by

$$u=Au.$$

Next we prove that A is a contraction mapping in a closed ball $B_R \subset \mathcal{C} = C([0,t_0];H^1)$, where sufficiently small t_0.
For this consider

$$\begin{aligned} \|Au(t)\|_{H^1} &\leq \|u_0\|_{H^1} + \int_0^t \left\| K * (Cu + \frac{1}{2}u^2) \right\|_{H^1} d\tau \\ &\leq \|u_0\|_{H^1} + \int_0^t \left\| Cu + \frac{1}{2}u^2) \right\|_{L^2} d\tau. \end{aligned}$$

And hence

$$\|Au_1(t) - Au_2(t)\|_{H^1} \leq t \|u_1 - u_2\|_{H^1} [|C|_\infty + R].$$

Now take the supremum $t \in [0,t_0]$ on both sides we have

$$\|Au_1(t) - Au_2(t)\|_{\mathcal{C}} \leq t_0 \{|C|_\infty + R\} \|U_1 - U_2\|_{\mathcal{C}}.$$

From the above equality it can be conformed that the operator A is a continuous mapping of the space $\mathcal{C}$ into itself. We can show that the mapping of the ball $\|U\|_{\mathcal{C}} \leq R$ satisfies Lipschitz condition with Lipschitz constant $\Theta < 1$.
we let $\|u_0\|_{\mathcal{C}} \leq d$, then

$$\begin{aligned} \|AU\|_{\mathcal{C}} &\leq \|AU - A0\|_{\mathcal{C}} + d \\ &\leq t_0 \{|C|_\infty + R\} \|U\|_{\mathcal{C}} + d \\ = \Theta R + d, \end{aligned}$$

where $\Theta = t_0 \{|C|_\infty + R\}$
If we choose $R = 2d$ and $t_0 = \frac{0.5}{\{|C|_\infty + R\}}$, then $\Theta = 0.5 < 1$, and

$$\|AU\|_{\mathcal{C}} \leq \frac{1}{2}R + \frac{R}{2} = R.$$

Which shows that A is a contraction mapping on B. Therefore A has a fixed point u in the ball B_R. This show that we have a solution $u \in C([0,t_0];H^1)$.
□

2.3.6 Examination of global existence results

Having in hand a solution u of (2.54) on a time interval $[0,t_0]$, we now examine to the global existence of u. From the equation (2.53), it shows immediately

that $u_t \in C([0,t_0];H^1)$. Since $u \in H^1(\mathbb{R})$, after some simple arguments we can find that $u_{xxt} \in L^2(\mathbb{R})$.
Multiply each term in (2.52) by u and integrate between $x=-R$ to $x=R$, we get that

$$\int_{-R}^{R} uu_t\,dx - \int_{-R}^{R} uu_{xxt}\,dx = -\int_{-R}^{R} u(C(x)u)_x\,dx \quad - \quad \frac{1}{3}\int_{-R}^{R} \frac{\partial}{\partial x}(u^3)\,dx$$

$$\begin{aligned}\int_{-R}^{R} \frac{1}{2}\frac{d}{dt}\left[u^2\right]\,dx - [uu_{xt}]_{-R}^{R} + \int_{-R}^{R} u_x u_{xt}\,dx \quad &= \quad \int_{-R}^{R} C(x)u\frac{\partial}{\partial x}(u)\,dx \\ &\quad - \frac{1}{3}\left[u^3\right]_{-R}^{R} - \left[Cu^2\right]\big]_{-R}^{R}\end{aligned}$$

Let $R \to \infty$, we use the fact that functions in $H^1(\mathbb{R})$ are vanish at infinity. Now we use Riemann-Lebesgue lemma and dominated convergence theorem to establish

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{2}\frac{d}{dt}\left(u^2+u_x^2\right)\,dx \quad &= \quad \int_{-\infty}^{\infty} C(x)\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\,dx \\ \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left(u^2+u_x^2\right)\,dx \quad &= \quad -\int_{-\infty}^{\infty} C'(x)\left(\frac{u^2}{2}\right)\,dx\end{aligned}$$

It yields that

$$\begin{aligned}\frac{dE}{dt} \quad &\geq \quad 0 \text{ if } h'(x) \leq 0 \\ \text{since } C(x) \quad &= \quad \sqrt{gh(x)}.\end{aligned} \tag{2.55}$$

And also

$$\frac{dE}{dt} \quad \leq \quad 0 \text{ if } h'(x) \geq 0 \tag{2.56}$$

Form (2.55), we can say that for decreasing height profile, we have increasing energy E. But for the previous model (2.40), we get exactly opposite results (see (2.44)). In the next section we check these results numerically.

Chapter 3

Numerical simulation

3.1 Numerical scheme

When we solving the partial differential equations, we rarely obtain an analytical solution. We use numerical method to find an approximate solution. The main idea behind numerical methods is to discretize the partial differential equations and use a computer to solve the discrete version of the system. There are several numerical methods exist, and which to choose depends on the properties of the equations we wish to solve, and the sought properties for the approximate solution. The famous common numerical methods are the finite difference methods (FDM), the finite element methods (FEM) and spectral methods.

The Finite difference method replace derivatives in a differential equation with differences, hence leading to a difference equation which is easily solved for one-dimensional domains. However, the accuracy is in general not good.

The finite element method approximates the solution as a linear combination of piecewise functions that are non-zero on small sub-domains. The FEM is suitable for complex and multidimensional domains, but the convergence is not always sufficient.

The spectral method approximates the solution as linear combination of continuous functions that are generally non-zero over the domain of solution. If we apply spectral method, we have excellent error properties in the form of an exponential convergence rate. Therefore the spectral method takes on a global approach while the finite element method is a local approach. Because of this reason, spectral method works best when the solution is smooth. The following presentation of spectral methods is based on the book of (Trefethen [23]) and paper of (Hussaini, Kopriva, Patera [11]).

3.1.1 Introduction to pseudo-spectral methods

The pseudo-spectral/collocation method is the method which used to solve the PDE, in particular, the collocation approximation consists of finding the polynomial which takes on the exact value of the original function at a finite number of grid (or collocation) points. Spectral collocation methods form an efficient and highly accurate class of techniques for the solution of nonlinear partial differential equations which are characterized by the expansion of the solution in terms of global basis functions, where the expansion coefficients are computed so that the differential equation is satisfied exactly at a set of so-called collocation points (Hussaini, Kopriva, Patera [11]). For example consider the PDE

$$\begin{cases} Lu(x) = f(x) & \text{if } x \in V \\ \text{Boundary condition: } Bu(y) = 0 & \text{if } y \in \partial V. \end{cases} \tag{3.1}$$

Here L is a spatial (linear) differential operator, B is a linear boundary operator and V is a spatial domain with boundary ∂V. We approximate the solution $u(x)$ by a sum of (N+1) basis functions $\phi_i(x)$ which span the space where the approximate solution exists

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} \hat{u}_i \phi_i(x), \tag{3.2}$$

where $\phi_j,\ j = 0,..N$ a finite set of trial functions. We want the numerical solution $u_N(x)$ with the coefficients $\hat{u}_i$ such that the residual R defined by

$$R(x) = Lu_N(x) - f(x)$$

must be minimized. For this, we choose a set of test functions $\chi_n = \delta(x - x_n),\quad n = 0, 1, 2, ...N$ and demand that

$$(\chi_n, R) = 0 \text{ for } n = 0, 1, 2, ...N, \tag{3.3}$$

where the x_n (n=0,1,2....N) are the spacial points, called the collocation points. From (3.3) we get that

$$0 = (\chi_n, R) = (\delta(x - x_n), R) = R(x_n) = Lu_N(x_n) - s(x_n).$$

That is

$$\sum_{i=0}^{N} \hat{u}_i L\phi_i(x_n) - f(x_n) = 0, \quad n = 0, 1, ...N.$$

Here we have N+1 equations to determine the unknown N+1 coefficients $\hat{u}_i$. We can now solve the N+1 system of equations, which gives the approximated solution u_N in the nodes x_i.
In our applications, we assume periodic boundary conditions and the problem is translated to the interval $[0, 2\pi]$ by some suitable scaling factor. When we work with spectral methods, the choice of basis functions is very important because the wrong choice of basis functions may give poor convergence properties. There are many choices possible, in particular trigonometric (Fourier) functions, Chebyshev and Legendre polynomials, but also lower-order Lagrange polynomials with local support (finite element method) or b-splines. However, we focus on the Fourier modes. Fourier series are particularly suited for the discretization of periodic functions u(x) = u(x + L) (Joseph Fourier 1768–1830). The Fourier expansion of a function u(x) is given by the infinite series

$$u(x) = \sum_{k=-\infty}^{\infty} a_k \; exp(ikx).$$

The collocation (pseudo-spectral approximation (Orszag(1971)) is defined by

$$P_N u = \sum_{k=-N/2}^{\frac{N}{2}-1} \hat{u}_k \; exp(ikx)$$

for even N and $\hat{u}_k$ dependent only on the N values of $u(x)$ at the collocation points. Here

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) \; exp(-ikx_j), \qquad k = -\frac{N}{2}, -\frac{N}{2}+1, ...\frac{N}{2}-1.$$

The collocation points x_i are uniform on the interval $[0, 2\pi]$

$$x_i = \frac{2\pi i}{N}, \quad i = 0, 1, ...N-1.$$

Differentiation in physical spaces based upon the values of the function $u(x)$ at the collocation points:

$$\hat{D}_N u(x) = \frac{d}{dx} \left(P_N u(x) \right) \neq P_N \frac{d}{dx} \left(u(x) \right).$$

The Fourier collocation derivative function can be represented as

$$\frac{d^l}{dx^l} \left(P_N u(x) \right) = \sum_{k=-N/2}^{\frac{N}{2}-1} (ik)^l e^{ikx} \hat{u}_k.$$

3.1.2 Introduction to Four-stage Runge-kutta method

After applying Fourier collocation method we have a set of ODEs, we use a four-stage explicit Runge-Kutta method (RK-4) to solve the system because it is quite accurate, stable and easy to program. Consider the ODE

$$\frac{dx}{dt} = f(x,t),$$

and if x_n is the approximate value of $x(t)$ at $n\Delta t$ where Δt is small time-step, then we approximate x_{n+1} by the following formula

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{6}\Delta t\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ \text{where } k_1 &= f(n\Delta t, x_n), \\ k_2 &= f((n+\frac{1}{2})\Delta t, x_n + \frac{1}{2}\Delta t k_1), \\ k_3 &= f((n+\frac{1}{2})\Delta t, x_n + \frac{1}{2}\Delta t k_2), \\ k_4 &= f((n+1)\Delta t, x_n + \Delta t k_3). \end{aligned}$$

The truncation error in RK-4 method is $O(\Delta t)^5$. The global error would be $O(\Delta t)^4$. Therefore this method is called fourth order method. When we half the time step the error will be $O\left(\frac{(\Delta t)^4}{16}\right)$. That is halving the time step results 16 times protection of error in RK-4 method.

3.2 Numerical results

For the purpose of numerical computations, We use the problem with periodic boundary conditions on the domain $x \in [0, L]$, where L=200 was observed that sufficient enough for the computations shown in this work. The problem is translated to the interval $[0, 2\pi]$ using the scaling $\eta(bx,t) = v(x,t)$, where $b = \frac{L}{2\pi}$. The initial-value problem

$$\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0 \tag{3.4}$$

is then

$$\begin{aligned} v_t + \frac{1}{b}\left[C(bx) + \frac{3}{2}v(x,t)\right]v_x - \frac{1}{6}\frac{1}{b^2}v_{xxt} &= 0, \quad x \in [0,2\pi], t > 0 \\ v(x,0) &= u(bx,0), \\ v(0,t) &= v(2\pi,t) \quad \text{for } t \geq 0. \end{aligned}$$

We consider Fourier transform of the above equation, we get

$$\begin{aligned} b^2\hat{v}_t + b\mathcal{F}\left(C(bx)v_x\right) + b\mathcal{F}\left(\frac{3}{2}vv_x\right) + \frac{1}{6}k^2\hat{v}_t &= 0, t > 0 \\ \hat{v}(k, t=0) &= \mathcal{F}(v(x,0),k), \quad t = 0 \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform operator. We can discretize the above problem in the following forms:

$$\begin{aligned} \hat{v}_t = \frac{-b}{b^2+\frac{1}{6}k^2}\mathcal{F}\left[C(bx)\mathcal{F}^{-1}\left(ik\hat{v}\right)\right] &- \frac{3ikb}{4(b^2+\frac{1}{6}k^2)}\left[\mathcal{F}\left(\left[\mathcal{F}^{-1}\left(\hat{v}\right)\right]^2, k\right)\right], \\ & k = -\frac{N}{2}+1, ..\frac{N}{2}, \quad t > 0, \\ \hat{v}_t(k,t) &= 0, \quad k = \frac{N}{2}, \quad t > 0, \\ \hat{v}(k, t=0) &= \mathcal{F}(v(x,0),k) = \frac{2\pi}{N}\sum_{i=1}^{N} e^{-ikx_i}v(x_i, 0), \\ & k = -\frac{N}{2}+1, ..\frac{N}{2}, \quad t = 0. \end{aligned}$$

This is a system of N ordinary differential equations for the discrete Fourier coefficients $\hat{v}_N(k,t)$, for $k = -\frac{N}{2}+1, ...\frac{N}{2}$. We solve the system by using a fourth order explicit Runge-Kutta scheme with time step h.
Here $\mathcal{F}^{-1}(\hat{v}, x_j) = \frac{1}{2\pi}\sum_{k=-\frac{N}{2}+1}^{N/2} e^{ikx_j}\hat{v}(k,t)$ is defined as the discrete Fourier transform of $\hat{v}$ at the grid points $x_j = \frac{2\pi j}{N}$ for $j = 1, 2, ...N$, where N is assumed to be an even number.

Similarly for the following Long wave equation

$$\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0, \tag{3.5}$$

we get the system

$$\begin{aligned} \hat{v}_t = \frac{-bik}{b^2+\frac{1}{6}k^2}\mathcal{F}\left[C(bx)\mathcal{F}^{-1}\left(\hat{v}\right)\right] &- \frac{3ikb}{4(b^2+\frac{1}{6}k^2)}\left[\mathcal{F}\left(\left[\mathcal{F}^{-1}\left(\hat{v}\right)\right]^2, k\right)\right], \\ & k = -\frac{N}{2}+1, ..\frac{N}{2}, \quad t > 0, \\ \hat{v}_t(k,t) &= 0 \quad k = \frac{N}{2}, \quad t > 0, \\ \hat{v}(k, t=0) &= \mathcal{F}(v(x,0),k) = \frac{2\pi}{N}\sum_{i=1}^{N} e^{-ikx_i}v(x_i, 0), \\ & k = -\frac{N}{2}+1, ..\frac{N}{2}, \quad t = 0. \end{aligned}$$

We use discrete L^2- norm to test the convergence of the algorithm and the numerical implementation. The L^2- norm is defined by

$$\|v\|_{N,2}^2 = \frac{1}{N}\sum_{j=1}^{N} |v(x_j)|^2 .$$

The relative L^2 - error is then defined to be

$$\frac{\|v - v_N\|_{N,2}}{\|v\|_{N,2}},$$

where $v_N(x_j)$ is the approximated numerical solution and $v(x_j)$ is the exact solution at a time T, for $j = 1, 2, ...N$.

3.2.1 Experiments with solitary waves

For the case $\mathbf{C(x)=1}$, we consider the solitary wave solution $\eta(x,t) = \phi(x - ct)$, here c is the speed of propagation of the solitary wave, then the above two long wave equations (3.4) and (3.5) becomes

$$\frac{c}{6}\phi'' + (1-c)\phi + \frac{3}{4}\phi^2 = 0.$$

We can check that the particular solution of above one is

$$\phi(x) = 2(c-1)\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{6(c-1)}{c}}\,x\right). \tag{3.6}$$

We use (3.6) as the exact form of the solitary waves with various values of c, both positive and negative, and we find the relative L^2 - error for various time step $h = 0.1/(2n)$, for $n = 1, 2, 3, ...$, and various $N = m \times 512$ for $m = 1, 2, 3, ...$

When we consider the solitary wave speed c, we have the following results which are proved by Kalisch, Nguyen [13]. If $c < 1$, then the waves are strictly positive progressive that is waves which propagate to the right in the direction of increasing values of x without changing their profile over time. If $0 < c < 1$, then there are no solitary waves and if c is negative then the waves are strictly negative progressive waves propagating to the left (in the direction of decreasing values of x).

In the computations shown in table 3.1 we approximate the solution from $t = 0$ to $t = 10$ and $L = 200$, 1024 Fourier modes are used. The time convergence of the scheme is apparent up to $h = 0.0008$. We see that the

Table 3.1: BBM equation; error due to temporal discretization.

h	n	L^2 error	Ratio
0.0500	200.0	6.316e-06	
0.0250	400	3.772e-07	16.74
0.0125	800	2.302e-08	16.39
0.0063	1600	1.421e-09	16.20
0.0031	3200	8.827e-11	16.10
0.0016	6400	5.500e-12	16.06
0.0008	12800	3.400e-13	16.13

rates mostly lie close to 16, which corresponds to fourth order convergence. This shows that we have reached the maximum precision for this value of Fourier modes, i.e, the spatial errors are dominating. Table 3.2 shows the spatial convergence rate for a calculation with the time step $h = 0.001$. Here we observe exponential convergence before reaching the limit set by the size of the time step. We find similar results for all other trials.

Table 3.2: BBM equation; error due to spatial discretization.

N	h	L^2 error	Ratio
512	0.001	9.871	
1024	0.001	0.831	11.89
2048	0.001	1.6e-04	5273.12
4096	0.001	5.9e-10	265503.99
8192	0.001	1.1e-13	5608.17
16384	0.001	3.4e-13	0.31
32768	0.001	5.0e-14	6.45
65536	0.001	3.0e-13	0.17
131072	0.001	2.3e-13	1.32
262144	0.001	1.7e-13	1.35

Chapter 4

Solitary waves in channels of decreasing depth

In this chapter we examine how solitary waves response for non-uniform depth. Here we study the solution of the both models with variable depth profile for both positive and negative wave speed. We first study solitary wave solution corresponding to the equation (3.4). We suppose that the depth of the water is slowly varying. That is initial depth of the bottom is $h1$ and it decreased to $h2$, suppose the amplitude of initial wave is a_1 and the amplitude of the wave at decreased height $h2$ is a_2 which is shown in figure 4.1.

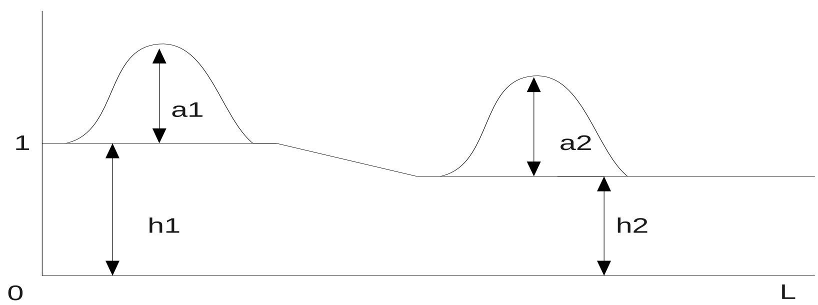


Figure 4.1: Height profile of the system.

We run the solitary wave solution corresponding to the equation $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ for both positive wave speed c and negative c, we obtain that for positive c, decreasing amplitude solitary wave profile and also find that the energy which is defined by

$$E(\eta) = \frac{1}{2}\int\limits_{-\infty}^{\infty}\left(\eta^2 + \frac{1}{6}\eta_x^2\right)\,dx$$

is decreasing function of t. We also note that the wavelength is less when we compare with initial wavelength. These results are shown in the figure 4.2. And for negative c, we find that the energy is increasing function of t which is shown in the figure 4.3. We also checked facts of the theoretical solution against numerical result. It actually coincide with our theoretical result (see (2.44)).

Table 4.1: Amplitude variations of the solitary wave solutions of the equation $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ for decreasing depth and positive wave speed.

a_1	a_2/a_1								
	Δh =0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.997	0.993	0.987	0.978	0.964	0.947	0.911	0.858	0.742
0.2	0.994	0.984	0.973	0.958	0.938	0.914	0.872	0.815	0.712
0.3	0.99	0.980	0.967	0.949	0.928	0.898	0.861	0.806	0.712
0.4	0.989	0.977	0.961	0.945	0.922	0.892	0.858	0.808	0.728
0.5	0.988	0.977	0.959	0.943	0.922	0.898	0.864	0.817	0.747
0.6	0.989	0.977	0.962	0.944	0.925	0.901	0.868	0.826	0.764
0.7	0.989	0.976	0.961	0.945	0.928	0.904	0.876	0.837	0.777
0.8	0.989	0.977	0.963	0.947	0.928	0.908	0.881	0.844	0.793
0.9	0.989	0.977	0.963	0.947	0.934	0.909	0.886	0.854	0.802
1	0.987	0.979	0.966	0.952	0.933	0.914	0.893	0.859	0.813

Further we consider the solitary wave solution of the equation $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$. Here we deduce that the solitary wave profile with increasing wave amplitude for positive c and the results is opposite for negative c. Because the energy is increasing with positive c and decreasing with negative c. It also confirms (2.55).
For each values of $\Delta h = h1 - h2$, we calculate the amplitudes $a1$ and a_2. These results are shown in the tables 4.1, 4.2, 4.3 and 4.4.

Table 4.2: Amplitude variations of the solitary wave solutions of the equation $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ for for decreasing $h(x)$ and negative wave speed.

a_1	a_2/a_1								
	Δh =0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	1.026	1.044	1.059	1.075	1.079	1.123	1.142	1.146	1.179
3.1	1.026	1.031	1.034	1.072	1.085	1.114	1.123	1.152	1.164
3.2	1.024	1.041	1.053	1.068	1.089	1.102	1.126	1.143	1.165
3.3	1.014	1.038	1.048	1.066	1.082	1.097	1.119	1.136	1.161
3.4	1.011	1.019	1.025	1.049	1.061	1.092	1.098	1.131	1.152
3.5	1.014	1.031	1.041	1.046	1.059	1.086	1.087	1.109	1.129
3.6	1.015	1.028	1.045	1.057	1.06	1.073	1.103	1.120	1.141
3.7	1.008	1.028	1.039	1.042	1.062	1.071	1.085	1.106	1.130
3.8	1.016	1.024	1.028	1.033	1.046	1.077	1.081	1.097	1.113
3.9	1.012	1.021	1.033	1.050	1.065	1.071	1.091	1.088	1.125

Table 4.3: Amplitude variations of the solitary wave solutions of the equation $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ for uneven bottom with decreasing depth and positive wave speed.

a_1	a_2/a_1								
	Δh =0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.050	1.107	1.177	1.259	1.363	1.499	1.683	1.964	2.467
0.2	1.046	1.099	1.159	1.234	1.324	1.437	1.588	1.802	2.159
0.3	1.043	1.091	1.146	1.213	1.292	1.389	1.514	1.687	1.967
0.4	1.04	1.085	1.136	1.196	1.267	1.354	1.459	1.615	1.832
0.5	1.036	1.079	1.127	1.181	1.244	1.324	1.418	1.551	1.749
0.6	1.035	1.072	1.117	1.167	1.228	1.299	1.388	1.501	1.677
0.7	1.033	1.070	1.111	1.157	1.211	1.278	1.359	1.457	1.612
0.8	1.03	1.064	1.105	1.144	1.200	1.261	1.330	1.429	1.568
0.9	1.03	1.063	1.099	1.139	1.188	1.243	1.313	1.401	1.519
1	1.026	1.059	1.094	1.128	1.171	1.231	1.295	1.366	1.481

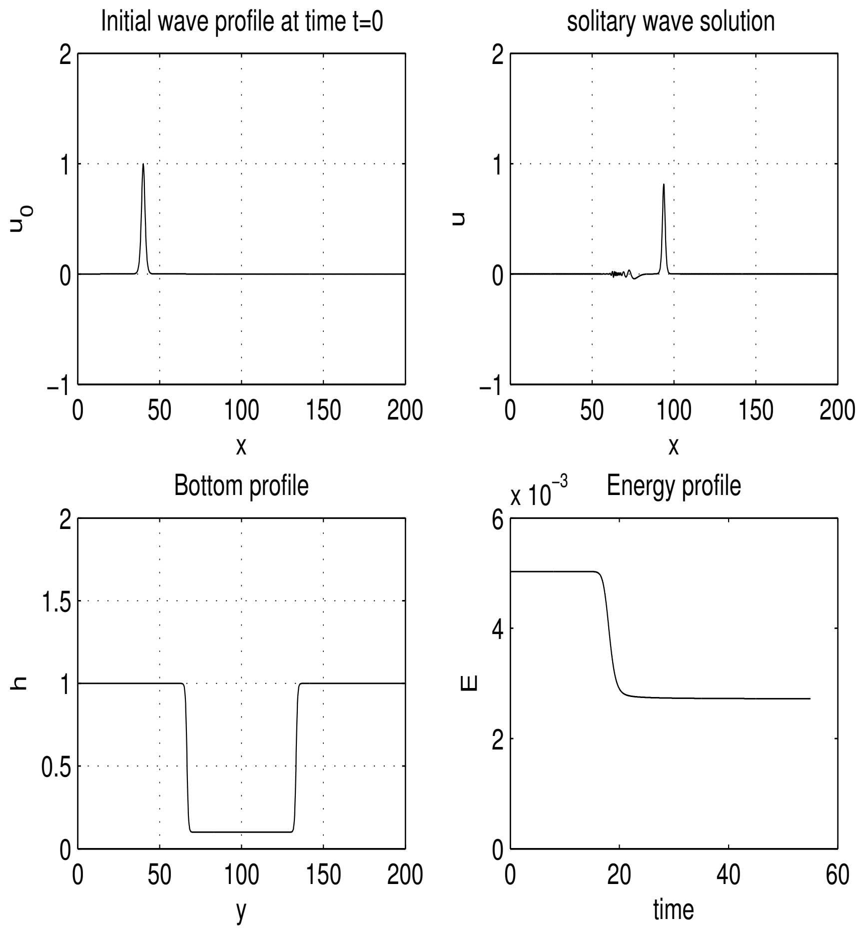


Figure 4.2: Solitary wave solution for $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ with positive wave speed. Solitary waves travelling in the direction of increasing values of x over an decreasing $h(x)$. We see that the energy is also decreasing.

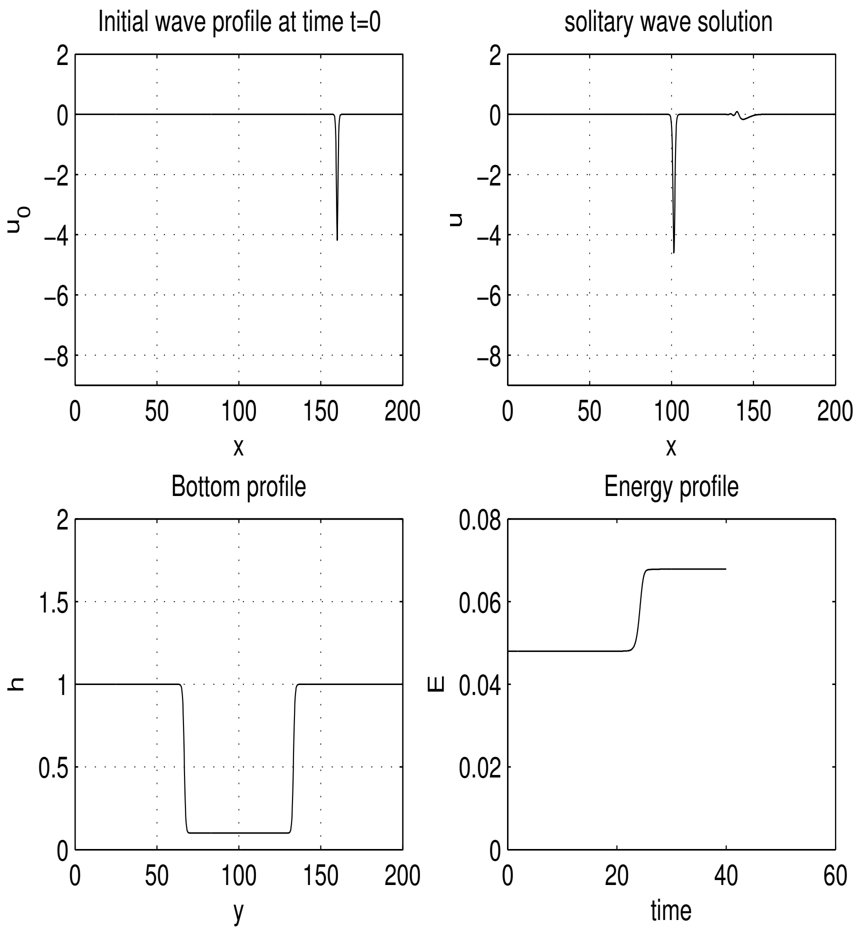


Figure 4.3: Solitary wave solution for $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ with negative wave speed. Negative progressive waves propagates to left over an decreasing height profile. We observe that the energy is increasing.

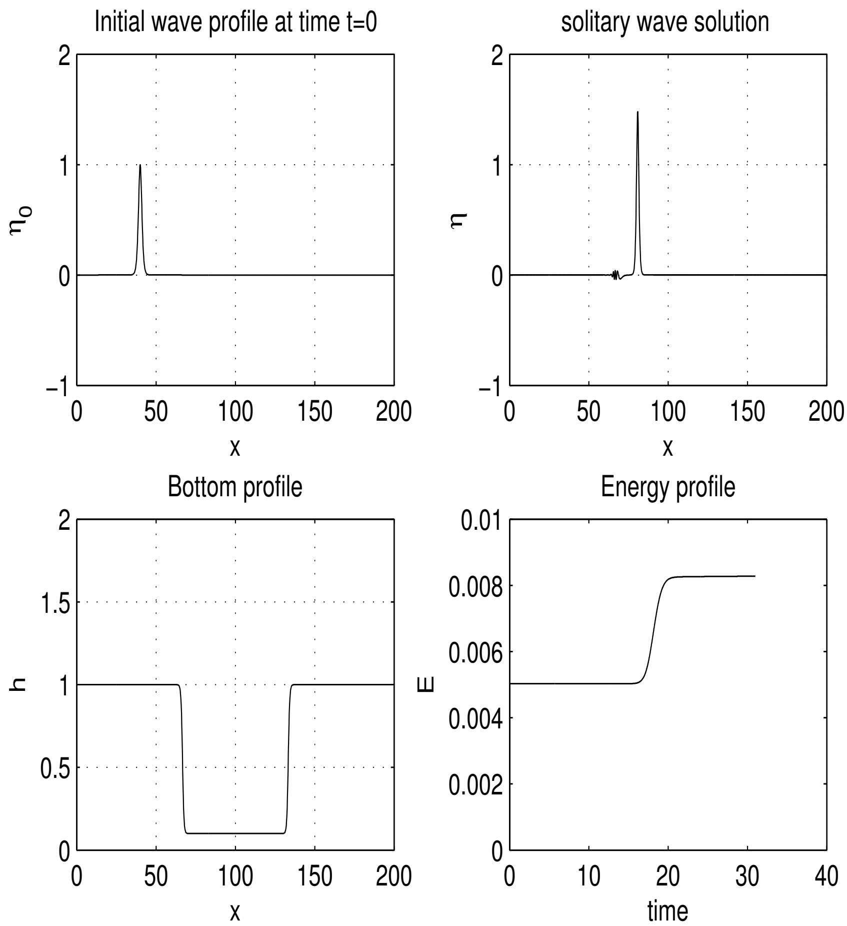


Figure 4.4: Solitary wave solution for $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ with positive wave speed. Positive progressive solitary waves propagates to right over an decreasing height profile. We see that the energy is increasing.

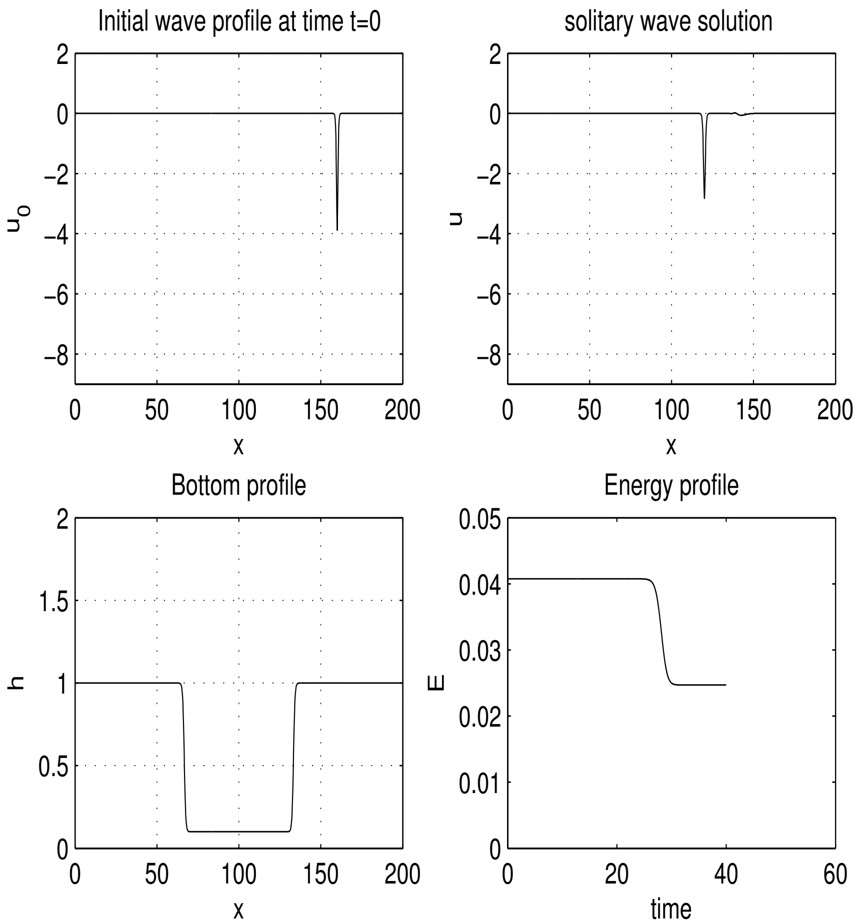


Figure 4.5: Solitary wave solution for $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta_x - \frac{1}{6}\eta_{xxt} = 0$ with negative wave speed. Solitary waves travelling in the direction of decreasing values of x over an decreasing $h(x)$. We see that the energy is also decreasing.

Table 4.4: Amplitude variations of solitary wave solutions of the equation $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ for decreasing depth and negative wave speed.

a_1	a_2/a_1								
	Δh =0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.960	0.913	0.894	0.855	0.800	0.762	0.720	0.663	0.593
3.1	0.969	0.941	0.884	0.847	0.809	0.776	0.729	0.682	0.622
3.2	0.976	0.923	0.911	0.872	0.823	0.797	0.754	0.726	0.638
3.3	0.955	0.943	0.889	0.872	0.846	0.808	0.755	0.721	0.652
3.4	0.978	0.926	0.912	0.884	0.853	0.802	0.779	0.731	0.676
3.5	0.976	0.932	0.914	0.891	0.859	0.820	0.785	0.735	0.691
3.6	0.977	0.951	0.909	0.877	0.859	0.832	0.788	0.756	0.695
3.7	0.980	0.939	0.929	0.893	0.869	0.831	0.802	0.760	0.731
3.8	0.980	0.953	0.929	0.886	0.866	0.844	0.809	0.774	0.719
3.9	0.972	0.944	0.933	0.909	0.882	0.851	0.808	0.783	0.729

Chapter 5

Transformation of solitary waves on a slope

Next we study the solitary wave transformation on a slope. We consider here the following model, the bottom depicted in figure (5.1) with a small but finite h_1 say $h_1 = 0.1h_0$.

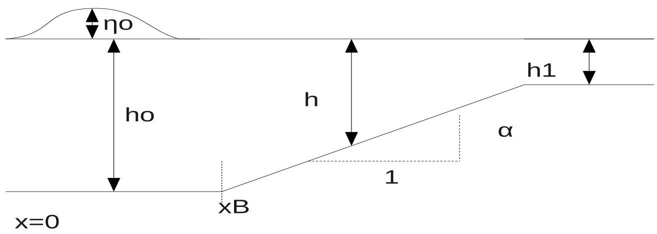


Figure 5.1: Geometry of the transformation of solitary waves.

Here let η_0 is the solitary wave amplitude at $t = 0$ and the solitary wave is supposed to pass $x = 0$ far from the slope, the distance between the point $x = 0$ and the bottom of the slope x_B is sufficiently large as long as $\eta(x_B, 0)/\eta_0 < 0.1$. In this study we consider the solitary wave solution of the equation $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$. We study the solitary wave transformation with different initial amplitudes for the slope $\alpha = 1/20$. Figure (5.2) shows that the transformation of the solitary wave on the slope for time to time with initial amplitude 0.2. It shows that the features one would expect the grows in amplitude.

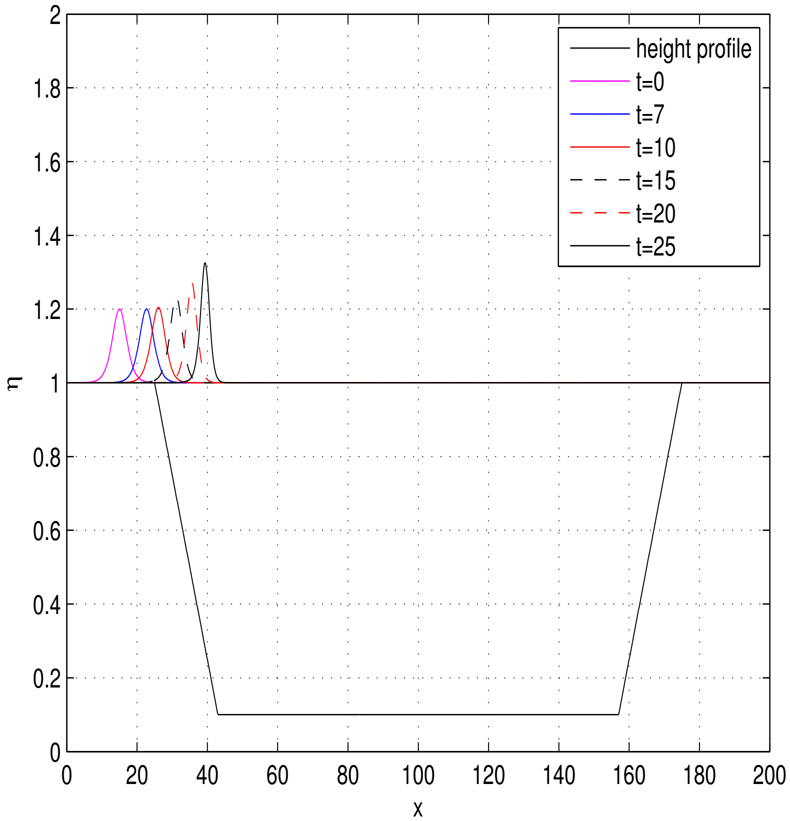


Figure 5.2: Solitary wave transformation on a slope. Here our physical domain is the left half of the numerical domain.

Results for the variation in amplitude of the waves are plotted for a variety of initial wave amplitudes with the same slope $\alpha = 1/20$. Here we found some variation which is not systematic. In the following we plot the largest values of η_{max}/η_0 with depth.

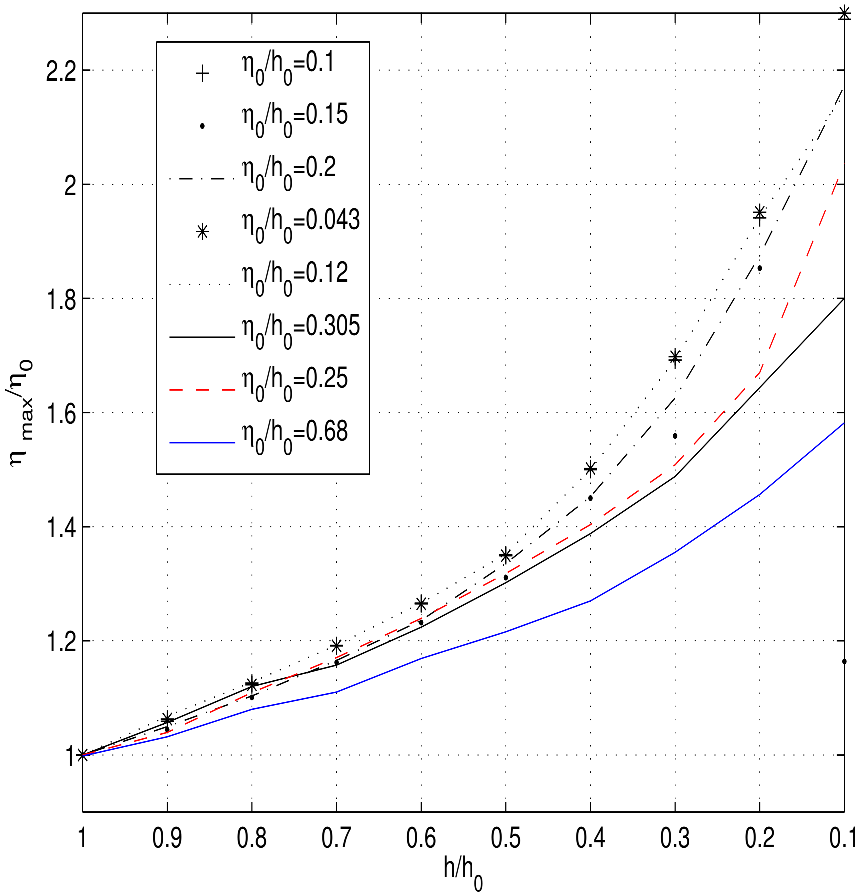


Figure 5.3: Variation of amplitude of a solitary wave with depth for the slope $\alpha = 1/20$ with different initial amplitudes.

5.1 Comparison of results

There were many peoples studied the same problem, a solitary wave on a beach. Here the work is based on the papers Madsen and Mei [16] and Peregrine [19]. They made same arrangement of experimental measurements. In 1966 Peregrine made the same experiment with the initial water wave profile was

$$\eta = \lambda_0 \operatorname{sech}^2 \frac{1}{2}(3\lambda_0)^{1/2}(x - \alpha^{-1}). \tag{5.1}$$

He calculated the results for a number of different beach slopes and initial wave amplitude. He found that when the wave approaches the shore it gets higher and steeper and the equation (1.61) is not valid for all time since the wave will ultimately break (Madsen and Mei [16]). The following figure (5.4) shows the comparison of our results with Peregrine results for the same slope 1/20.

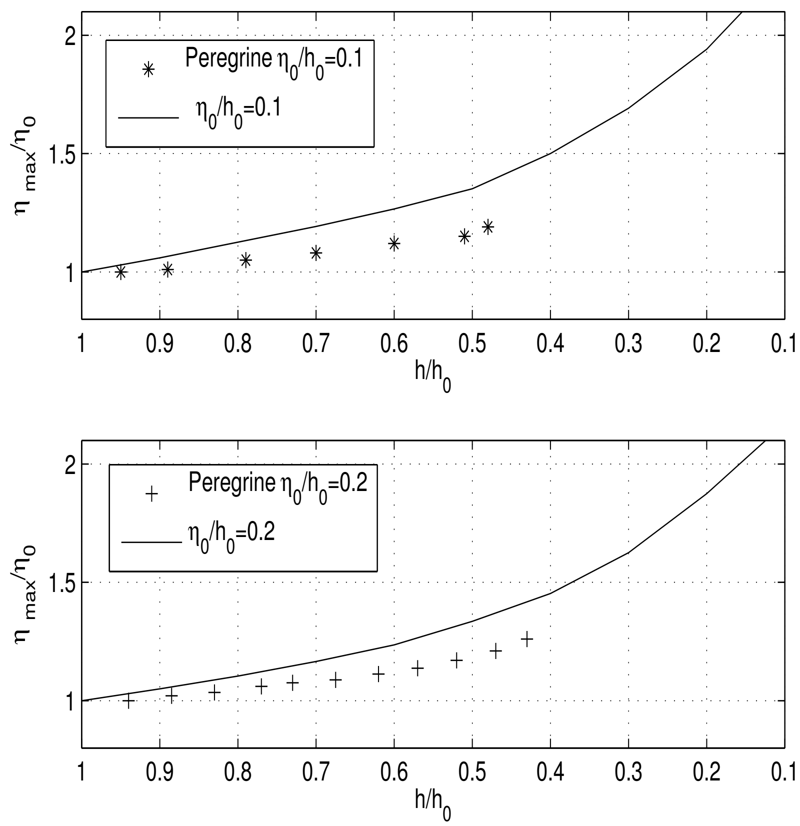


Figure 5.4: Comparison of our results with Peregrine for $\eta_0/h_0 = 0.1$ and $\eta_0/h_0 = 0.2$ with depth for slope 1/20. We observe less than 20% deviation in our results.

The same experimental set-up with the slope 1/20, Ippen and Kulin (1955) found that the amplitude decreases from its initial value measured some distance away from the slope, to a smaller value as the crest reach the slope. But when we compare our results with Ippen and Kulin results we get opposite trend of results. Both Ippen and kulin, Peregrine reported a non-systematic change in amplitude variation with the initial amplitude.

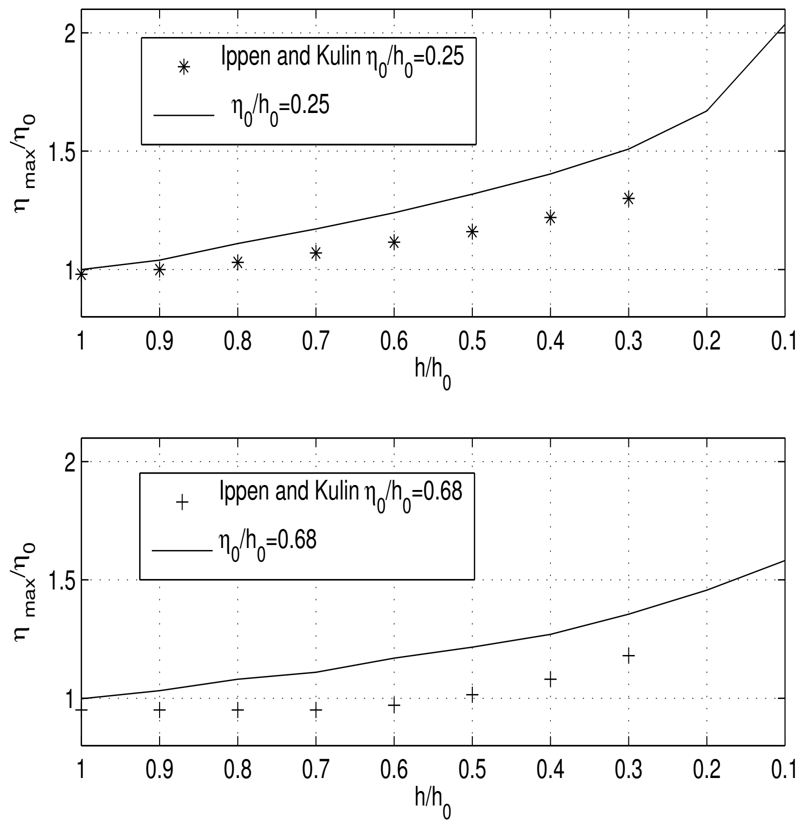


Figure 5.5: Comparison of our results with Ippen and Kulin for $\eta_0/h_0 = 0.25$ and $\eta_0/h_0 = 0.68$ with slope 1/20. Qualitative agreement in lower panel is not good as the amplitude decreases for some distance in Ippen and Kulin results.

However the experiments made by Madsen and Mei (1969), Peregrine (1967) with a bottom slope 1/20 shows the similar trends as our results. Madsen and Mei found that because of the rough bottom the frictional effects are pronounced and the amplitude does not increase appreciably over the first part of the slope and as the amplitude to depth ratio becomes large the amplitude increase is no longer cancelled by the frictional attenuation.

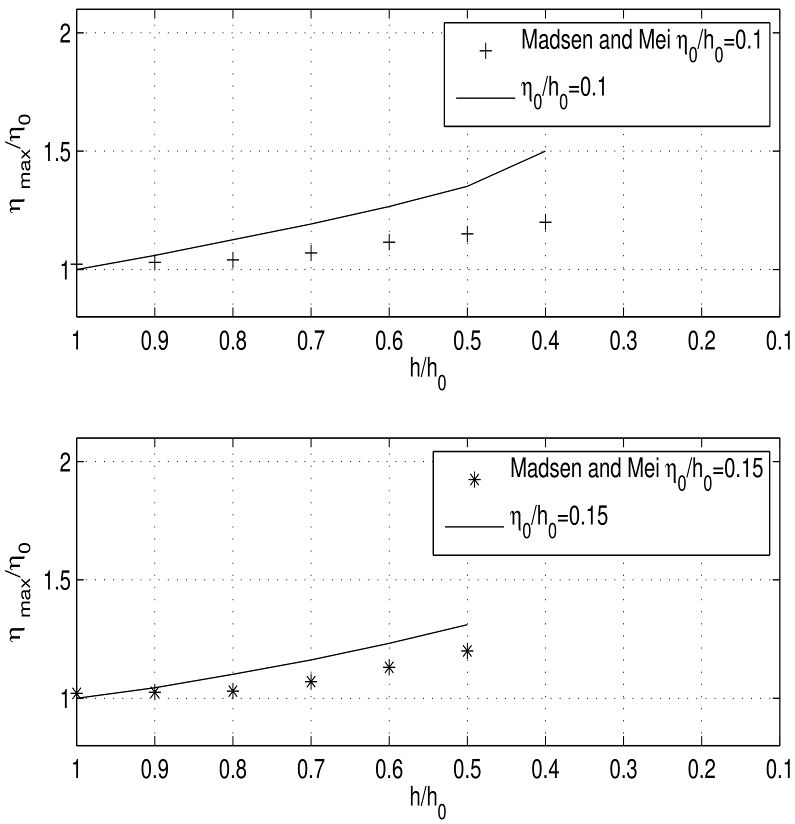
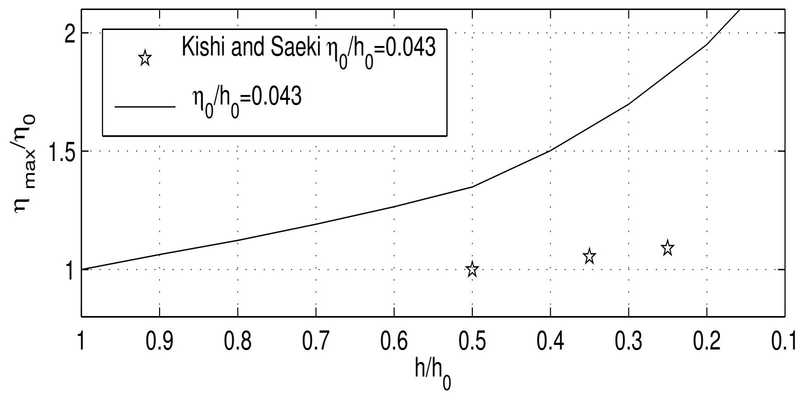


Figure 5.6: Comparison of our results with Madsen and Mei for $\eta_0/h_0 = 0.1$ and $\eta_0/h_0 = 0.15$ with depth for slope 1/20. The lower panel results are quantitatively better than upper panel.



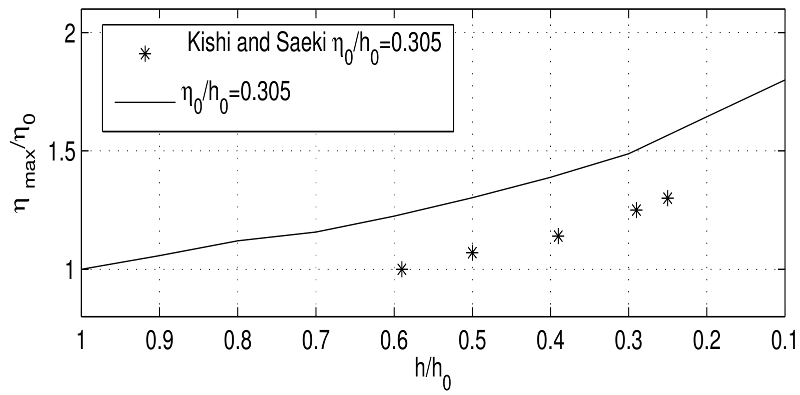


Figure 5.7: Comparison of our results with Kishi and Saeki for $\eta_0/h_0 = 0.043$ and $\eta_0/h_0 = 0,305$ with slope 1/20. We observe good qualitative agreement in our results.

Chapter 6

Summary and conclusions

BBM equation is a model equation for surface water waves of small amplitude, uni-directional and long waves in two-dimensional. In addition this equation was found to have solitary wave solutions. We reviewed dispersion relation for the BBM equation. Theory is presented for non-breaking waves. Further, this model can be extended to an uneven bottom (varying depth).

The well-posedness in spaces of classical solutions, results of the initial and boundary-value problem for BBM equation were reviewed. Also we found new local well-posedness results for the BBM equation supplemented with an initial and following various types of boundary conditions:

- Neumann boundary conditions
- mixed Dirichlet and Neumann boundary conditions.

The global well-posedness of the initial and boundary-value BBM equation were also studied in the Sobolev space $H^1(0, L)$. Further we analysed both local and global existence results for the initial-value BBM equation with the models of uneven bottom in the Sobolev space $H^1(\mathbb{R})$.

We studied the evolution of solitary waves over an uneven bottom. Amplitude variation of the both models are tabulated with various initial amplitudes. The amplitude of the solitary waves decreases with positive wave speed in decreasing height profile of the first model $\eta_t + C(x)\eta_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$ and it was opposite for second model $\eta_t + (C(x)\eta)_x + \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxt} = 0$.

Finally we studied shoaling of solitary waves on a slope with help of the second model because we are interested in increasing amplitude wave model. Our results show that as a solitary waves climbs a slope the rate of amplitude increase depends on the initial amplitude. We compared our results with existing experimental results. We found good qualitative agreement in most cases and also quantitatively good in some cases.

Appendix A: Contraction mapping theorem

Definition .1. *A mapping $F : X \to X$, where X is a subset of a normed space N, is called a contraction mapping, if there is a positive number $a < 1$ such that*

$$\|Fx - Fy\| \leq a \|x - y\| \quad \textit{for all } x, y \in X \tag{1}$$

Theorem .2. *If $F : X \to X$ is a contraction mapping of a closed subset X of a Banach space, then there is exactly one $x \in X$ such that $Fx = x$. For any $x_0 \in X$, the sequence (x_n) defined by $x_{n+1} = Fx_n$ converges to x.*

Proof: For any $x_0 \in X$, set $x_n = F^n x_0$. Let a be as in previous definition (1), then

$$\|x_{n+1} - x_n\| \leq a \|x_n - x_{n-1}\| \leq ... \leq a^n \|x_1 - x_0\|.$$

Hence for any $m > n$, the triangle inequality gives

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + ... + \|x_{n+1} - x_n\| \\ &\leq (a^{m-1} + a^{m-2} + ... + a^n) \|x_1 - x_0\| \\ &\leq (a^n/1 - a) \|x_1 - x_0\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence we obtain that for any $\epsilon > 0$ there is an N such that $\|x_m - x_n\| < \epsilon$ whenever $m > n > N$, (x_n) is a cauchy sequence. Since the space is complete, (x_n) is convergent to the limit $\bar{x}$ (say). Since X is closed, $\bar{x} \in X$.
Now we have to prove that $F\bar{x} = \bar{x}$.
We have for any n,

$$\begin{aligned} \|F\bar{x} - \bar{x}\| &\leq \|F\bar{x} - Fx_n\| + \|Fx_n - \bar{x}\| \\ &\leq a \|\bar{x} - x_n\| + \|\bar{x} - x_{n+1}\| \end{aligned} \tag{2}$$

We know that (2) tends to 0 as $n \to \infty$. That is $\|F\bar{x} - \bar{x}\|$ is less than every member of a sequence which tends to 0, and therefore must be 0. Hence we conclude $F\bar{x} - \bar{x} = 0$.
Finally, to prove uniqueness suppose that $F\bar{x} = \bar{x}$ and $F\bar{y} = \bar{y}$.then consider

$$\|\bar{x} - \bar{y}\| = \|F\bar{x} - F\bar{y}\| \leq a \|\bar{x} - \bar{y}\|$$

Which is a contraction unless $\bar{x} = \bar{y}$. □

Appendix B: Inequalities

a) **Cauchy-Schwarz inequality:**

$$|x.y| \leq |x|\,|y| \quad (x,y) \in \mathbb{R}^n \tag{3}$$

b) **Hölder's inequality (integral form):**

Assume $1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then if $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |fg|\,dx \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{4}$$

In particular if p=q=2, then the above equality is called Cauchy-Schwartz inequality.

c) **Gronwall's inequality (integral form):** Let $\eta(t)$ be a non-negative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality

$$\eta(t) \leq C_1 \int_0^t \eta(s)\,ds$$

for constant C_1 then

$$\eta(t) = 0 \quad \text{for a.e.} 0 \leq t \leq T.$$

d) **Gronwall's integral inequality (differential form):** Let $\eta(t)$ be a non-negative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on [0,T]. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)\,ds} \left[\eta(0) + \int_0^t \psi(s)\,ds \right] \quad \text{for a.e.} 0 \leq t \leq T.$$

e) **Minkowski's integral inequality:**

Minkowski's inequality on a measure space (X,μ) is simply the triangle inequality for $L^p(X)$, saying that the norm of a sum is bounded by the sum of the norms

$$\left\| \sum_j f_j v_j \right\|_{L^p(X)} \leq \sum_j \|f_j\|_{L^p(X)} v_j.$$

Whenever $f_j \in L^p(X)$ and the constants v_j are non-negative. Similarly we have the following theorem

Theorem .3. *Suppose (X, μ) and (Y, ν) are $\sigma-$ finite measure spaces and that $f(x, y)$ is measurable on the product space $X \times Y$. If $1 \le p < \infty$, then*

$$\left\| \int_Y f(x, y)\, dv(y) \right\|_{L^p(X)} \le \int_Y \|f(x, y)\|_{L^p(X)}\, dv(y),$$

whenever the right side is finite.

Proof: Take q to be conjugate exponent with $\frac{1}{p} + \frac{1}{q} = 1$ and use Hölder's inequality. Then for all $g \in L^q(X)$,

$$\begin{aligned} \left| \int_X \left(\int_Y f(x, y)\, dv(y) \right) g(x)\, d\mu(x) \right| &\le \int_Y \int_X |f(x, y)|\, |g(x)|\, d\mu(x)\, dv(y) \\ &\le \int_Y \left(\int_X |f(x, y)|^p\, d\mu(x) \right)^{1/p} \|g\|_{L^q(X)}\, dv(y) \\ &= \int_Y \|f(x, y)\|_{L^p(X)}\, dv(y)\, \|g\|_{L^q(X)}\,. \end{aligned}$$

Now the remaining proof follows from the dual characterization of the norm on $L^p(X)$. (see Folland [9] thm. 6.14). □

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