Nonholonomic geometry on finite and infinite dimensional Lie groups and rolling manifolds

Dissertation for the degree of Philosophiae Doctor (Ph.D.)

Erlend Grong

Department of Mathematics
University of Bergen
Norway

January 2012

Preface

The structure of the thesis is as follows:

Part I We start by giving some background on the topics discussed in this thesis.

The main topic of the thesis is nonholonomic geometry. In Chapter 1 we give an introduction of nonholonomic geometry in the context of geometric control theory. In a brief exposition, we try to give an overview of the areas of sub-Riemannian and sub-Lorentzian geometry, stating several of the most important results in this area. A historical account concludes this chapter.

Chapters 2 and 3 consist of mathematical prerequisits for the later presented results. However, these chapters mainly focus on certain selected facts, rather than trying to give an overview of a whole topic. Chapter 2 contains some results from differential geometry related to submersions and geodesic curvatures. Chapter 3 gives introductory remarks on the convenient calculus of infinite dimensional manifolds.

Chapter 4, the last chapter in part I, gives a short presentation and summary of the main results of the papers included in Part II. We first present the results of Paper B, regarding sub-Riemannian and sub-Lorentzian geometry on the universal cover of $\mathrm{SU}(1,1)$. The results in Papers C, D and F are then considered, which concern the nonholonomic dynamical system of two manifolds rolling on each other without twisting or slipping. Finally, we present some results in infinite dimensional manifolds in Paper A and Paper F. In particular, Paper F contains a generalization of sub-Riemannian geometry to the infinite dimensional setting.

Part I ends with the bibliography of the 4 first chapters.

Part II Here, six papers are included, Papers A to F. Papers are listed in chronological order according to their date of completion. Two of them are published, one is accepted for publication and three are submitted.

Acknowledgement

First of all, I would like to thank my main supervisor Alexander Vasiliev for his guidance and keen insight when I am stuck at a problem, and my co-supervisor Irina Markina for her patience and ability to ground my intuitive ideas into solid mathematics. Together, they make a perfect team, but what shines through more than anything else, is their true dedication for the best of their students. In everything from conferences and getting in touch with the international mathematical community to writing papers, they have worked to give me every opportunity to evolve as a mathematician.

I would also like to thank my co-authors Fátima Silva Leite and Mauricio Godoy Molina. To Fátima, I would like to thank you for introducing me to rolling manifolds, and for your warm hospitality when I visited Portugal. To Mauricio, I am grateful for having you as a friend, a source of necessary distraction from work and 'living library' about mathematics and everything else.

A big thank you goes to everyone at the department for making these four years so enjoyable. Georgy, Kristoffer, Mirjam, Henning, Anastasia and many, many others are responsible for truly making this the best workplace anyone can have. Former members Ksju, Anja, Martin and Pasha should also be mentioned. I would like to thank each one of you for your support and friendship, and I wish you all the very best.

Finally, I would like to thank Christian Autenried, Alexander Lundervold and Maria Vik Søfteland for assisting me in the making of this thesis.

I especially would like to thank Maria for making me happy, and for believing in me. A special gratitude also belongs to my family for all their support.

Contents

Part I.

Introduction and background

1. Optimal control theory and nonholonomic geometry

This chapter considers sub-Riemannian and sub-Lorentzian geometry from the point of view of geometric control theory. We will assume that the reader is familiar with topics from standard differential topology, such as manifolds, fiber bundles, vector bundles and metrics on them. Some typical references are [Lee02, Lee09, Ram05].

In this chapter, the term manifold means a smooth, finite dimensional manifold, that is always assumed to be Hausdorff and second countable. If $f : M \to Q$ is a smooth map between manifolds, then the tangent map or differential is denoted by $f_* : TM \to TQ$. We write $f|_m : T_mM \to T_{f(m)}Q$ for the restriction to the tangent space at the point $m \in M$. The space of all such smooth function is denoted by $C^\infty(M, Q)$ or just $C^\infty(M)$, if $Q = \mathbb{R}$.

If X is a vector field on M, then we will write $X|_m$ rather than $X(m)$, and we also adopt this convention for other types of sections. Furthermore, for one-forms (and also other tensors) we will in general write $\alpha(v)$ rather than $\alpha|_m(v)$ if it is clear from the context that $v \in T_mM$.

1.1. Optimal control and the Pontryagin Maximum Principle

There are many definitions and generalizations describing problems of optimal control. The definition we will follow, can be found in [Agr08], although we will also use much of the material from [AgSa04].

Let M and U be two manifolds of respective dimensions n and k, and let $\pi : \mathcal{U} \to M$ be a fiber bundle with fiber U. Let f be a fiber preserving map

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & TM \arrow[dl, "\mathrm{pr}_M"] \\ & M & \end{tikzcd} .$$

The map $\mathrm{pr}_M : TM \to M$ in the above formula, is just the natural projection. The pair $(\mathcal{U}, f)$ is called a control system on M. If u is an element in $\mathcal{U}$, we will alternate between writing this element simply as u or as a pair (m, u) with its footprint $\pi(u) = m$. Which one of these we choose will depend on the context. We will adopt this convention for the rest of this thesis whenever we have points or curves in a fiber bundle. Consequently, if $u\colon [0,\tau] \to \mathcal{U}$ is a curve in $\mathcal{U}$, we will also write this curve as a pair $t \mapsto (\gamma_u(t), u(t))$, where $\pi(u(t)) = \gamma_u(t)$. Then (γ_u, u) is called an admissible pair, if γ_u is Lipschitz and for almost every t,

$$\dot{\gamma}_u = f(\gamma_u(t), u(t)).$$

The space

$$\mathcal{C}_\tau = \{u \in L^\infty([0,\tau],\mathcal{U}) : \ (\gamma_u, u) \text{ is an admissible control}\},$$

is a Banach submanifold of $L^\infty([0,\tau],\mathcal{U})$, with the latter being a Banach manifold modeled on $L^\infty([0,\tau],\mathbb{R}^{n+k})$.

A fixed-time optimal control system is a triple $(\mathcal{U}, f, \Phi)$, where $(\mathcal{U}, f)$ is a control system, and $\Phi : \mathcal{C}_\tau \to \mathbb{R}$ is a functional of the form

$$\Phi(\gamma_u, u) = \int_0^\tau \varphi\big(\gamma_u(t), u(t)\big)\, dt.$$

Here, φ is a smooth function on $\mathcal{U}$. Φ is called the cost functional, while φ is referred to as the cost function. It is called a free-time optimal control problem if τ is allowed to vary.

We look at the following fixed-time optimal control problem. For a given pair of points $m_0, m_1 \in M$, let $\mathcal{C}_\tau(m_0, m_1)$ be the elements in $\mathcal{C}_\tau$ satisfying $\gamma_u(0) = m_0$ and $\gamma_u(\tau) = m_1$. We look for an element $(\gamma_{\widehat{u}}, \widehat{u}) \in \mathcal{C}_\tau(m_0, m_1)$, such that $\Phi(\gamma_{\widehat{u}}, \widehat{u}) \leq \Phi(\gamma_u, u)$ for any $(\gamma_u, u) \in \mathcal{C}_\tau(m_0, m_1)$. $\widehat{u}$ is then referred to as an optimal control, while $\gamma_{\widehat{u}}$ is called an optimal trajectory. The main tool to solve such problems is the first order condition given by the Pontryagin Maximum Principle (PMP).

We call an element $H \in C^\infty(T^*M)$ a Hamiltonian function. Corresponding to this, we write $\vec{H}$ for the Hamiltonian vector field on T^*M, uniquely determined by the property $H_*(\widetilde{X}) = \sigma(\vec{H}, \widetilde{X})$, where $\widetilde{X}$ is any vector field on T^*M and σ is the canonical symplectic form on the same space. We have chosen to mark the vector field with a tilde, to emphasize that it is defined on T^*M rather than on M. We will use this convention on all fiber bundles.

An element $\mathscr{H} \in C^\infty(\mathcal{U} \times_M T^*M)$ is called a pseudo-Hamiltonian function. In order to present the Pontryagin maximum principle in a more standard way, let us first assume that $\mathcal{U}$ trivializes, such that we can identify $\mathcal{U}$ with $M \times U$, and hence also identify $\mathcal{U} \times_M T^*M$ and $U \times T^*M$. This always holds locally. We will later make the appropriate changes to make the statement valid for more general control systems. For a pseudo-Hamiltonian function, let $\vec{\mathscr{H}}$ be defined so that for a fixed $u \in U$, $(m,p) \mapsto \vec{\mathscr{H}}(m,u,p)$ is the Hamiltonian vector field associated to the Hamiltonian $(m,p) \mapsto \mathscr{H}(m,u,p)$. Then we have the following result.

Theorem 1. PMP for Optimal Control Problem with fixed time τ [AgSa04, Theorem 12.3]: *For a given value of τ, let $(\gamma_{\widehat{u}}, \widehat{u}) \in \mathcal{C}_\tau(m_0, m_1)$ be a solution to the above problem. For each $\nu \in \mathbb{R}$, consider a pseudo-Hamiltonian function defined by*

$$\mathscr{H}_\nu(m,u,p) = p(f(m,u)) + \nu\varphi(m,u), \qquad u \in U, \quad p \in T^*_m M.$$

Then there exists a curve $\lambda : [0,\tau] \to T^*M$, and a number $\nu \leq 0$ such that *(i)* $\mathrm{pr}_M \lambda(t) = \gamma_{\widehat{u}}(t)$.

(ii) $\dot{\lambda}(t) = \vec{\mathscr{H}}_\nu(\gamma_{\widehat{u}}(t), \widehat{u}(t), \lambda(t))$ *for almost every* t,

(iii) $\mathscr{H}_\nu(\gamma_{\widehat{u}}(t), \widehat{u}(t), \lambda(t)) = \max_{u \in U} \mathscr{H}_\nu(\gamma_{\widehat{u}}(t), u, \lambda(t))$ *for a.e.* $t \in [0, \tau]$.

Moreover, if $\nu = 0$*, then* λ *never intersects the zero section of* T^*M*.*

If $\nu < 0$, then the solution is called normal. We can always normalize it to the case when $\nu = -1$. If $\nu = 0$, the solution is called abnormal.

Remark 1.
- For the problem of the maximum of Φ, the above theorem has the same formulation, changing only the requirement $\nu \leq 0$ to $\nu \geq 0$.
- If we consider a free-time problem, then we also require $\mathscr{H}_\nu(\gamma_{\widehat{u}}(t), \widehat{u}(t), \lambda(t)) \equiv 0$.
- If $H_\nu(m, p) = \max_{u \in U} \mathscr{H}_\nu(m, u, p)$ is defined and if it is C^2 on $T^*M \setminus \underline{0}(M)$, where $\underline{0} : M \to T^*M$ is the zero-section, then $\dot{\lambda}(t) = \vec{H}_\nu(\lambda(t))$ [AgSa04, Proposition 12.1].
- We can also write this theorem without choosing a local trivialization of $\mathcal{U}$. Consider the projection
$$\mathrm{pr}_2 : \mathcal{U} \times_M T^*M \to T^*M.$$
Then the requirement in (ii) must be replaced by
$$(\mathrm{pr}_2^* \sigma)\Big|_{(\widehat{u}(t), \lambda(t))} \left(\widetilde{X}, \dot{\lambda}\right) = (\mathcal{H}_\nu)_* \Big|_{(\widehat{u}(t), \lambda(t))} (\widetilde{X}),$$
for any vector field $\widetilde{X}$ on $\mathcal{U} \times_M T^*M \to T^*M$. In (iii), the maximum over all elements in U must be changed to the maximum over all elements in $\mathcal{U}_{\gamma_{\widehat{u}}}$.

1.2. The Orbit theorem

While the Pontryagin maximum principle might be the most useful tool for finding optimal solution, the same can be said about the orbit theorem and its corollaries when it comes to the question of controllability. Let $\mathrm{Vect}(M)$ be the collection of all vector fields on M, and let $\mathfrak{F}$ be a subset of $\mathrm{Vect}(M)$. For any vector field X, write ψ^X for the (local) flow of X. For practical purposes, we will use the notation $\psi_t^X(m)$ rather than $\psi^X(t, m)$. Then the orbit of $\mathfrak{F}$ through $m_0 \in M$ is given by

$$\mathcal{O}_{m_0} = \left\{ \psi_{t_l}^{X_l} \circ \psi_{t_{l-1}}^{X_{l-1}} \circ \cdots \circ \psi_{t_1}^{X_1}(m) : \ t_j \in \mathbb{R}, X_j \in \mathfrak{F}, l \in \mathbb{N} \right\}, \tag{1.1}$$

where each t_j must be chosen such that the flow is well defined.

Let $(\mathcal{U}, f)$ be a control system and assume that any local section of $\pi : \mathcal{U} \to M$ can be extended to a global section. Let $\Gamma(\mathcal{U})$ denote all global sections of $\pi : \mathcal{U} \to M$ and let $\mathfrak{F}$ be its image under f. Given a point $m_0 \in M$, we are interested in all points in M that can be reached from m_0 by a curve γ_u such that (γ_u, u) is part of an admissible pair.

These points form what is called the attainable set from m_0, denoted by $\mathcal{A}_{m_0}$. This can be defined similarly to the orbit, only adding the requirement that each t_j is non-negative in (1.1). We say that we have local controllability at m_0 if there is a neighborhood of m_0 contained in $\mathcal{A}_{m_0}$, that is, m_0 is in the interior of $\mathcal{A}_{m_0}$. A system $(\mathcal{U}, f)$ is called controllable or completely controllable if $\mathcal{A}_{m_0} = M$ for one (and hence any) $m_0 \in M$.

The attainable set from m_0 coincides with the orbit of $\mathfrak{F}$ through m_0, if $\mathfrak{F} = -\mathfrak{F}$. Orbits are easier to study, as they have nicer structure by the following theorem.

Theorem 2 (The orbit theorem)**.** [Sus73]

- *$\mathcal{O}_{m_0}$ is a connected immersed submanifold of M.*
- *Define* $\operatorname{Lie}\mathfrak{F}$ *as*

$$\operatorname{Lie}\mathfrak{F} = \{[X_l\,[X_{l-1}[\cdots[X_2, X_1]]\cdots]] \,:\, X_j \in \mathfrak{F}, l \in \mathbb{N}\},$$

and define $\operatorname{Lie}_m \mathfrak{F}$ *as the subset of T_mM obtained by evaluating all elements from* $\operatorname{Lie}\mathfrak{F}$ *at m. Then*

$$\operatorname{Lie}_m \mathfrak{F} \subseteq T_m\, \mathcal{O}_{m_0} \tag{1.2}$$

for any $m \in \mathcal{O}_{m_0}$.

Remark 2. The actual orbit theorem is more general than the one we have presented here. A good reference is [AgSa04, Chapter 5]. For the original formulation, see [Sus73]. Note that the latter reference also contains a formulation for the case when $\mathfrak{F}$ consist of only local vector fields, which means that it can be applied to control systems where it is not possible to extend local sections of $\pi : \mathcal{U} \to M$.

As a corollary of this theorem, we come to a result that was already proved in the late 1930s. We say that $\mathfrak{F}$ is bracket-generating if $\operatorname{Lie}_m \mathfrak{F} = T_mM$ for every $m \in M$. In particular, let D be a sub-bundle of the tangent bundle. We will say that D is bracket generating if $\Gamma(D)$, the space of all sections of D, is bracket generating. An absolutely continuous curve γ is called horizontal or D-horizontal if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for almost every t. It is clear that (γ_u, u) is admissible with respect to the optimal control system (D, inc), where $\mathrm{inc} : D \to TM$ is just the inclusion, if and only if it is Lipschitz and D-horizontal.

Theorem 3 (Rashevskiĭ-Chow Theorem)**.** [Cho39, Ras38] *If $\mathfrak{F}$ is a bracket generating family of vector fields, then for any $m_0 \in M$, we have $\mathcal{O}_{m_0} = M$.*

In particular, if D is a bracket generating sub-bundle of TM, then any pair of points can be connected by a piecewise smooth, immersed D-horizontal curve.

That we can choose the curve to be piecewise smooth follows from the definition of the orbit. The inclusion (1.2), in general, only give us a lower bound on the dimension of the orbit, but in some special cases, we are ensured equality. Let $\mathfrak{V}$ be a $C^\infty(M)$-submodule of $\operatorname{Vect}(M)$. Then $\mathfrak{V}$ is called locally finitely generated, if any point m has a neighborhood N such that $\mathfrak{V}|_N$ is spanned by a finite number of vector fields (the span is over $C^\infty(N)$).

Theorem 4. *Suppose $\mathfrak{F} \subseteq \operatorname{Vect}(M)$ is such that $\operatorname{Lie}\mathfrak{F}$ is a locally finitely generated $C^\infty(M)$-submodule of $\operatorname{Vect}(M)$. Then*

$$\operatorname{Lie}_m \mathfrak{F} = T_m\, \mathcal{O}_{m_0} \text{ for any } m \in \mathcal{O}_{m_0}, m_0 \in M.$$

1.3. Sub-Riemannian geometry and optimal curves in finite dimensions

Here, we will introduce sub-Riemannian manifolds and discuss how minimal curves in this geometry can be considered as solutions of an optimal control system.

Definition 1. *A sub-Riemannian manifold is a triple $(M, D, \mathbf{h})$, where M is a connected n-dimensional manifold, D is a sub-bundle of TM and $\mathbf{h}$ is a metric tensor on D.*

The pair $(D, \mathbf{h})$ is called a sub-Riemannian structure on M. D is called the horizontal sub-bundle or horizontal distribution. As we mentioned before, an absolutely continuous curve γ is said to be horizontal or D-horizontal if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for almost every t. In order to be able to have a meaningful notion of distance in this geometry, we want that every pair of points can be connected by a horizontal curve, i.e., we want the control system (D, inc) to be controllable, where inc is the inclusion $\mathrm{inc} : D \to TM$. The most common way to show that this indeed holds is to prove that D is bracket generating, which gives a sufficient condition for controllability by the Rashevskiĭ-Chow Theorem. Sometimes, the requirement that D is bracket generating is even included in the definition of a sub-Riemannian manifold. We will not follow this convention. For a necessary and sufficient condition of controllability, see [SuJu72].

Remark 3. There is a generalization of sub-Riemannian manifolds called rank-varying sub-Riemannian manifolds [ABS08]. Here, we still consider a general vector bundle D with a metric $\mathbf{h}$, but instead of the inclusion, we use a general linear bundle map $f : D \to TM$ that need not be injective on fibers. The only requirement is that the map $f_* : \Gamma(D) \to \Gamma(TM)$ is injective. Note that $E := f(D)$ can vary in rank. If $f_*\Gamma(D)$ is bracket generating, then the control system (D, f) is controllable.

The simplest nontrivial example of a rank varying sub-Riemannian manifold is the Grushin plane. Let $M = \mathbb{R}^2$ with coordinates (x, y), and define $D = M \times \mathbb{R}^2$, where we use u_1, u_2 for the coordinates of the fibers. The metric is given by $\mathbf{h}|_{(x,y)}(u_1, u_2) = u_1^2 + u_2^2$. Finally, we define f by

$$f(x, y, u_1, u_2) = u_1\partial_x|_{(x,y)} + u_2 x\partial_y|_{(x,y)}.$$

We will continue to construct a distance function relative to the sub-Riemannian structure $(D, \mathbf{h})$. For a pair of points $m_0, m_1 \in M$, let $\mathrm{AC}_D(m_0, m_1)$ denote the collection of all horizontal absolutely continuous curves $\gamma : I = [0, 1] \to M$ with square integrable derivative, that satisfy $\gamma(0) = m_0$ and $\gamma(1) = m_1$. Here, square integrability has to be defined relative to the metric $\mathbf{h}$, however, any other choice of metric on D gives us the same set of curves. Hence, $\mathrm{AC}_D(m_0, m_1)$ depends only on D. The associated distance function on M corresponding to the sub-Riemannian structure $(D, \mathbf{h})$ is given by

$$d_{C-C}(m_0, m_1) = \inf \left\{ \int_0^1 \mathbf{h}(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\, dt :\ \gamma \in \mathrm{AC}_D(m_0, m_1) \right\}.$$

This distance is called the Carnot-Carathéodory distance. If this distance admits only finite values, i.e., if $\mathrm{AC}_D(m_0, m_1)$ is nonempty for every pair of points $m_0, m_1 \in M$, then

(M, d_{C-C}) forms a metric space. The metric topology induced by the Carnot-Carathéodory distance coincides with the manifold topology when D is bracket generating. However, in contrast to usual Riemannian geometry, the map $m \mapsto d_{C-C}(m, m_0)$ is not smooth in general, and the Hausdorff dimension of the metric space (M, d_{C-C}) can be greater than the topological dimension n of the manifold.

Remark 4. The set $\mathrm{AC}_D(m_0, m_1)$ is bigger than the set $\mathcal{C}_1(m_0, m_1)$ defined relative to the control system (D, inc) in Section 1.1, as we only require the derivatives to be in L^2 rather than L^∞. Since we want to ensure finite values of the cost functional with respect to any cost function φ, L^∞ is preferred for a general control system. However, we only need L^2 for a D-horizontal curve to have finite length. Some authors do, however, prefer to require horizontal curves in sub-Riemannian geometry to be Lipschitz.

Sometimes we can extract the information of the Hausdorff dimension of d_{C-C} from the brackets of D. Assume that D is bracket generating. Let $D^1 = \Gamma(D)$ be the sections of D. Iteratively, define the following collections of vector fields

$$D^{j+1} = D^j + [D, D^j], \qquad j = 1, 2, \dots$$

and write D^j_m for the sub-space of T_mM obtained by evaluating the elements from D^j at m. The minimal integer r, such that $D^r_m = T_mM$, is called the step of D at m. The vector

$$(k_1(m), \dots, k_r(m)) := (\operatorname{rank} D^1_m, \dots, \operatorname{rank} D^r_m).$$

is called the growth vector at m. A distribution is called regular if the growth vector is independent of m.

Theorem 5 (Mitchell's measure theorem). [Mon02, Theorem 2.17] *Let $(D, \mathbf{h})$ be a sub-Riemannian structure, where D is a bracket generating, regular horizontal distribution with growth vector $(k_1, \dots, k_r)$. Then the Hausdorff dimension of d_{C-C} is equal to $k_1 + \sum_{j=2}^r j(k_j - k_{j-1})$.*

1.4. Sub-Riemannian length minimizers

Let us now look for sub-Riemannian length minimizers. We can view this as an optimal control problem where we try to minimize the functional

$$\mathrm{Length}(\gamma) = \int_0^1 \mathbf{h}(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt.$$

Equivalently, we can consider minimizing curves with respect to the energy functional $E(\gamma) = \frac{1}{2}\int_0^1 \mathbf{h}(\dot\gamma(t), \dot\gamma(t))\, dt$. Let us apply Pontryagin Maximum Principle to this problem. We will first look at the normal solutions corresponding to the pseudo-Hamiltonian function

$$\mathscr{H}_{-1}(m, u, p) = p(u) - \tfrac{1}{2}\mathbf{h}(u, u). \qquad u \in D_m, p \in T^*_mM. \tag{1.3}$$

Consider the unique linear bundle map $\beta : T^*M \to D$ determined by the condition $p(u) = \mathbf{h}(u, \beta_m p)$ for any $u \in D_m$. Use this to introduce a cometic on T^*M given by

$\mathbf{h}^*(p_1, p_2) = p_1(\beta_m p_2)$ for any $p_1, p_2 \in T_m^*M$. Note that this metric degenerates on the sub-bundle

$$\mathrm{Ann}(D) = \ker \beta = \{p \in T_m^*M : p(u) = 0 \text{ for any } u \in D_m, m \in M\}.$$

Using this, we can rewrite (1.3) as $\mathscr{H}_{-1}(m, u, p) = \mathbf{h}(u, \beta_m p - \frac{1}{2}u)$, which attains its maximum for $u = \beta_m p$. Define

$$H_{sR} = \mathscr{H}_{-1}(m, \beta_m p, p) = \tfrac{1}{2}\mathbf{h}^*(p, p).$$

By Remark 1, we can look for integral curves corresponding to this Hamiltonian.

Definition 2.

- *H_{sR} is called the sub-Riemannian Hamiltonian.*
- *Projections of solutions of the Hamiltonian system corresponding to H_{sR} are called normal minimizers or normal geodesics.*
- *Projections of solutions to the (time-dependant) Hamiltonian system corresponding to $\mathscr{H}_0(m, u, p) = p(u)$ for some control $t \mapsto u(t)$ satisfying the requirements of Theorem 1 are called abnormal curves.*

Notice in the case of abnormal curves, (γ_u, λ) must be in $\mathrm{Ann}(D)$ for almost all t in order to satisfy (iii). We call a horizontal curve from m_0 to m_1 a length minimizer if its length is equal to $d_{C-C}(m_0, m_1)$. It is called a local length minimizer if any sufficiently short arc is a length minimizer. Any length minimizer is a normal geodesic or an abnormal curve, but the converse does not hold in general.

Proposition 1. *(a) Normal geodesics are always local length minimizers. For small enough values of t_0, γ is the unique length minimizer connecting $\gamma(0)$ with $\gamma(t_0)$.* [Mon02, Theorem 1.14]

(b) A normal geodeisc is a smooth curve. [Ham90, Lemma 4.1]

(c) If D is bracket generating, then every $m_0 \in M$ has a neighborhood N, such that there is a minimizing curve connecting m_0 to any $m_1 \in N$. [Mon02, Theorem 1.17]

(d) If D is bracket generating, and M is complete relative to the metric d_{C-C} then any two points can be joined by a length minimizer. [Mon02, Theorem 1.18]

Remark 5. Proposition 1(c) does not mention anything about the existance of a unique geodesics locally. One might be misled by Proposition 1(a) to think that there are no arbitrarily close points connected by more than one normal geodesic, but in fact, any neighborhood N of a point m_0 may contain points that can be connected by more than one or even an infinite number of normal geodesics.

Remark 6. Let us describe the sub-Riemannian Hamiltonian formulation locally. For any vector field X, define the Hamiltonian function P_X by $P_X : p \mapsto p(X|_m), p \in T_m^*M$. Let N be a neighborhood such that D trivialize over N, and pick an orthonormal frame of vector fields $X_1, \ldots, X_k$ of D.

Then the sub-Riemannian Hamiltonian on N, from which the normal solutions can be obtained, can then be written as

$$H_{sR} = \frac{1}{2}\sum_{j=1}^{k} P_{X_j}^2.$$

An abnormal sub-Riemannian curve must be a solution to the system $\mathscr{H}_0(m, u(t), p) = \sum_{j=1}^k u_j(t)P_{X_j}(p)$ for some control $u(t) = \sum_{j=1}^k u_j(t)X_j$.

As can be seen from Proposition 1, normal sub-Riemannian geodesics have similar behavior as Riemannian geodesics. We will next focus on abnormal sub-Riemannian curves, and try to explain why they appear.

1.5. Abnormal curves

In order to give broader perspective of why there are abnormal curves in the sub-Riemannian case, let us introduce some other types of "bad curves" that might occur on M and on T^*M.

Let $\mathrm{AC}_D(m_0)$ be the collection of all horizontal absolutely continuous curves $\gamma : I = [0, 1] \to M$, which are square integrable and satisfy $\gamma(0) = m_0$. This is a Hilbert manifold modeled on $L^2(I, \mathbb{R}^k)$, where k is the rank of D (see [Mon02, Chapter 5.1] or [Mon95]). $\mathrm{AC}_D(m_0, m_1)$ can then be identified with the preimage $(\mathrm{end}_{m_0})^{-1}(m_1)$ of the mapping

$$\begin{array}{rccc} \mathrm{end}_{m_0} : & \mathrm{AC}_D(m_0) & \to & M \\ & \gamma & \mapsto & \gamma(1) \end{array}.$$

Hence, if γ is a regular point of end_{m_0}, by the implicit function theorem, $\mathrm{AC}_D(m_0, m_1)$ has the structure of a Hilbert submanifold of codimension n locally around γ. However, for critical or singular points of the endpoint map, that is, points where the differential is not surjective, this does not necessarily hold.

Definition 3. *An absolutely continuous horizontal curve γ with $\gamma(0) = m_0$ is called singular, if it is a singular point of* end_{m_0}.

As we can see from the definition, singular curves depend only on the sub-bundle D, not on the metric $\mathbf{h}$. When $D = TM$, there are no singular curves, and the space of all absolutely continuous curves connecting two points is a Hilbert manifold. In general, this does not hold.

We define another type of curves, which related to the fact that the canonical symplectic form σ may degenerate when restricted to $\mathrm{Ann}(D)$. We say that an absolutely continuous curve $t \mapsto (\gamma(t), \lambda(t))$ in $\mathrm{Ann}(D)$ is a characteristic of $\mathrm{Ann}(D)$ if it never intersects the zero section and satisfies

$$\sigma(\dot{\lambda}(t)), \widetilde{v}) = 0, \qquad \text{for any } t \text{ and any } \widetilde{v} \in T_{\lambda(t)}(\mathrm{Ann}(D)).$$

Hence, characteristics are curves that are horizontal to the sub-bundle of $T(\mathrm{Ann}(D))$ formed by the kernel of the map $\widetilde{v} \mapsto \sigma(\widetilde{v}, \cdot)|_{\mathrm{Ann}(D)}$.

Theorem 6. [Mon02, Theorem 5.3, Proposition 5.7] *The following are equivalent.*

- γ *is an abnormal curve.*
- γ *is a singular curve.*
- γ *is the projection of a characteristic of* $\mathrm{Ann}(D)$ *and its derivative is square integrable.*

Hence, we have three ways of viewing abnormal curve. Recall the definition of the step of D found before Theorem 5.

Theorem 7.
- *D is called strongly bracket generating if, for any point $m \in M$, and any $X \in D^1 = \Gamma(D)$ that does not vanish at m, we have*

$$T_m M = D_m + [X, D^1]|_m.$$

If D is strongly bracket generating, then there are no abnormal curves. [Mon02, Section 5.6]

- (The Goh condition) *An abnormal curve is calledis called strictly abnormal if it is not also normal. Let (γ, λ) be an abnormal solution to the optimal control problem, such that γ is a strictly abnormal length minimizer. Then for any $X_1, X_2 \in \Gamma(D)$, λ must satisfy*

$$\lambda(t)\left([X_1, X_2]\big|_{\gamma(t)}\right) = 0.$$

Consequently, if D is bracket generating of step 2 at every point, there are no strictly abnormal length minimizing curves [AgSa04, Chapter 20: 4.3, 5.2].

It is still an open question if all abnormal curves which are also minimizers, are smooth. Some results in this direction can be found, e. g., in [CJT06, GoKa95]. For more on the theory of singular curves, see [AgSar96, BrHs93, BoTr01, Mon02, Mon95].

1.6. Some examples of sub-Riemannian manifolds

Let us illustrate the theory of sub-Riemannian geometry with two examples. We omit most of the calculations, explaining only the general ideas. In both examples, $M = \mathbb{R}^3$ with coordinates x, y, z.

Example 1. Introduce a group structure on $M = \mathbb{R}^3$ by the formula

$$(x, y, z) \cdot (x_0, y_0, z_0) = \Big(x + x_0, y + y_0, z + z_0 + \tfrac{1}{2}(xy_0 - yx_0)\Big).$$

The resulting group is called the (first) Heisenberg group. A basis of left invariant vector fields is given by

$$X = \partial_x - \tfrac{1}{2} y \partial_z, \qquad Y = \partial_y - \tfrac{1}{2} x \partial_z \qquad Z = \partial_z.$$

Let D be the span of X, Y and define $\mathbf{h}$ such that X and Y form an orthonormal basis at each point. Since $[X, Y] = Z$, D is bracket generating, regular, with growth vector $(2, 3)$ and is even strongly bracket generating. Hence, there are no abnormal curves, so we only need to look for solutions of the system corresponding to the sub-Riemannian Hamiltonian

$$H_{sR} = \frac{1}{2}\left(P_X^2 + P_Y^2\right).$$

Note that if $\{\cdot, \cdot\}$ denotes the Poisson bracket, then $\{P_{X_1}, P_{X_2}\} = -P_{[X_1,X_2]}$. If (γ, λ) is a curve in T^*M, then it can be written as

$$\lambda(t) = P_X(t)\alpha_X|_{\gamma(t)} + P_Y(t)\alpha_Y|_{\gamma(t)} + P_Z(t)\alpha_Z|_{\gamma(t)}.$$

where $P_X(t) := P_X(\lambda(t))$, α_X satisfies

$$\alpha_X(X) = 1, \qquad \alpha_X(Y) = \alpha_Y(Z) = 0,$$

and we use similar definitions when X is replaced by Y or Z. To find the integral curves of H_{sR}, we need to solve the system

$$\dot{\gamma}(t) = P_X(t)X|_{\gamma(t)} + P_Y(t)Y|_{\gamma(t)},$$

$$\dot{P}_X = -P_Y P_Z, \quad \dot{P}_Y = P_X P_Z, \quad \dot{P}_Z = 0.$$

Then the normal geodesic with initial condition $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ and $(P_X(0), P_Y(0), P_Z(0)) = (r\cos\theta, r\sin\theta, c)$ is given by

$$\gamma(t) = (x(t), y(t), z(t)),$$

$$x(t) = \frac{r}{c}(\sin(ct+\theta) - \sin\theta) + x_0, \qquad y(t) = -\frac{r}{c}(\cos(ct+\theta) - \cos\theta) + y_0$$

$$z(t) = \frac{r}{2c}(rt - \frac{r}{c}\sin ct - y_0(\sin(ct+\theta) - \sin\theta) - x_0\cos(ct+\theta)) + z_0.$$

or if $c = 0$, by $(x(t), y(t), z(t)) = (tr\cos\theta + x_0, tr\sin\theta + y_0, z_0)$. Notice that for any arbitrary small value of $z_1 > 0$, there is an uncountable family of normal geodesics connecting the point $\gamma(0) = (0,0,0)$ with the point $\gamma(1) = (0,0,z_1)$, which are obtained by choosing $r = 2\sqrt{\pi z_1}$ and $c = 2\pi$. Explicitly,

$$\gamma(t) = \sqrt{\frac{z_1}{\pi}}\Big(\sin(2\pi t+\theta) - \sin\theta, \cos(2\pi t+\theta) - \cos\theta, \sqrt{\pi z_1}\left(t - \tfrac{1}{2\pi}\sin 2\pi t\right)\Big).$$

Example 2. Let us consider D as the sub-bundle spanned by the vector fields

$$X = \partial_x - \tfrac{1}{2}y^2\partial_z, \qquad Y = \partial_y.$$

This is called the Martinet distribution and it is bracket generating, but not strongly bracket generating. It is not regular, since the growth vector is $(2,2,3)$ at points where $y = 0$, and $(2,3)$ otherwise. $\mathrm{Ann}(D)$ is spanned by the one-form

$$\theta = dz - \tfrac{1}{2}y^2 dx.$$

If we give $\mathrm{Ann}(D)$ coordinates by letting (x, y, z, p_0) represent an element p that is on the form $p = p_0\theta|_{(x,y,z)}$, then the Louville one-form ϑ on T^*M restricted to $\mathrm{Ann}(D)$ is simply $p_0\theta$. Hence, we have

$$\sigma|_{\mathrm{Ann}(D)} = -d\vartheta|_{\mathrm{Ann}(D)} = -dp_0 \wedge \theta - p_0 d\theta.$$

and so

$$\begin{aligned}\sigma|_{\mathrm{Ann}(D)}(\dot\lambda) &= -\dot p_0\theta + (\dot z - \tfrac{1}{2}y^2\dot x)dp_0 - p_0 y(\dot y dx + \dot x dy) \\ &= y(p_0\dot y + \tfrac{1}{2}y\dot p_0)dx - p_0 y\dot x dy + \dot p_0 dz + (\dot z - \tfrac{1}{2}y^2\dot x)dp_0.\end{aligned}$$

We immediately observe that $\dot p_0 = 0$ and $\dot z - \frac{1}{2}y^2\dot x = 0$, the latter being the horizontality condition. Furthermore, if $y \neq 0$, then $\dot x$ and $\dot y$ must be equal to 0, which gives us just a constant curve. If $y = 0$ along the curve, then $\dot y(t) = 0$ also, but there are no restrictions to $\dot x$, so any abnormal curves can be written on the form

$$(x(t), 0, z_0).$$

It actually holds that such a non-constant curve will be a local length minimizer relative to $(D, \mathbf{h})$, where D is the Martinet distribution and $\mathbf{h}$ is any metric on D, see [Mon02, Theorem 3.3]

1.7. Sub-Lorentzian geometry

Let M be a connected manifold and let D be a sub-bundle of TM. We will still use the term horizontal to refer to an absolutely continuous curve γ satisfying $\dot\gamma(t) \in D_{\gamma(t)}$. Then we call the triple $(M, D, \mathbf{h})$ a sub-Lorentzian manifold, if $\mathbf{h}$ is a pseudo-metric on D of index 1. Such a metric divides D into 3 disjoint sets. We call an element $v \in D$,

- timelike if $\mathbf{h}(v, v) < 0$,
- lightlike or null if $\mathbf{h}(v, v) = 0$,
- spacelike if $\mathbf{h}(v, v) > 0$,
- causal or nonspacelike if $\mathbf{h}(v, v) \leq 0$.

We will use the same terminology for horizontal curves, in the sense that a horizontal curve γ is called timelike (resp. light like, space like, causal) if $\dot\gamma(t)$ is a timelike (resp. light like, space like, causal) vector for almost every t.

A time-orientation on M is a chosen vector field $Z \in \Gamma(D)$, such that $Z|_m$ is timelike for every $m \in M$. Not every sub-Lorentzian manifold permits a time-orientation. This is possible if and only if the space

$$\{v \in D : \ v \text{ is timelike}\},$$

has two components. A causal vector $v \in T_mM$ is called future directed if $\mathbf{h}(Z|_m, v) < 0$, and past directed if $\mathbf{h}(Z|_m, v) > 0$. We use similar definitions for horizontal curves.

For a given point $m_0 \in M$, we define sets of casual and timelike futures and pasts relative to the metric $\mathbf{h}$. The timelike future $\mathcal{I}^+(m_0, \mathbf{h})$ (resp. the timelike past $\mathcal{I}^-(m_0, \mathbf{h})$) of m_0 is the set of all points $m_1 \in M$, such that there is a horizontal, timelike future directed (resp. past directed) curve γ, with $\gamma(0) = m_0$ and $\gamma(1) = m_1$. Similarly, the causal future $\mathcal{J}^+(m_0, \mathbf{h})$ or causal past $\mathcal{J}^-(m_0, \mathbf{h})$ consist of all points which can be reached by a future or past directed causal curve, respectively.

We define the length of a horizontal causal curve $\gamma : [0, 1] \to M$ by

$$\text{Length}(\gamma) = \int_0^1 |\mathbf{h}(\dot\gamma(t), \dot\gamma(t))|^{1/2}\, dt.$$

The sub-Lorentzian distance is defined by

$$d(m_0, m_1) = \begin{cases} \sup\limits_{\gamma} \text{Length}(\gamma), & \text{if } m_1 \in \mathcal{J}^+(m_0, \mathbf{h}), \\ 0, & \text{otherwise.} \end{cases}$$

The supremum is taken over all horizontal future directed causal curves from m_0 to m_1. Similarly to the Lorentzian distance, the sub-Lorentzian distance satisfies the reverse triangle inequality, and may not be very well behaving. For instance, if there is a timelike loop trough a point $m \in M$, then $d(m, m) = \infty$.

A curve $\gamma : [0, 1] \to M$ is called a length maximizer, if $\text{Length}(\gamma) = d(\gamma(0); \gamma(1))$. Similarly, a curve γ is called a relative maximizer with respect to an open set N in M, if $\gamma([0, 1]) \subseteq N$ and $\text{length}(\gamma) = \sup_{\tilde\gamma} \ell(\tilde\gamma)$, where the supremum is taken over all horizontal future directed causal curves contained in N, connecting $\gamma(0)$ and $\gamma(1)$.

By using the maximum principle, for D bracket generating, we know that all relative maximizers are either abnormal in the sense of Section 1.5 and normal sub-Lorentzian geodesics [Gro04], and that the relative maximizers always exist locally. By normal sub-Lorentzian geodesics γ we mean projections of solutions to the Hamiltonian system with Hamiltonian function $H_{sL}(p) = \frac{1}{2}\mathbf{h}^*(p, p)$, where $\mathbf{h}^*$ is the cometic of $\mathbf{h}$, defined similarly as in Section 1.4. If N is a neighborhood such that there exist local vector fields $X_1, \dots, X_{k-1}$ along with the time orientation Z form an orthonormal basis for $\mathcal{H}\,|_N$, then H_{sL} can be written as $H_{sL} = -\frac{1}{2}P_Z^2 + \frac{1}{2}\sum_{j=1}^k P_{X_j}$, locally on N.

It is much more complicated to obtain general results about the existence of length maximizers in sub-Lorentzian geometry, than for length minimizers in Riemannian or sub-Riemannian geometries. The most common sufficient condition for the global existence of maximizing curves is the following

Definition 4. *A sub-Lorentzian manifold $(M, D, \mathbf{h})$ is called globally hyperbolic if*

- $(M, D, \mathbf{h})$ *is strongly causal, that is, any point m has a neighborhood N, such that any timelike curve that leaves N never returns.*
- *for any $m_0, m_1 \in M$, $\mathcal{J}^+(m_0, \mathbf{h}) \cap \mathcal{J}^-(m_1, \mathbf{h})$ is compact.*

If $(M, D, \mathbf{h})$ is globally hyperbolic, then for any pair of points $m_0, m_1 \in M$ with $0 < d(m_0, m_1) < \infty$, there exists a length maximizer from m_0 to m_1.

1.8. Historical notes

The Maximum Principle in control theory was the results of an effort by a group in automatic control at the Steklov Mathematical Institute headed by Lev Semenovich Pontyagin. It was first published in [PBGM61], and later translated into English in [PBGM62]. Being part of the Steklov Institue mission at the time to deliver applied research, in particular results that could be useful for aircraft dynamics, the Maximum Principle has since then been considered as the birth of modern optimal control theory. It is still one of the most important results for applications of mathematics to real world problems (see e.g. [CLS03, GCFT08, SeSy87]).

Control systems were originally described through differential equations, but a geometric point of view has in later years been very fruitful. One of the most important papers for introducing geometry to a wider class of control systems is by Roger W. Brockett [Bro84] called quasi-Riemannian control system. Later, Robert S. Strichartz introduced sub-Riemannian geometry [Str86, Str86cor], which corresponds to input linear systems without drift, that is systems locally on the form

$$\dot{\gamma}_u = \sum_{j=1}^{k} u_j f_j|_{\gamma_u(t)}. \tag{1.4}$$

Here, f_j are linearly independent vector fields and k is generally less than the dimension of our state space M. Strichartz, in his original definition, also required that the distribution spanned by the vector fields $f_1, \ldots, f_k$ should be bracket generating, which was already known to be a sufficient condition for controllability of this optimal control system by earlier results obtained independently by Wei-Liang Chow [Cho39] and Petr Konstantinovich Rashevskiĭ [Ras38].

A geometric interpretation of systems where the vector fields $f_1, \ldots, f_k$ in (1.4) are not linearly independent can be found in [ABS08], where a definition of rank-varying sub-Riemannian manifolds can be found. Also, input linear systems with a drift term, that is systems that can locally on the form

$$\dot{\gamma}_u = \sum_{j=1}^{k} u_j f_j|_{\gamma_u(t)} + f_0|_{\gamma_u(t)},$$

can be considered in the sub-Lorentzian framework, where we let the drift term f_0 represent the time-orientation. For details, see [Gro09, Section 6].

In addition to the interest from the point of view of control theory, part of the motivation for sub-Riemannian geometry comes from a result in 1967 by Lars Hörmander which connects bracket generating property and hypoellipticity. Let us consider a collection of k vector fields $f_1, \ldots, f_k$ and an associated second order operator

$$\Delta = \sum_{j=1}^{k} f_j^2 + L,$$

where L is some first order differential operator. Then, if $\mathfrak{F} = \text{span}\{f_1, \ldots, f_k\}$ is bracket generating, Δ is hypoelliptic [Hör67, Theorem 1.1]. It was hoped that the relationship between sub-Riemannian geometry and the "sub-Laplacian" Δ might be similar to that of Riemannian geometry and Laplacians. It was indeed shown by Rémi Léandre that if $p_t(m_1, m_2)$ is the heat kernel corresponding to the operator $(\partial_t - \Delta)$, then, if the limit exists, we have

$$-\lim_{t\to 0+} 2t \log p_t(m_1, m_2) = d_{C-C}(m_1, m_2),$$

where $d_{C-C}(m_1, m_2)$ is the Carnot-Carathéodory distance. See [Léa97a] for the upper bound and [Léa87b] for the lower bound.

An intrinsic formulation for the sub-Laplacian associated to a sub-Riemannian structure was not formulated until [ABGR09], where the heat kernel in terms of eigenfunctions was given for a wide class of sub-Riemannian structures on Lie groups. There also exist integral representations of heat kernels in some special cases (see e.g. [BGG00, BaBo09, Bon11]), but no general approach is known so far.

2. Selected topics of differential geometry

This chapter we will deals with selected topics of differential geometry, which will become useful when presenting the results in Chapter 4. All manifolds in this chapter are finite dimensional and we will continue the notation and conventions from the previous chapter.

In what follows, we will deal with several types of connections. We will reserve the term affine connection on M for a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, which is $C^\infty(M)$-linear in the first coordinate, $\mathbb{R}$-linear in the second and satisfies the Leibnitz condition $\nabla_X fY = X(f)Y + f\nabla_X Y$, for $f \in C^\infty(M), X, Y \in \Gamma(TM)$.

For more details on this material, see [Jos05, Sha97] and [Spi99].

2.1. Submersions and Riemannian submersions

Consider a submersion $\pi : Q \to M$ between (finite dimensional) manifolds Q and M, that is, a map such that $\pi_*|_q$ is surjective for any $q \in Q$. The vertical bundle corresponding to this submersion is defined as $\mathcal{V} = \ker \pi_*$. An Ehresmann connection $\mathcal{H}$ is a sub-bundle of TQ, such that

$$TQ = \mathcal{H} \oplus \mathcal{V}\,.$$

Relative to $\mathcal{H}$, we can define horizontal lifts since the map $\pi_*|_{\mathcal{H}_q} : \mathcal{H}_q \to T_{\pi(q)}M$ is invertible. A horizontal lift of a vector $v \in T_mM$ to $q \in \pi^{-1}(m)$, which we will denote by $h_q v$, is the unique vector in $\mathcal{H}_q$ that is projected to v under π_*. Also, for any vector field X on M, the horizontal lift hX is given by

$$hX|_q = h_q X|_{\pi(q)}.$$

Finally, we say that $\widetilde{\gamma}$ is a horizontal lift of an absolutely continuous curve γ if $\widetilde{\gamma}$ is $\mathcal{H}$-horizontal and is projected to γ by π. Clearly, this curve is uniquely determined by its initial condition.

The connection form corresponding to $\mathcal{H}$ is the projection $\mathrm{pr}_{\mathcal{V}}$ to $\mathcal{V}$ with respect to the splitting $\mathcal{H} \oplus \mathcal{V}$. For any section $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathcal{H})$, the curvature of $\mathcal{H}$ is determined by $\mathcal{R}(\widetilde{X}, \widetilde{Y}) = \mathrm{pr}_{\mathcal{V}}[\widetilde{X}, \widetilde{Y}]$ (some authors prefer to define the curvature as the negative of this expression). The definition is usually extended to all vector fields on Q by the formula $\mathcal{R}(\widetilde{X}, \widetilde{Y}) = \mathrm{pr}_{\mathcal{V}}[\mathrm{pr}_{\mathcal{H}} \widetilde{X}, \mathrm{pr}_{\mathcal{H}} \widetilde{Y}]$. Notice the relation

$$[hX, hY] = h[X, Y] + \mathcal{R}(hX, hY) \text{ for any } X, Y \in \mathrm{Vect}(M).$$

Consider the case when Q and M are furnished with respective Riemannian metrics $\widetilde{\mathbf{g}}$ and $\mathbf{g}$. We require that $\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to $\widetilde{\mathbf{g}}$. Then $\pi : (Q, \widetilde{\mathbf{g}}) \to (M, \mathbf{g})$ is called a Riemannian submersion, if

$$\widetilde{\mathbf{g}}(\widetilde{v}_1, \widetilde{v}_2) = \mathbf{g}(\pi_* \widetilde{v}_1, \pi_* \widetilde{v}_2) \text{ for any } \widetilde{v}_1, \widetilde{v}_2 \in \mathcal{H}_q \,.$$

or equivalently

$$\mathbf{g}(v_1, v_2) = \widetilde{\mathbf{g}}(h_q v_1, h_q v_2) \text{ for any } v_1, v_2 \in T_m M, q \in \pi^{-1}(m).$$

Let us use $\widetilde{\nabla}$ and ∇ for the respective affine Levi-Civita connections of $\widetilde{\mathbf{g}}$ and $\mathbf{g}$. Then for any $X, Y \in \mathrm{Vect}(M)$, we have [O'Ne66]

$$\widetilde{\nabla}_{hX} hY = h \nabla_X Y + \frac{1}{2} \mathcal{R}(hX, hY).$$

In particular, a curve γ is a (Riemannian) geodesic on M, if and only if, any horizontal lift is a Riemannian geodesic in Q that is also $\mathcal{H}$-horizontal.

2.2. Principal Ehresmann connections

Let $\pi : Q \to M$ be a principal G-bundle, where the action of the Lie group G is on the right. Denote this action by r_a for $a \in G$. We again denote the vertical bundle by $\mathcal{V} = \ker \pi_*$. It is spanned by the vector fields corresponding to the infinitesimal action of the Lie algebra $\mathfrak{g}$ of G. These vector fields are defined such that for any $A \in \mathfrak{g}$, the associated vector fields is given by formula

$$v(A)|_q = \left. \frac{d}{dt} \right|_{t=0} r_{\exp(At)}(q).$$

An Ehresmann connection $\mathcal{H}$ on $\pi : Q \to M$ is called principal, if it is invariant under the action of G, that is, $r_{a*}|_q \mathcal{H}_q = \mathcal{H}_{qa}$. The corresponding principal connection form ω, is the $\mathfrak{g}$-valued one-form determined uniquely by the two properties

$$\ker \omega = \mathcal{H}, \qquad \text{and} \qquad \omega(v(A)|_q) = A \text{ for any } A \in \mathfrak{g}, q \in Q.$$

It allows us to define a corresponding curvature form as the two-form by $\Omega(\widetilde{X}, \widetilde{Y}) := d\omega(\widetilde{X}, \widetilde{Y}) + [\omega(\widetilde{X}), \omega(\widetilde{Y})]$, or alternatively $\Omega(\widetilde{X}, \widetilde{Y}) = -\omega(\mathcal{R}(\widetilde{X}, \widetilde{Y}))$, where $\mathcal{R}$ is the curvature of $\mathcal{H}$ defined as in Section 2.1.

2.3. Cartan's moving frame

For two finite dimensional vector spaces V and $\widehat{V}$, we will use the notation $\mathrm{GL}(V, \widehat{V})$ for the space of all invertible linear maps between these vector spaces. For an n-dimensional manifold M, define the frame bundle $\mathcal{F}(M) \to M$ as the principal $GL(n)$-bundle, such that the fiber over $m \in M$, is

$$\mathcal{F}_m(M) = \mathrm{GL}(\mathbb{R}^n, T_m M).$$

Any map $f \in \mathrm{GL}(\mathbb{R}^n, T_m M)$ can be identified with a choice of basis $\{f_1, \dots, f_n\}$ of $T_m M$. The correspondence is given

$$f(\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{1 \text{ in the } j\text{-th coordinate}}) = f_j. \tag{2.1}$$

Let us choose a principle Ehresmann connection $\mathcal{H}$ on the principal bundle $\pi : \mathcal{F}(M) \to M$. The correspondence is given by declaring the curve $t \mapsto f(t)$ in $\mathcal{F}(M)$ to be $\mathcal{H}$-horizontal if $f_1(t), \dots, f_n(t)$ is a parallel frame along $\gamma(t) = \pi(f(t))$. Because we have more structure on the frame bundle than on a general principal bundle, we also have a special $\mathbb{R}^n$-valued one-form θ called the solder form or tautological one-form. It is defined by

$$\theta|_f(\widetilde{v}) = f^{-1}(\pi_* \widetilde{v}), \qquad \widetilde{v} \in T_f\, \mathcal{F}(M).$$

Hence, we have two one-forms θ and ω on $\mathcal{F}(M)$ that have kernels $\mathcal{V}$ and $\mathcal{H}$ respectively. They are connected by the equations

$$\Theta = d\theta + \omega \wedge \theta, \qquad \Omega = d\omega + \omega \wedge \omega.$$

The two-form Θ is called torsion.

2.4. Geodesic curvatures

Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension n and let $\gamma : [0, \tau] \to M$ be a curve of class C^{n+1}. We say that γ is C^k-regular, where $1 \leq k \leq n$, if for every t,

$$\left\{ \dot{\gamma}(t), \frac{D}{dt}\dot{\gamma}(t), \dots, \frac{D^{k-1}}{dt^{k-1}}\dot{\gamma}(t) \right\}.$$

Here, $\frac{D}{dt}$ is the covariant derivative along γ with respect to the Levi-Civita connection on M. Assume that γ is C^1-regular. Then a reparametrization of γ with respect to arc length will still be of class C^{n+1}. Let us, therefore, assume that γ is parametrized by arc length. We can then define the Frenet frame and geodesic curvatures of $\gamma(t)$ by the following procedure.

- Define the unit vector field $v_1(t) = \dot{\gamma}(t)$ along $\gamma(t)$, and let $\kappa_1(t) = \mathbf{g}\left(\frac{D}{dt} v_1(t), \frac{D}{dt} v_1(t)\right)^{1/2}$.
- Assuming $\kappa_1(t)$ never vanishes, there is a unique unit vector field $v_2(t)$ along $\gamma(t)$ satisfying $\frac{D}{dt} v_1(t) = \kappa_1(t) v_2(t)$.

- Inductively, assume that $\kappa_i(t)$ and $v_{i+1}(t)$ are well defined for $i < j$, where $j \leq n$ is fixed. Denote

$$\kappa_j(t) = \mathbf{g}\left(\tfrac{D}{dt} v_j + \kappa_{j-1}(t)v_{j-1}, \tfrac{D}{dt} v_j + \kappa_{j-1}(t)v_{j-1}\right)^{1/2}$$

If $\kappa_j(t)$ never vanishes, define v_{j+1} to be the unit vector field along $x(t)$ satisfying

$$\tfrac{D}{dt} v_j(t) + \kappa_{j-1}(t)v_{j-1}(t) = \kappa_j(t)v_{j+1}(t). \tag{2.2}$$

It is easy to check that $\mathbf{g}(v_i(t), v_j(t)) = \delta_{i,j}$ for all i, j.

The unit vector field $v_j(t)$ in (2.2) is called the j-th Frenet vector field of γ. The function $\kappa_j(t)$ is called the $j-$th geodesic curvature of γ. Clearly, the j-th Frenet vector field is well defined, if and only, if γ is C^j-regular.

If M is orientated, we can define all Frenet vectors along γ, only requiring that the curve is C^n and C^{n-1}-regular. We keep the definition of geodesic curvatures $\kappa_1, \ldots, \kappa_{n-2}$ and the Frenet vector fields $v_1, \ldots, v_{n-1}$ defined above. However, for the last Frenet vector field, we define v_n as the unique unit vector field such that $v_1(t), \ldots, v_n(t)$ is a positively oriented orthonormal basis for every t. The last geodesic curvature κ_{n-1} is subsequently defined as

$$\kappa_{n-1}(t) = \mathbf{g}\left(v_n(t), \tfrac{D}{dt} v_{n-1}(t) + \kappa_{n-2}(t)v_{n-2}(t)\right).$$

This curvature may admit both positive and negative values. An advantage of this definition is that it includes the orientation of M into the definition, since the sign of κ_n changes if we switch orientation. For this reason, we will use the term oriented geodesic curvatures, when we use this definition.

Given an initial point m_0 in a manifold M and geodesic curvatures $(\kappa_1, \ldots, \kappa_{n-1})$, a curve in M is uniquely determined by the starting point m_0, the curvatures and an initial configuration of the Frenet frames. Given any choice of initial conditions, a curve with the given curvatures always exists for a short time, and for all time if M is complete. Furthermore, we have the following result

Theorem 8. [Spi99, Corollary 4]) *Let M be a complete, simply connected manifold of constant sectional curvature. Then the geodesic curvatures $\kappa_1, \ldots, \kappa_{n-1}$ determine a curve in M uniquely up to isometry.*

If we pick an orientation of M and use it to define oriented geodesic curvatures, then they define a curve up to an orientation preserving isometry.

3. Infinite dimensional manifolds and the convenient calculus

The introduction of convenient vector spaces to have a well defined calculus on a wider class of infinite dimensional locally convex vector spaces, is an idea proposed by Andreas Kriegl and Peter W. Michor. The entire calculus is based on the following two ideas

- The concept of a differentiable curve $\gamma : \mathbb{R} \to V$ into a locally convex vector space, is without any difficulties.
- The convergence of the limit

$$\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h},$$

  really depends on the bornology of the vector space, that is the collection of its bounded sets.

Constructing their calculus on smooth curves, they where able to introduce the convenient calculus, which generalizes the concepts of Banach and Frechét spaces.

An overview of most of the aspects of the theory is found in [KrMi97b]. For shorter introductions to the theory, see [KrMi97a] or [Mic06].

3.1. Convenient vector spaces

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A topological vector space V over $\mathbb{K}$ is called locally convex if it is Hausdorff, addition and scalar multiplications are continuous, and 0 has a basis of absolutely convex absorbent sets. A set B in V is called bounded, if any neighborhood U of 0 absorbs B. This means that for every U there is some scalar $\lambda \in \mathbb{K}$ such that $B \subset \lambda U$.

A curve $\gamma : \mathbb{R} \to V$ is called smooth if all its derivatives exist and are continuous. We write $C^\infty(\mathbb{R}, V)$ for the space of all smooth curves on V. This set depend on the bornology of V rather than the topology, in the sense that if we change the topology of V to a different one with the same bounded sets, the same curves into V are smooth [KrMi97b, Chapter I, Section 1.2].

Definition 5. [KrMi97b, Chapter I, Theorem 2.14] *A locally convex vector space is called convenient or c^∞-complete if the following equivalent definitions hold.*

(i) For any curve $\gamma : \mathbb{R} \to V$, the Riemann integral $\int_0^1 \gamma(t)\, dt$ exists in V.

(ii) *For any* $\gamma_1 \in C^\infty(\mathbb{R}, V)$*, there is a curve* $\gamma_2 \in C^\infty(\mathbb{R}, V)$ *satisfying* $\dot{\gamma}_2 = \gamma_1$*.*

(iii) V *is* c^∞*-closed (see definition below) in any locally convex vector space.*

(iv) *Let* V^* *be the space of continuous functionals on* V*. Then a curve* $\gamma : \mathbb{R} \to V$ *is smooth if and only if* $\alpha \circ \gamma : \mathbb{R} \to \mathbb{R}$ *is smooth for any* $\alpha \in V^*$*.*

(v) *A sequence* $\{v_n\}$ *is called Mackey-Cauchy if there is some sequence* $t_{nm} \to \infty$ *such that* $t_{nm}(v_n - v_m) \to 0$*. Any such sequence converges in* V*.*

A convenient vector space remains convenient if we change the topology to a different topology with the same bornology. One of the possibilities is the c^∞-topology which is the finest topology such that the maps in $C^\infty(\mathbb{R}, V)$ remain continuous. If V is a Frechét space, then the usual topology and the c^∞-topology coincide. In general, this is not the case, and it may even happen that V with the c^∞-topology is not a topological vector space. An alternative is the bornologification, which is the finest locally convex topology on V with the same bounded sets.

Let V and W be two convenient vector spaces. If U is a c^∞-open set in V, then a function $f : U \to W$ is smooth or C^∞, if for any $\gamma \in C^\infty(\mathbb{R}, U)$ we have $f \circ \gamma \in C^\infty(\mathbb{R}, W)$.

Proposition 2. *(a) In the case of Frechét spaces, this definition of smoothness coincides with the definition given by the Gâteaux derivative.*

(b) *Multilinear maps are smooth if and only if they are bounded.*

(c) *If* U *is a* c^∞*-open subset of* V *and* $f : U \subseteq V \to W$ *is a smooth map, then the derivative* $f_* : U \times V \to W$ *and the mapping* $f_* : U \to L(V, W)$ *are smooth. Here,* $L(V, W)$ *is the space of all bounded linear maps from* V *to* W*.*

(d) *The chain rule holds.*

(e) $C^\infty(U, W)$ *is also a convenient vector space. The structure is given by the inclusions*

$$\begin{array}{ccccc} C^\infty(U, W) & \to & \prod_{\gamma \in C^\infty(\mathbb{R}, U)} C^\infty(\mathbb{R}, W) & \to & \prod_{\gamma \in C^\infty(\mathbb{R}, U), \alpha \in W^*} C^\infty(\mathbb{R}, \mathbb{R}) \\ f & \mapsto & (f \circ \gamma)_\gamma & \mapsto & (\alpha \circ f \circ \gamma)_{\gamma, \alpha} \end{array}.$$

(f) *If* $U_j \subseteq V_j, j = 1, 2$ *are* c^∞*-open subsets, then identification*

$$C^\infty(U_1, C^\infty(U_2, W)) \cong C^\infty(U_1 \times U_2, W),$$

is a linear diffeomorphism of convenient vector spaces.

(g) *Any linear map* $f : V_1 \to C^\infty(U_2, W)$ *is smooth (which is equivalent to bounded by (b)), if and only if,* $ev_v \circ f : V_1 \to W$ *is smooth for every* $v \in U_2$*, where* ev_v *is the evaluation map.*

We remark that (f) is sometimes referred to as the exponential law, which is fundament for variational calculus, as it allows us to identify a smooth curve $s \mapsto \gamma^s$ into the space $C^\infty(\mathbb{R}, W)$ with an element $(t, s) \mapsto \gamma^s(t)$ in $C^\infty(\mathbb{R}^2, W)$.

3.2. Manifolds modeled on infinite dimensional vector spaces

We consider manifolds M modeled on c^∞-open subsets of convenient vector spaces, defined similarly to finite dimensional manifolds in terms of charts and atlases. By this we mean the following. Let M be a set.

- For some subset $U \subseteq M$, a chart (ξ, U) is a bijective map $\xi : U \to \xi(U) \subseteq V_U$, where V_U is a convenient vector space, and $\xi(U)$ is a c^∞-open subset of V_U.
- A C^∞-atlas is a collection of charts (ξ_α, U_α) such that $\{U_\alpha\}_\alpha$ cover M, and the mappings
$$\xi_\beta \circ \xi_\alpha^{-1} \colon \xi_\alpha(U_\alpha \cap U_\beta) \to \xi_\beta(U_\alpha \cap U_\beta),$$
are smooth.
- Two C^∞-atlases are equivalent if their union is also an C^∞-atlas. An equivalent class of C^∞-atlases is called a smooth structure. M furnished with a smooth structure is a manifold.
- A map $f : M \to Q$ between two manifolds is called smooth if for any chart (ξ, U) on M and $(\widetilde{\xi}, \widetilde{U})$ on Q the map $\widetilde{\xi} \circ f \circ \xi^{-1}$ is smooth. From this definition, it follows that f is smooth if and only if $f \circ \gamma$ is a smooth curve in Q for any $\gamma \in C^\infty(\mathbb{R}, M)$.

We introduce a topology on M by requiring all charts to be homeomorphisms. We also want to require M to be Hausdorff. However, there are three concepts of a Hausdorff space, that coincide in finite dimensions, but not necessarily in an infinite dimensional manifold.

(a) The diagonal is closed in the manifold $M \times M$ (the topology induced from the product manifold structure can be weaker than the product topology).

(b) M is a Hausdorff topological space, that is, the diagonal is closed in the product topology on $M \times M$.

(c) Elements in $C^\infty(M, \mathbb{R})$ separate points.

We have implications (a) $\Leftarrow$ (b) $\Leftarrow$ (c). We will assume that manifolds satisfy property (c) which is called smoothly Hausdorff. All infinite dimensional manifolds that we will work with, will be of this type.

Let TM denote the tangent bundle consisting of equivalence classes of smooth curves constructed similarly as in finite dimensions. This bundle is sometimes referred to as the kinematic tangent bundle, due to the fact that we will get a different, in general larger, bundle, if we construct a tangent bundle from the point of view of derivations of function germs. See [KrMi97b, Chapter VI, Section 28] for details.

3.3. Regular Lie groups

We will use the term Lie group, for a (finite or infinite dimensional) manifold G modeled on c^∞-open sets in convenient vector spaces, with a group structure such that multiplication and inversion are smooth. Use $\mathbf{1}$ for the group identity. Note that for infinite dimensional manifolds, it still holds true that if X_i is f-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$. Hence, brackets of left- and right-invariant vector fields remain respectively left- or right-invariant. Since these types of vector fields are uniquely determined by their value at $\mathbf{1}$, we can use either the left- or the right-invariant vector fields to induce a Lie algebra structure on $\mathfrak{g} := T_{\mathbf{1}}G$. We will use the left-invariant structure, as is most common.

A difficulty one has when working with manifolds modeled on c^∞-open subsets of convenience vector spaces, is that it is not always possible to define a local flow of a vector field. Because of this, we cannot be sure that we have a well defined exponential map. We therefore need an addition requirement for the Lie group G.

We use the symbol ℓ_a to denote the left multiplication by an element $a \in G$. Associated to the left multiplication, we define a $\mathfrak{g}$-valued one-form on G. This one-from is called the left Maurer-Cartan form κ^ℓ, given by the formula

$$\kappa^\ell(v) = (\ell_{a^{-1}})_* |_a v, \qquad v \in T_a G.$$

To any smooth curve $\gamma \colon \mathbb{R} \to G$ we associate a smooth curve $u \in C^\infty(\mathbb{R}, \mathfrak{g})$ given by

$$u(t) = \kappa^\ell(\dot{\gamma}(t)), \qquad t \in \mathbb{R}\,.$$

called the left logarithmic derivative of γ. If the correspondence also goes the other way around, that is, if any curve $u \in C^\infty(\mathbb{R}, \mathfrak{g})$ can be integrated to a smooth curve in G, we call the group regular

Definition 6. [KrMi97a, Section 5.3] [Mil84, Definition 7.6] *A Lie group G is called regular if*

(a) Any smooth curve $u \in C^\infty(\mathbb{R}, \mathfrak{g})$, is the left logarithmic derivative of some curve $\gamma \colon \mathbb{R} \to G$, with $\gamma(0) = \mathbf{1}$,

(b) The mapping

$$\begin{array}{ccc} C^\infty(\mathbb{R}, \mathfrak{g}) & \to & G \\ u & \mapsto & \gamma(1) \end{array}$$

is smooth. Here γ is a solution to the equation $\kappa^\ell(\dot{\gamma}(t)) = u(t), t \in \mathbb{R}$ with the initial data $\gamma(0) = \mathbf{1}$.

So far, there has been no known examples of non-regular Lie groups. We remark the following about the definition of regular Lie groups.

- In any Lie group, not necessarily regular, a solution to the initial value problem

$$\kappa^\ell(\dot{\gamma}(t)) = u(t), \qquad \gamma(0) = a, \tag{3.1}$$

is unique, if it exists. Hence, the mapping in Definition 6 (b) is well defined. Clearly, (a) holds if and only if (3.1) always has a solution, since we can use left multiplication by a to move the solution starting at the identity.

- Here we have used the left multiplication to define regular Lie groups. However, we could have used right multiplication r_a and the right Maurer-Cartan form $\kappa^r(v) = r_{-a*}|_a v$ instead. The definition we then obtain is equivalent to the one using left multiplication.

- By identifying elements of $\mathfrak{g}$, with the constant curves in $C^\infty(\mathbb{R}, \mathfrak{g})$, we always know that there is a smooth exponential map $\exp_G : \mathfrak{g} \to G$ in regular Lie groups by (b). However, the exponential map is not necessarily locally surjective, and it does not need to satisfy the Baker-Campbell-Hausdorff formula.

We list some properties of regular Lie groups

Theorem 9. *(a) Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $L : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then, if G is connected, then there exist at most one group homomorphism $f : G \to H$ with $f_*|_\mathbf{1} = L$. If G is also simply connected and H is regular, then there is exactly one group homomorphism.* [Mil84, Lemma 7.1, Theorem 8.1]

(b) A Lie group which is connected, simply connected and regular, is uniquely determined by its Lie algebra. However, not every Lie algebra on a convenient vector space is the Lie algebra of some Lie group. Furthermore, if G is a Lie group with Lie algebra $\mathfrak{g}$, then there may be Lie sub-algebras of $\mathfrak{g}$ that do not correspond to any sub-group of G. [Mil84, Corollary 8.2, Warning 8.3 and 8.5]

Remark 7. The term “regular Lie groups” was first used for groups modeled on Fréchet spaces in a series of seven papers by the four authors Hideki Omori, Yoshiaki Maeda, Akira Yoshioka, and Osamu Kobayashi from 1980 to 1985. The definition appeared in the fourth paper [OMYK82], (see also [KAMO85]). The definition was somewhat stricter than the one presented here, but it ensures that we were able integrate any curve in the Lie algebra to the Lie group. It was realized by John Milnor [Mil84] that many of the same properties hold for Lie-Fréchet groups satisfying the definition we have presented here. Regular Lie groups in the framework of groups modeled on convenient vector spaces were first considered in [KrMi97a].

4. Presentation of main results

4.1. Paper B: Nonholonomic geometry on $\mathrm{SU}(1,1)$ and its universal cover

Motivation

Although, as we have mentioned, sub-Riemannian geometry was started in 1986, it is still quite a young topic. Many of the results deal mainly with sub-Riemannian structures on nilpotent groups, in particular the Heisenberg group. We wanted to look at a concrete example where the underlying manifold is a semi-simple Lie group. A Lie algebra $\mathfrak{g}$ is called simple if it is not abelian and contains no proper, nontrivial ideals. $\mathfrak{g}$ is called semi-simple if it is a direct sum of simple Lie algebras. A Lie group is semi simple if it is connected and has a semi-simple Lie algebra. We want to consider a noncompact Lie group G, by the following reason.

A Lie algebra is semi-simple if and only if the Killing form

$$\mathrm{Kil}(X,Y) = \mathrm{tr}\left(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right), \qquad X, Y \in \mathfrak{g}$$

is a non-degenerate bilinear form. If the Lie group is noncompact (or more precisely, if the group quotiented out by its center is noncompact), then we can choose a maximal sub-space $\mathfrak{p} \subseteq \mathfrak{g}$, such that the restriction of the Killing form to this subspace is positive definite. A distribution obtained by left translation of $\mathfrak{p}$ will always be bracket generating of step 2, so there are no abnormal minimizers. For more details on this topic, see [Kna96, Chapter IV.2] and [BCG02, Appendix A].

The simplest choice of G in the previously mentioned case, is the noncompact semi-simple $\mathrm{SU}(1,1)$. This is a real sub-group of $\mathrm{GL}(2,\mathbb{C})$ of matrices

$$g = \begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix}, \qquad \det g = 1.$$

We also have that $\mathrm{SU}(1,1)$ (or more precisely its universal cover group) with the metric induced by the Killing form is a Lorentzian manifold which in physics is known as the Anti-de Sitter space. If we restrict this metric to the appropriate distribution, we get a sub-Lorentzian manifold which is not globally hyperbolic.

Summary

Consider $\mathrm{SU}(1,1)$ with the metric ρ induced by the Killing form on the Lie algebra. This is a bi-invariant Lorentzian metric. As this space contains timelike loops, it is better to consider the universal cover group of $\widetilde{\mathrm{SU}}(1,1)$, with the lifted metric $\tilde{\rho}$. We pick two left invariant distributions $\tilde{D}$ and $\tilde{E}$ of rank 2. The sub-bundle $\tilde{D}$ is a left invariant sub-bundle on $\widetilde{\mathrm{SU}}(1,1)$ such that the restriction of ρ to $\tilde{D}$ is positive definite, while E is chosen so that $\rho|_{\tilde{E}}$ is a sub-Lorentzian metric. We study the sub-Riemannian and sub-Lorentzian geometry with respect to the structures $(\tilde{D}, \tilde{\rho})$ and $(\tilde{E}, \tilde{\rho})$. Both distributions are strongly bracket generating, so there are no abnormal curves.

We want to describe the structure of the (normal) sub-Riemannian geodesics. As the distribution is left invariant, it is sufficient to do this at the identity $\tilde{1}$. We obtain the following results.

- We give a complete description of the number of geodesics and the number of them connecting a given point $\tilde{g} \in \widetilde{\mathrm{SU}}(1,1)$ and $\tilde{1}$ (Paper B, Proposition 2). We connect these results to the conjugate locus, that is, the critical points of the sub-Riemannian exponential map (Paper B, Proposition 3).

- We give a formula for the Carnot-Carathéodory distance on $\widetilde{\mathrm{SU}}(1,1)$ and on $\mathrm{SU}(1,1)$ (Paper B, Corollary 1 and 2).

When it comes to the sub-Lorentzian structure, things become more complicated. We manage to give a complete description of the number of (normal) sub-Lorentzian geodesics connecting an arbitrary point $\tilde{g}$ with $\tilde{1}$, but the results are quite complicated (Paper B, Proposition 5). However, we remark that there are points that can be connected to $\tilde{1}$ by a sub-Lorentzian geodesic, that cannot be connected to $\tilde{1}$ by a Lorentzian geodesic and vice-versa (compare with Paper B, Proposition 6).

In addition to discuss the geodesics, we find the time-like future of $\tilde{1}$ with respect to the sub-Lorentzian (and Lorentzian) structure (Paper B, Proposition 7). We give no result on the sub-Lorentzian distance, as it is difficult to find a restriction to a globally hyperbolic set, which would allow us to compute the distance from the geodesics.

4.2. Papers C, D and E: Rolling without twisting or slipping

Motivation

Sub-Riemannian geometry can be seen as dynamics in a Riemannian manifold where we have nonholonomic constraints given by a sub-bundle D. One classical example of a nonholonomic dynamical system, is a sphere rolling on a plane without slipping or twisting which can be traced as far back as Euler, see [Cha1903, Introduction]. An intrinsic definition for two general 2-dimensional manifolds rolling on each other without twisting or slipping can be found in [BrHs93, Section 4.4] or [AgSa04, Chapter 24]. Here, it is proven that we have local controllability if the connecting points have different Gaussian curvature. Hence, the system where a sphere rolls on the plane is completely controllable, while this is not the case when a cylinder rolls on the plane.

There has lately been an interest in rolling manifolds in higher dimension, in particular from the engineering community. Part of the reason is the possibility of using rolling without twisting or slipping as a tool in interpolation theory, see [HüSi07]. We want to consider this problem from the point of view of geometric control theory.

Definition of rolling without twisting or slipping

If we generalize the intrinsic definition given in two dimensions, we obtain the following formulation. Let us first adopt the convention, that in the rest of Section 4.2, whenever we write $\mathbb{R}^n$, it will always come furnished with the standard orientation and the Euclidean metric. For an n-dimensional oriented Riemannian manifold M, define the oriented frame bundle $F(M)$ as the principal $\mathrm{SO}(n)$-bundle, whose fiber over $m \in M$ is the space of all linear orientation preserving isometries

$$F_m(M) = \mathrm{SO}(\mathbb{R}^n, T_m M).$$

An element $f \in \mathrm{SO}(\mathbb{R}, T_m M)$ can be considered as a choice of a positively oriented orthonormal frame $f_1, \ldots, f_n$ by the correspondence (2.1).

Let M and $\widehat{M}$ be two connected, n-dimensional oriented Riemannian manifolds. Define a fiber bundle over $M \times \widehat{M}$, by

$$Q := (F(M) \times F(\widehat{M}))/\,\mathrm{SO}(n) = \left\{ q \in \mathrm{SO}(T_m M, T_{\widehat{m}} \widehat{M}) :\, m \in M, \widehat{m} \in \widehat{M} \right\}.$$

Here, the quotient is taken with respect to the diagonal action of $\mathrm{SO}(n)$ on $F(M) \times F(\widehat{M})$. In general, $Q \to M \times \widehat{M}$ is not a principal bundle for $n > 2$.

The space Q can be thought of as the space of all ways that M can lie tangent to $\widehat{M}$. An element $q : T_m M \to T_{\widehat{m}} \widehat{M}$ then represents a configuration where M at a point m lies tangent to $\widehat{M}$ at $\widehat{m}$, and the way the tangent spaces at m and $\widehat{m}$ connect is given by q. Define the natural projections

$$\pi : Q \to M, \qquad \widehat{\pi} : Q \to \widehat{M}, \qquad \overline{\pi} : Q \to M \times \widehat{M}.$$

Then a rolling without twisting or slipping is a curve in Q, satisfying the following properties.

Definition 7. *Consider an absolutely continuous curve $q : [0, \tau] \to Q$, and write*

$$\pi(q(t)) = m(t), \qquad \widehat{\pi}(q(t)) = \widehat{m}(t).$$

Then $q(t)$ is called a rolling without twisting or slipping if it satisfies the following conditions for almost every t

(i) (no slipping-condition) $\dot{m}(t) = q(t)\dot{\widehat{m}}(t)$.

(ii) (no twisting-condition) for any vector field $Z(t)$ along $m(t)$, we have

$$q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)\dot{\widehat{m}}(t).$$

In (ii), $\frac{D}{dt}$ is the covariant derivative along $m(t)$ or $\widehat{m}(t)$ respectively, corresponding to the Levi-Civita connection of the respective manifolds. An equivalent way of formulating (ii) is to require that $q(t)$ sends parallel vector fields to parallel vector fields.

Summary: Paper C

The first thing we need to do in order study this problem, is to justify our the previous definition. Before Paper C, it had only previously appeared in literature for 2 dimensional manifolds. There did exist a definition for higher dimensions [Sha97, Appendix B], but only for manifolds embedded into Euclidean space. The main achievement of Paper C is splitting the definition of "the embedded rolling" into two parts, one intrinsic and one extrinsic (Paper C, Proposition 1). The intrinsic part satisfies our previous definition and uniquely determines the extrinsic part up to initial condition (Paper C, Theorem 2).

Summary: Paper D

The purpose of this paper is to address the question of controllability of the system of two n dimensional manifolds M and $\widehat{M}$ rolling on each other without twisting or slipping for the case $n > 2$. Our main goal is to find a way to answer this question in terms of the curvature of the manifolds involved, similar to the result in [BrHs93, Section 4.4] and [AgSa04, Chapter 24] for the 2 dimensional case.

First of all, we make the observation that rather than considering our rolling $q(t)$ as a curve in the configuration space Q, we can lift it to the product of oriented orthonormal frame bundles $F(M) \times F(\widehat{M})$ by the following result.

Theorem 10 (Paper D, Corollary 1). *Let $q(t)$ be an absolutely continuous curve in Q, with*

$$\overline{\pi}(q(t)) = (m(t), \widehat{m}(t)).$$

Then $q(t)$ is rolling without slipping or twisting if and only if there exists a curve $t \mapsto (f(t), \widehat{f}(t))$ in $F(M) \times F(\widehat{M})$, satisfying

- $\mathrm{pr}_M f(t) = m(t), \mathrm{pr}_{\widehat{M}} \widehat{f}(t) = \widehat{m}(t),$
- $q(t) = \widehat{f}(t) \circ f^{-1}(t),$
- *(no slipping-condition)* $f^{-1}(t)(\dot{m}(t)) = \widehat{f}^{-1}(t)(\dot{\widehat{m}}(t)).$
- *(no twisting-condition) $f_j(t)$ is parallel along $m(t)$ and $\widehat{f}_j(t)$ is parallel along $\widehat{m}(t)$ for any $1 \leq j \leq n$.*

Let D be the distribution of dimension n on Q formed by the tangent vectors of all rollings without twisting or slipping (see Paper C, Proposition 3). Write $\mathcal{D}$ for the n dimensional distribution on $F(M) \times F(\widehat{M})$ formed by tangent vectors of curves $(f(t), \widehat{f}(t))$ satisfying the previous theorem. It turns out that we can determine if D is bracket generating (and thereby show controllability) by computing brackets of $\mathcal{D}$. This simplifies calculations, leading to the following sufficient conditions.

Let R be the usual curvature tensor on M defined by

$$R(Y_1, Y_2, Y_3, Y_4) = \left\langle \left(\nabla_{Y_1}\nabla_{Y_2} - \nabla_{Y_2}\nabla_{Y_1} - \nabla_{[Y_1,Y_2]}\right) Y_3, Y_4 \right\rangle,$$

where ∇ is the affine Levi-Civita connection on M corresponding to the Riemannian metric $\langle \cdot, \cdot \rangle$. Define $\widehat{R}$ similary on $\widehat{M}$.

Theorem 11 (Paper D, Theorem 3). *Let $q : T_m M \to T_{\widehat{m}}\widehat{M}$ be any element of Q. Then we have local controllability at q if for one (and hence any) orthonormal basis $v_1, \ldots, v_n$ of $T_m M$, the determinant*

$$\det \Big(R(v_1, v_2, v_\alpha, v_\beta) - \widehat{R}(qv_1, qv_2, qv_\alpha, qv_\beta) \Big)_{1 \le i < j \le n}^{1 \le \alpha < \beta \le n},$$

does not vanish. Here, (i, j) is the row index and (α, β) is the column index.

Corollary 1 (Paper D, Corollary 4). *For any 2-dimensional plane L in D_q, that is, an element of $Gr_2(D_q)$, define an operator $\overline{\varkappa}_q$ by*

$$\overline{\varkappa}_q(L) = \varkappa_{\pi(q)}(\pi_* L) - \widehat{\varkappa}_{\widehat{\pi}(q)}(\widehat{\pi}_* L).$$

Here, $\varkappa_m$ and $\widehat{\varkappa}_{\widehat{m}}$ is the sectional curvature at $m \in M$ and $\widehat{m} \in \widehat{M}$ respectively. Then, if $\overline{\varkappa}_q$ never vanishes for any L, we have local controllability at q.

Summary: Paper E

Let $q(t)$ be a rolling (an intrinsic one) without slipping or twisting, and consider curves $m(t) = \pi(q(t))$ and $\widehat{m}(t) = \widehat{\pi}(q(t))$ in respectively M and $\widehat{M}$. Then, by the following theorem, $\widehat{m}(t)$ is uniquely determined by $m(t)$ up to an initial condition.

Theorem 12. *Let $[0, \tau] \mapsto M, t \mapsto m(t)$ be a curve in M with $m(0) = m_0$. For a given point $\widehat{m}_0$, let q_0 be a given orientation preserving linear isometry*

$$q_0 : T_{m_0} M \to T_{\widehat{m}_0}\widehat{M}.$$

(a) For sufficiently short time, there exists a unique rolling $q(t)$ satisfying $q(0) = q_0$ and $\pi(q(t)) = m(t)$. [Sha97, Proposition 2.4]

(b) If $\widehat{M}$ is complete, then such a solution exists for all time $t \in [0, \tau]$. (Paper D, Lemma 6)

We want to understand how the geometric properties of $m(t)$ affect the curve $\widehat{m}(t)$. Let us first assume that $m(t)$ is parametrized by arc length and is C^n and C^{n-1}-regular.

Theorem 13 (Paper E, Lemma 2 and Theorem 2). *If $m(t)$ is a C^n curve, that is C^{n-1}-regular and parametrized by arc length, with oriented geodesic curvatures $\kappa_1, \ldots, \kappa_{n_1}$, then there exists a rolling $q(t)$ with*

$$\pi(q(t)) = m(t), \qquad \widehat{\pi}(q(t)) = \widehat{m}(t),$$

if and only if, $\widehat{m}(t)$ is also a C^n curve which is C^{n-1}-regular, parametrized by arc length and whose length and oriented geodesic curvature coincide with the ones of $m(t)$.

If $m(t)$ is C^n and C^{n-1}-regular, but not parametrized by arc length, then we need to add the requirement that $\widehat{m}(t)$ has the same speed as well. However, we argue a rolling without twisting or slipping preserves even more structure than can be detected by geodesic curvatures. In order to do this, we have to make sense of the term "local structure" for the general absolutely continuous curve.

Definition 8. *For a given absolutely continuous curve in $M, t \mapsto m(t)$, an absolutely continuous curve $y(t)$ in $\mathbb{R}^n$ with $y(0) = 0$ is called an anti-development of $m(t)$, if there is a curve $f(t)$ in M, with $\operatorname{pr}_M f(t) = m(t)$, each f_j is parallel along $m(t)$ and*

$$\dot{m}(t) = f(t)(\dot{y}(t)),$$

for almost every t. The curve $m(t)$ is called a development of $y(t)$.

Clearly, any anti-development is defined uniquely up to rotation. By (Paper D, Corollary 1), an anti-development curve is nothing more than the result of rolling M on $\mathbb{R}^n$ along $m(t)$. Conversely, we obtain a development by rolling $\mathbb{R}^n$ on M along $y(t)$. Since we can split any rolling into an anti-development and a development, a rolling exists between two curves if and only if they have the same set of anti-developments (Paper E, Corollary 3).

We argue that an anti-development can be seen as a generalization of the oriented geodesic curvatures, because of the following properties. If the oriented geodesic curvatures of $m(t)$ are well defined, they, along with the speed of $m(t)$, are encoded into its anti-development, which will have the same oriented geodesic curvatures and speed. A development of a given curve $y(t)$ in M is uniquely determined up to the initial conditions of $m(t)$ and $f(t)$. It always exists for short time and for all time if M is complete. Finally, if M is a complete, simply connected manifold of constant sectional curvature, then an anti-development determines the curve uniquely up to an orientation preserving isometry (Paper E, Corollary 4). Compare this with the properties of the geodesic curvatures given in Section 2.4.

4.3. Papers A and F: Infinite dimensional sub-Riemannian geometry

Motivation

The main idea underlying geometric control theory, is that one can get information of the control system by using geometric tools. For many mechanical systems, the optimal curves are geodesics on Riemannian or sub-Riemannian manifolds. For example, the motion of a force-free rotating rigid body is governed by a left-invariant Riemannian metric on the Lie group SO(3). We have just discussed how optimal curves of rolling without twisting or slipping can be seen as geodesics in a sub-Riemannian manifold.

In 1966, Vladimir Igorevich Arnol'd proved that Euler's equations on a compact Riemannian manifold M can be considered as geodesics in the infinite dimensional Lie group of volume preserving diffeomorphisms, thereby showing that geometric intuition could also be applied to fluid dynamics and other mechanical systems involving PDEs [Arn66]. This was later generalized in [EbMa69] to compact manifolds with boundary. There has since been several results connecting PDEs to different infinite dimensional Riemannian manifolds (see e.g. [MiRa98, Anc08]).

It has also become interesting to study infinite dimensional geometry for other reasons, in particular for finding good metrics to measure the distance between two infinite dimensional objects. Two particular topics of interest has been geometry on the space of all Riemannian metrics on a manifold [Ebi70, ClRu11] and the space of shapes [MiMu07].

In light if the success of Riemannian geometry in the infinite dimensional setting, we want to be able to have an appropriate definition for the sub-Riemannian geometry as well.

Summary Paper A

Let $\operatorname{Diff}_+ S^1$ denote the Lie-Frechét group of orientation preserving diffeomorphisms of the unit circle S^1. By a result of Aleksandr Aleksandrovich Kirillov [Kir87], we can identify the symmetric space $\operatorname{Diff}_+ S^1 / \operatorname{Rot} S^1$ with the space of normalized univalent functions $\mathcal{F}_0$. Here, $\operatorname{Rot} S^1$ is the subgroup of rotations, while $\mathcal{F}_0$ is the space of all injective, holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ with smooth extensions to the boundary, normalized by requirements $f(0) = 0$ and $f'(0) = 1$.

The Lie algebra of $\operatorname{Diff}_+ S^1$ is the space $\operatorname{Vect} S^1$ of vector fields on S^1 with brackets $[x\partial_\theta, y\partial_\theta] = (x'y - xy')\partial_\theta$. This space $\operatorname{Vect} S^1$ can be identified with $C^\infty(S^1)$. For any $f \in \mathcal{F}_0$, we can identify $T_f \mathcal{F}_0$ with the space of all holomorphic functions $F : \mathbb{D} \to \mathbb{C}$ that have a smooth extension to the boundary and satisfy $F(0) = F'(0) = 0$. Let ϕ be an element of $\operatorname{Diff}_+ S^1$ and let r_ϕ be the right multiplication by ϕ. Use $\pi : \operatorname{Diff} S^1 \to \mathcal{F}_0$ for the projection. Then the map $(\pi \circ r_\phi)_*|_{\mathrm{id}_{S^1}}$, has the formula [Kir98],

$$x \mapsto i f^2 I_f[x], \qquad \pi(\phi) = f, \qquad x \in C^\infty(S^1).$$

$$I_f[x](z) = \frac{1}{2\pi} \int_{S^1} \left(\frac{\zeta f(\zeta)}{f'(\zeta)} \right)^2 \frac{x(\arg \zeta)}{f(z) - f(\zeta)} d\zeta.$$

We extend the definition of I_f to a map from the space $\operatorname{Lip}_\alpha(S^1)$ of α-Hölder continuous functions, $\alpha \in (0,1)$, into the space of holomorphic functions $F : \mathbb{D} \to \mathbb{C}$ whose extension to the boundary satisfy $F|_{S^1} \in \operatorname{Lip}_\alpha(S^1)$. We prove that the kernel of this is one-dimensional, and describe it explicitly (Paper A, Theorem 1).

Let $g : \mathbb{D}_- \to \mathbb{C}$ be an injective holomorphic map from the exterior of the unit disk $\mathbb{D}_-$ which satisfies $g(\infty) = \infty$. Then g is said to match $f \in \mathcal{F}_0$ if the boundaries of $f(\mathbb{D})$ and $g(\mathbb{D}_-)$ coincide. For any f, such a function g exist and is unique up to a rotation on the right by the Riemann mapping theorem. As a corollary of Paper A, Theorem 1 we describe how this function can be obtained as a solution of a first order differential equation, depending on f and the kernel of I_f. We give some concrete examples of this.

Summary Paper F

We define an infinite dimensional sub-Riemannian manifold in the following way.

Definition 9. *A sub-Riemannian manifold is a triple $(M, \mathcal{H}, \mathbf{h})$, where*

- *M is a connected (smoothly Hausdorff) manifold modeled on c^∞ open subsets of convenient vector spaces.*
- *$\mathcal{H}$ is a splitting sub-bundle of TM, that is, there exist another sub-bundle $\mathcal{V}$ such that*
$$TM = \mathcal{H} \oplus \mathcal{V}.$$
- *$\mathbf{h}$ is a (weak) metric on $\mathcal{H}$.*

We use the term 'weak metric' as we do not require that the map $v \mapsto \mathbf{h}(v, \cdot) \in \mathcal{H}_m^*$ to be surjective, only injective. We consider horizontal curves as smooth curves (not absolutely continuous) that are tangent to $\mathcal{H}$.

Our previous definition of abnormal curves or "bad points" in the space of curves, can not be used any longer. The reason is that we no longer have an inverse function theorem, which means that we cannot search for "bad curves" by looking for singular points of the endpoint map. Instead, we introduce a new class of curves called semi-rigid. See Paper F, Section 3.3 for the definition. These curves are always abnormal when M is finite dimensional.

We are not ready to describe a general theory for finding geodesics in infinite dimensional sub-Riemannian manifolds, but we describe a particular case. By a sub-Riemannian geodesic, we mean a critical value of the sub-Riemannian energy functional $E(\gamma) = \int_0^1 \mathbf{h}(\dot\gamma(t), \dot\gamma(t))\, dt$. We can no longer use the Pontryagin's Maximum Principle to find these, but we can use calculus of variation. Let us view any bundle chart of the tangent bundle defined on a neighborhood $U \subset M$,

$$TU \to U \times V, \qquad v \in T_m U \mapsto (m, \theta(v)),$$

as a V valued one-form on U, where V is a convenient vector space. For a sufficiently small neighborhood, we can pick a bundle chart such there is a splitting $V = \mathcal{H}_0 \oplus V_0$, satisfying

$$\theta^{-1}(\mathcal{H}_0) = \mathcal{H} \cap TU, \qquad \theta^{-1}(\mathcal{V}_0) = \mathcal{V} \cap TU.$$

Furthermore, assume that there is an inner product $\langle \cdot, \cdot \rangle$, such that $\mathcal{H}_0$ and $\mathcal{V}_0$ are orthogonal with respect to this inner product and $\mathbf{h}(v_1, v_2) = \langle \theta(v_1), \theta(v_2) \rangle$, for any $v_1, v_2 \in \mathcal{H}_m, m \in U$. Extend this sub-Riemannian metric to a Riemannian metric $\mathbf{g}$ by the same formula, and assume that there exist sa map $a^\top : V_\theta \times V_\theta \to V_\theta$, satisfying

$$\langle d\theta(v_1, v_2), u \rangle = \langle \theta(v_1), a^\top(\theta(v_2), u) \rangle.$$

Assuming some additional minor technical conditions, we have the following result.

Theorem 14 (Paper F, Theorem 1)**.** *Assume that γ is a sub-Riemannian geodesic. Then either γ is semi-rigid or there is a curve $\lambda \in C^\infty(I, \mathcal{V}_0)$, such that γ satisfies*

$$\theta(\dot{\gamma}) = u, \qquad \dot{u} = -\operatorname{pr}_{\mathcal{H}_0} a^\top(u, u+\lambda) \qquad \dot{\lambda} = -\operatorname{pr}_{\mathcal{V}_0} a^\top(u, u+\lambda). \tag{4.1}$$

All solutions to (4.1) *are sub-Riemannian geodesics.*

All finite dimensional manifolds can be described this way, and γ satisfies (4.1) if and only if it is a normal geodesic (Paper F, Proposition 2). All finite and infinite dimensional Lie groups with a left or right invariant sub-Riemannian structure can be described this way if $\operatorname{ad}(x)$ has an adjoint for any $x \in \mathfrak{g}$ (this is non-trivial in infinite dimensions).

In particular, let us choose θ to be the left Maurer-Cartan form κ^ℓ on a Lie group G, giving us a global bundle chart. Assume furthermore that $\mathcal{V}_0$ is the Lie algebra of some Lie group K. Let us define a Riemannian metric by left translations of an $\operatorname{Ad}(K)$-invariant inner product on the Lie algebra of G, and construct a sub-Riemannian metric by restricting it to $\mathcal{H}$. Then, any solution to (4.1) is of the form $\gamma_{sR}(t) = \gamma_R(t) \exp_G(-\lambda t)$, where γ_R is a Riemannian geodesic and $\lambda := \operatorname{pr}_{\mathcal{V}_0} \kappa^\ell(\dot{\gamma}_R(t))$ is a constant (Paper F, Theorem 4).

We give a concrete example by considering $G = \operatorname{Diff}_+ S^1$ and $K = \operatorname{Rot} S^1$. We also consider a sub-Riemannian structure on the Virasoro-Bott group. In spite of the fact that there is no Rashevskiĭ-Chow Theorem in infinite dimensions, we are able to prove complete controllability in these cases. We also write down the equations for the normal sub-Riemannian geodesics.

Bibliography

[AgSar96] A. Agrachev and A. V. Sarychev, *Abnormal sub-Riemannian geodesics: Morse index and rigidity.* Annales de l'Institut Henri Poincarẃ - Analyse Non linéaire, **13** (1996), no. 6, 635–690.

[AgSa04] A. Agrachev and Yu. Sachkov, *Control theory from the geometric viewpoint.* Encyclopaedia of Math. Sci., 87. Control Theory and Optimization, II. Springer-Verlag, Berlin, 2004.

[Agr08] A. A. Agrachev, *Geometry of optimal control problems and Hamiltonian systems.* Nonlinear and optimal control theory, 1–59, Lecture Notes in Math., 1932, Springer, Berlin, 2008.

[ABS08] A. Agrachev; U. Boscain and M. Sigalotti, *A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds.* Discrete Contin. Dyn. Syst. **20** (2008), no. 4, 801–822.

[ABGR09] A. A. Agrachev, U. Boscain, J. P. Gauthier and F. Rossi, *The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups.* J. Funct. Anal. **256** (2009), no. 8, 2621–2655.

[Anc08] S. C. Anco, *Hamiltonian curve flows in Lie groups $G \subset U(N)$ and vector NLS, mKdV, sine-Gordon soliton equations.* Symmetries and overdetermined systems of partial differential equations, 223–250, IMA Vol. Math. Appl., 144, Springer, New York, 2008.

[Arn66] V. Arnold, *Sur la gomtrie diffrentielle des groupes de Lie de dimension infinie et ses applications l'hydrodynamique des fluides parfaits.* Ann. Inst. Fourier (Grenoble) **16** (1966) no. 1, 319–361.

[BaBo09] F. Baudoin and M. Bonnefont, *The subelliptic heat kernel on SU(2): representations, asymptotics and gradient bounds.* Math. Z. **263** (2009), no. 3, 647–672.

[BGG00] R. Beals, B. Gaveau and P. C. Greiner, *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups.* J. Math. Pures Appl. **79** (2000), no. 7, 633–689.

[BoTr01] B. Bonnard, and E. Trélat, *On the role of abnormal minimizers in sub-Riemannian geometry.* Ann. Fac. Sci. Toulouse Math. **(6) 10** (2001), no. 3, 405–491.

[Bon11] M Bonnefont *The subelliptic heat kernels on SL(2,R) and on its universal covering $\widetilde{SL}(2,R)$: integral representations and some functional inequalities.* arXiv:0906.1977.

[BCG02] U. Boscain, T. Chambrion and J.-P. Gauthier, *On the K+P problem for a three-level quantum system: optimality implies resonance.* J. Dynam. Control Systems **8** (2002), no. 4, 547–572.

[Bro84] R. W. Brockett *Nonlinear control theory and differential geometry.* Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Warsaw 1983), p 1357-1368, PWN, Warsaw, 1984.

[BrHs93] R. Bryant and L. Hsu, *Rigidity of integral curves of rank* 2 *distributions.* Invent. Math. **114** (1993), no. 2, 435–461.

[Cha1903] S. A. Chaplygin, *On the rolling of a sphere on a horizontal plane.* (in Russian), Mat. Sbornik **XXIV** (1903), 139–168.

[CJT06] Y. Chitour, F. Jean, and E. Trélat, *Genericity results for singular curves.* J. Differential Geom. **73** (2006), no. 1, 45–73.

[CLS03] M. Chyba, N. E. Leonard and E. D. Sontag, *Singular trajectories in multi-input time-optimal problems: application to controlled mechanical systems.* J. Dynam. Control Systems **9** (2003), no. 1, 103–129.

[Cho39] W. L. Chow *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.* Math. Ann. **117** (1939), 98–105.

[ClRu11] B. Clarke and Y. A. Rubinstein, *Conformal deformations of the Ebin metric and a generalized Calabi metric on the space of Riemannian metrics.* arXiv:1104.4314v1.

[EbMa69] D. G. Ebin, J. E. Marsden, *Groups of diffeomorphisms and the solution of the classical Euler equations for a perfect fluid.* Bull. Amer. Math. Soc. **75** (1969) 962–967.

[Ebi70] D. G. Ebin, *The manifold of Riemannian metrics.* 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 11–40 Amer. Math. Soc., Providence, R.I.

[GoKa95] C. Golé and R. Karidi, *A note on Carnot geodesics in nilpotent Lie groups.* J. Dynam. Control Systems **1** (1995), no. 4, 535–549.

[GCFT08] D. Grass, J. P. Caulkins, G. Feichtinger, G. Tragler and D. A. Behrens, *Optimal control of nonlinear processes. With applications in drugs, corruption, and terror.* Springer-Verlag, Berlin, 2008.

[Gro02] M. Grochowski, *Geodesics in the sub-Lorentzian geometry.* Bull. Polish Acad. Sci. Math. **50** (2002), no. 2, 161–178.

[Gro04] M. Grochowski, *On the Heisenberg sub-Lorentzian metric on* $\mathbb{R}^3$, Geometric Singularity Theory, Banach Center Publ., Polish Acad. Sci. **65** (2004), 57–65.

[Gro09] M. Grochowski, *Properties of reachable sets in the sub-Lorentzian geometry.* J. Geom. Phys. **59** (2009), no. 7, 885–900.

[Ham90] U. Hamenstädt, *Some regularity theorems for Carnot-Carathodory metrics.* J. Differential Geom. **32** (1990), no. 3, 819–850.

[Hör67] L. Hörmander, *Hypoelliptic second order differential equations.* Acta Math. **119** (1967), 147–171.

[HüSi07] K. Hüper, F. Silva Leite, *On the geometry of rolling and interpolation curves on* S^n, SO_n, *and Grassmann manifolds.* J. Dyn. Control Syst. **13** (2007), no. 4, 467–502.

[Jos05] J. Jost *Riemannian Geometry and Geometric Analysis*, Springer-Verlag Berlin Heidelberg,2005

[Kir87] A. A. Kirillov, *Kähler structures on K-orbits of the group of diffeomorphisms of a circle*, Funct. Anal. Appl. **21** (1987), no. 2, 42–45.

[Kir98] A. A. Kirillov, *Geometric approach to discrete series of unirreps for Vir.* J. Math. Pures Appl. **77** (1998), 735–746.

[Kna96] A. W. Knapp, *Lie Groups, Beyond an Introduction.* Birkhäuser 1996.

[KAMO85] O. Kobayashi, Y. Akira, Y. Maeda, and H. Omori, *The theory of infinite-dimensional Lie groups and its applications.* Acta Appl. Math., **3**, no. 1, (1985), 71–106.

[KrMi97a] A. Kriegl and P. W. Michor, *Regular infinite-dimensional Lie groups.*, J. Lie Theory, **7** (1997), no. 1, 61–99.

[KrMi97b] A. Kriegl and P. W. Michor, *The convenient setting of global analysis.* Mathematical Surveys and Monographs Volume 53, Amer Mathematical Society 1997.

[Léa97a] R. Léandre, *Majoration en temps petit de la densité d'une diffusion dégénérée. (Small time upper bounds for the density of a degenerate diffusion)* Probab. Theory Related Fields **74** (1987), no. 2, 289–294.

[Léa87b] R. Léandre, *Minoration en temps petit de la densité d'une diffusion dégénérée. (Minorization in small time of the density of a degenerate diffusion)* J. Funct. Anal. **74** (1987), no. 2, 399–414.

[Lee02] J. M. Lee, *Introduction to Smooth Manifolds.* GTM Vol. 218, Springer 2002.

[Lee09] J. M. Lee, *Manifolds and Differential Geometry.* GSM Vol. 107, American Mathematical Society 2009.

[MiRa98] P. W. Michor and T. S. Ratiu, *On the Geometry of the Virasoro-Bott group*, J. Lie Theory, **8**, (1998), no. 2, 293–309.

[Mic06] P. W. Michor, *Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach.* In: Phase space analysis of Partial Differential Equations, Progress in Non Linear Differential Equations and Their Applications, Vol.69. Birkhauser Verlag 2006. Pages 133-215.

[MiMu07] P. W. Michor and D. Mumford, *An overview of the Riemannian metrics on the spaces of curves using the Hamiltonian approach.* Appl. Comput. Harmonic Analysis **23** (2007), 74–113.

[Mil84] J. Milnor, *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: 'Relativité, Groupes et Topologie II', B. DeWitt and R. Stora (Eds), North-Holland, Amsterdam, 1984.

[Mon94] R. Montgomery, *Abnormal minimizers.* SIAM J. Control Optim. **32** (1994), no. 6, 1605–1620.

[Mon95] R. Montgomery, *A survey of singular curves in sub-Riemannian geometry*, J. Dynam. Control Systems, **1** (1995), 49–90.

[Mon02] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, **91**. American Mathematical Society, Providence, RI, 2002.

[O'Ne66] B. O'Neill, *The fundamental equations of a submersion.* The Michigan Mathematical Journal, **13** (1966), 459–469.

[OMYK82] H. Omori, Y. Maeda, A. Yoshioka, O. Kobayashi, *On regular Frchet-Lie groups. IV. Definition and fundamental theorems.* Tokyo J. Math. **5** (1982), no. 2, 365–398.

[PBGM61] L. S. Pontrjagin, V. G. Boltjanskiĭ, R. V. Gamkrelidze and E. F. Miscenko, *Matematicheskaya teoriya optimal'nykh protsessov.* (Russian) [The mathematical theory of optimal processes] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow (1961) 391 pp.

[PBGM62] L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze and E. F. Mishchenko, E. F. *The mathematical theory of optimal processes.* (Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt) Interscience Publishers John Wiley & Sons, Inc. New York-London (1962) viii+360 pp.

[Ram05] S. Ramanan, *Global calculus*, GSM, Vol 65., American Mathematical Society 2005.

[Ras38] P. K. Rashevskiĭ, *About connecting two points of a completely nonholonomic space by admissible curve*, Uch. Zapiski Ped. Inst. K. Liebknecht **2** (1938), 83–94.

[SeSy87] A. Seierstad, K. Sydsæter, *Optimal control theory with economic applications.* Advanced Textbooks in Economics, 24. North-Holland Publishing Co., Amsterdam, 1987.

[Sha97] R. W. Sharpe, *Differential geometry.* GTM, 166. Springer-Verlag, New York, 1997.

[Spi99] M. Spivak, *A comprehensive Introduction to Differential Geometry, Volume IV.* Publish or Perish Inc., 1999.

[Str86] R. S. Strichartz, *Sub-Riemannian geometry.* J. Differential Geom. **24** (1986) no. 2, 221–263.

[Str86cor] R. S. Strichartz, Corrections to: "Sub-Riemannian geometry". J. Differential Geom. **30** (1989) no. 2, 594–596.

[SuJu72] H. J. Sussmann and V. Jurdjevic, *Controllability of nonlinear systems.* J. Diff. Equations **12** (1972), no. 3, 95–116.

[Sus73] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions.* Trans. Amer. Math. Soc. **180** (1973), 171–188.