Ruin probability and optimal dividend policy for models with investment

PhD Thesis

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Abstract

In most countries the authorities impose capital requirements on insurance companies in order to avoid the adverse consequences to society when insurance companies default on claims. Since holding capital is costly, this naturally leads to the problem of deciding how large the risk reserve needs to be, or what is a "safe" level of liquidity. A common answer is that the probability that the insurance company will default on policyholder claims should not be higher than a certain small level ϵ. An implementation of this policy requires reasonably accurate methods for determining this probability, known as the *ruin probability.*

Rigorous mathematical treatments of the ruin probability problem can be traced at least as far back as the acclaimed doctoral thesis of Filip Lundberg from 1903 with the title "Approximerad framställning af sannolikhetsfunktionen". Traditionally the focus has been on ruin probability on an infinite time horizon. In these models an insurance company can avoid ruin by allowing its risk reserve to grow toward infinity. At the 15th International Congress of Actuaries in 1957 Bruno de Finetti criticized this approach. In particular he couldn't see why an older company should hold more capital than a younger one bearing similar risks, only because it is older. As an alternative de Finetti formulated what is known as the "de Finetti´s dividend problem": Maximizing the expected sum of the discounted paid out dividends from time zero until ruin. Since then several papers have presented solutions to this problem for various risk processes. Two of the papers in this thesis, which we denote Paper A and Paper B, focus on de Finetti´s dividend problem, with the risk process following a general diffusion and a jump-diffusion process, respectively. These models are particularly relevant for insurance companies where the premium income is invested in assets with stochastic returns. In keeping with de Finetti´s original paper, where ruin probability played a central role, Paper A also discusses solutions of de Finetti´s dividend problem under solvency constraints.

In the last few decades a growing number of papers have focused on ruin probability on a finite time horizon. For short time spans the assumption that the risk reserve is allowed to grow freely is less spurious. An important tool for calculating the ruin probability on a finite horizon is solving certain partial integro-differential equations (PIDEs). The third paper, denoted Paper C, discusses how these PIDEs can be solved numerically. The last paper, denoted Paper D, discusses regularity properties for some of these PIDEs.

Papers

- **Paper A**: L. Bai, M. Hunting and J. Paulsen (2012). Optimal dividend policies for a class of of growth-restricted diffusion processes under transaction costs and solvency constraints. *To appear in Finance and Stochastics.*

- **Paper B**: M. Hunting and J. Paulsen (2012). Optimal dividend policies with transaction costs for a class of jump-diffusion processes. *To appear in and accepted by, Finance and Stochastics.*

- **Paper C**: M. Hunting (2012). A numerical approach to ruin probability in finite time for fitted models with investment. *Not submitted,*

- **Paper D**: M. Hunting (2012). Existence of a classical solution of a parabolic PIDE associated with ruin probability. *Not submitted.*

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1 Ruin probability

1.1 Cramér-Lundberg model

1.1.1 General theory

As explained in Chapter 2 in Mikosch (2004), the foundations of modern risk theory were laid in 1903 by the Swedish actuary Filip Lundberg in his acclaimed thesis, Lundberg (1903). Lundberg's major contribution was to introduce a simple model that is capable of describing the basic dynamics of a homogeneous insurance portfolio. There are three assumptions in Lundberg's model:

(i) Claims occur at the Poisson-distributed times τ_i, satisfying $0 \leq \tau_1 \leq \tau_2 \leq \cdots$. In this thesis we will refer to these times as *claim times.* and let λ be the parameter of the Poisson process.

(ii) The i-th claim, arriving at time τ_i, results in a claim of size S_i. The sequence $\{S_i\}$ constitutes an i.i.d. sequence of non-negative random variables. In this thesis we will denote the common distribution function of the claim sizes by $F(x)$.

(iii) The claim size process $\{S_i\}$ and the claim arrival process $\{\tau_i\}$ are *mutually independent.*

Based on the above we define the *claim number process*

$$N_t = \min \{i \in 0, 1, \cdots : \tau_{i+1} > t\}.$$

From the point of view of insurance companies it is common to assume a continuous premium income at a constant rate p. The risk process is then

$$Y_t = y + pt - S_t, \quad t > 0,$$

where y is the initial capital and S_t is the total claim amount process

$$S_t = \sum_{i=1}^{N_t} S_i.$$

Here we follow the convention that $\sum_{i=1}^{0} = 0$. If we assume that the waiting times between claims are i.i.d. then S_t is referred to as a *renewal process.* Generalizations of the Cramér-Lundberg model to general i.i.d. waiting times between claims are in the literature referred to as renewal models, or the

Sparre-Andersen model. The time τ when the process falls below zero for the first time is called ruin time,

$$\tau = \inf \{t > 0 : Y_t < 0\}. \tag{1.1.1}$$

The probability of eventual ruin is then

$$\psi(y) = P(\tau < \infty | Y_0 = y), \quad y > 0.$$

In Section 1.3 we consider the probability that $\tau \leq T$. An important result concerning renewal processes of the above type is given in, for example, Proposition 4.1.3 in Mikosch (2004). This result says that if we assume that

$$E\tau_1 < \infty$$

and

$$ES_1 < \infty,$$

then

$$ES_1 \geq pE\tau_1$$

implies that $\tau < \infty$ with probability 1 for every initial capital y. Any sensible premium policy would therefore satisfy the condition

$$ES_1 < pE\tau_1, \tag{1.1.2}$$

known as the net profit condition. In the following we will assume that this condition holds and let

$$\rho = p\frac{E\tau_1}{ES_1} - 1. \tag{1.1.3}$$

The quantity ρ is often referred to as the *safety loading*. In both Mikosch (2004) and Asmussen (2000) there are extensive discussions of ruin probability results in the Cramér-Lundberg model. To better understand these results we first review some of the definitions used in these two books.

Definition 1.1.1. *The survival probability (sometimes referred to as the* non-ruin probability*) is defined as*

$$\phi(y) = 1 - \psi(y).$$

Definition 1.1.2. *Let*

$$Z_1 = S_1 - p\tau_1,$$

and assume that the moment-generating function of Z_1 exists in some neighborhood around 0. If a unique positive solution h of the equation

$$Ee^{h(S_1 - p\tau_1)} = 1 \tag{1.1.4}$$

exists it is called the adjustment coefficient *or* Lundberg coefficient*.*

In the literature equation (1.1.4) is known as the *Lundberg equation*, and a distribution whose moment-generating function exists around the origin is generally referred to as being *light-tailed*. In the important special case of the exponential distribution with parameter β it is shown in Example 4.2.4 in Mikosch (2004) that the adjustment coefficient γ is given as

$$\gamma = \beta - \frac{\lambda}{p}. \tag{1.1.5}$$

Definition 1.1.3. *A function $L(x)$ is said to be* slowly varying *if*

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1, \quad \text{for all } c > 0.$$

Definition 1.1.4. *A positive random variable S and its distribution are said to be regularly varying with (tail) index α if for some $\alpha \geq 0$ the right tail of the distribution has the representation*

$$P\left(S > x\right) = L(x)x^{-\alpha},$$

where L is a slowly varying function.

Definition 1.1.5. *A positive random variable S and its distribution are said to be* subexponential *if, for a sequence S_i of i.i.d. random variables with the same distribution as S, the following relation holds: For all $n \geq 2$:*

$$P\left(\sum_{j=1}^{n} S_j > x\right) = P\left(\max_{i=1,\ldots,n} S_i > x\right)\left(1 + o\left(1\right)\right) \qquad \text{as } x \to \infty.$$

Definition 1.1.6. *Define $\bar{F}(x) = 1 - F(x)$,*

$$F_s(y) = \left(ES_1\right)^{-1} \int_0^y \bar{F}(x)dx, \tag{1.1.6}$$

and

$$\bar{F}_s(y) = 1 - F_s(y). \tag{1.1.7}$$

It is well known that all subexponential distributions are heavy-tailed. It is shown in Section 3.2.5 in Mikosch (2004) that every regulary varying distribution is a subexponential distribution. Furthermore, it is shown there that if a distribution has a density f, then a sufficient criterion for the distribution to be regulary varying is that, for some tail index $\delta > 0$,

$$\lim_{x\to\infty} \frac{f(cx)}{f(x)} = c^{\delta}, \quad \text{for all } c > 0.$$

For i.i.d. random variables $X_1, \dots, X_n$ with common distribution function $F(x)$ we will denote the cumulative distribution of the sum

$$\sum_{j=0}^{n} X_j$$

by

$$F^{*n}(x).$$

For general claim distributions no closed form formula is known for the ruin probability in the Cramér-Lundberg model. However, under some not very restrictive conditions, the ruin probability can be expressed as a solution of an integral equation. This is indicated in the result below, which is the same as Lemma 4.2.6 in Mikosch (2004).

Theorem 1.1.1. *Consider the Cramér-Lundberg model with safety loading $\rho > 0$ and expected claim size $ES_1 < \infty$. In addition assume that the claim size distribution F has a density. Then the survival probability satisfies the integral equation*

$$\phi(y) = \frac{\rho}{1+\rho} + \frac{1}{(1+\rho)\,ES_1} \int_0^y \bar{F}(x)\phi(y-x)dx. \tag{1.1.8}$$

In the above $\bar{F}(x) = 1 - F(x)$ is the common tail distribution of the claims. While for general claim distributions (1.1.8) does not give very much qualitative information, it (1.1.8) can be used as a basis for numerical computation. Moreover, for the case of exponential distributions with parameter β, it can be shown (see e.g. Example 4.2.9 in Mikosch (2004)) that the exact ruin probability is given by

$$\psi(y) = \frac{1}{1+\rho} e^{-\beta \frac{\rho}{1+\rho} y}. \tag{1.1.9}$$

In chapter VIII in Asmussen (2000) there is a discussion of ruin probability for a wider class of claims distributions, known as phase-type distributions. A distribution F is said to be of *phase-type* if F is the distribution of the lifetime of a terminating Markov process $\{J_t\}$ with finitely many states and time homogeneous transition rates. This class includes, for example, the exponential distribution, the hyper-exponential distribution (a mixture of a finite number of exponential distributions) and the Erlang distribution (Gamma distribution with an integer shape parameter) as special cases. The tail distribution $\bar{F}(x)$ of a phase-type distribution can be shown to be of the form $\bar{F}(x) \sim Cx^k e^{-\eta x}$, where C and η are positive constants and k is a non-negative integer. For claim distributions of this type an exact formula for the ruin probability is given in Theorem VIII.2.1 in Asmussen (2000). In the same chapter of that book an example is given on how that formula can be applied to a mixture of two exponential distributions.

For general light-tailed claim distributions the following result is well known (see e.g. Proposition II.1.1 in Asmussen (2000)).

Lemma 1.1.1. *Let*

$$\tilde{S}_t = \sum_{k=1}^{N_t} S_i - pt$$

and let

$$\xi(y) = \tilde{S}_\tau - y$$

be the overshoot at the time of ruin. Make the following assumptions:

(a) *For some $c > 0$, $\left\{e^{c\tilde{S}_t}\right\}_{t\geq 0}$ is a martingale.*

(b) $\tilde{S}_t \overset{a.s.}{\to} -\infty$ *on* $\{\tau = \infty\}$.

Then

$$\psi(y) = \frac{e^{-cy}}{E\left[e^{c\xi(y)}|\tau < \infty\right]}. \tag{1.1.10}$$

It can be shown (see e.g. Example II.1.2 in Asmussen (2000)) that if the adjustment coefficient γ exists, then under the Cramér-Lundberg model $\left\{e^{\gamma\tilde{S}_t}\right\}_{t\geq 0}$ is a martingale. Furthermore, since $\xi(y) \geq 0$ it then immediately follows that

$$\psi(y) \leq e^{-\gamma y}, \tag{1.1.11}$$

whenever the conditions of the Lemma hold. The formula (1.1.11) is known as the Cramér-Lundberg inequality. Moreover, this formula and the memoryless property of the exponential distribution provide an alternative method for deriving the identity (1.1.9). This is done in Example 1.3 in Asmussen (2000).

As mentioned above the exponential distribution is a light-tailed distribution. For many situations it is more appropriate to assume that the claim sizes follow a heavy-tail distribution. The most important heavy-tail distributions in insurance belong to the class of subexponential distributions, defined in Definition 1.1.5. For this class of distributions there is no known exact formula, but the asymptotic result below is given, for example, as Theorem IX.3.1 in Asmussen (2000).

Theorem 1.1.2. *Let* $\upsilon = \frac{ES_1}{E\tau_1}$. *Assume the Cramér-Lundberg model standardized such that* $p = 1$. *In addition assume that* $\upsilon < 1$, *that* $ES_1 < \infty$ *and that the integrated tail distribution* F_s *(defined in* (1.1.7)*) of the claim size distribution is a subexponential distribution. Then, asymptotically,*

$$\psi(y) \sim \frac{\upsilon}{1-\upsilon}\bar{F}_s(y). \tag{1.1.12}$$

1.1.2 Diffusion approximations

A rather different way of obtaining approximations of the ruin probability is by fitting diffusion processes to approximate the compound Poisson process in the Cramér-Lundberg model. In Section XI.1 in Asmussen (2000) there is a short discussion on ruin probability under a diffusion model with drift $\mu(y)$ and variance $\sigma^2(x) > 0$. This kind of model can be expressed as letting $Y_t = y + P_t$, where P_t is the continuous solution of the stochastic differential equation

$$dY_t = \mu\left(Y_t\right) + \sqrt{\sigma^2\left(Y_t\right)}dY_t.$$

The appeal of this approach is that under mild conditions the exact ruin probability has the closed-form solution stated in the result below, which is the same as Theorem XI.1.10 in Asmussen (2000).

Theorem 1.1.3. *Let*

$$s(x) = \exp\left(-2\int_0^x \frac{\mu(z)}{\sigma^2(z)}dz\right).$$

Assume that $\mu(x)$ *and* $\sigma^2(x)$ *are continuous functions, that* $\sigma^2(x) > 0$ *for* $x > 0$ *and that*

$$\int_0^\infty s(z)dz < \infty. \tag{1.1.13}$$

Then $0 < \psi(y) < 1$ *for all* $y > 0$ *and*

$$\psi(y) = 1 - \frac{\int_0^y s(z)dz}{\int_0^\infty s(z)dz}. \tag{1.1.14}$$

Conversely, if (1.1.13) *fails then* $\psi(y) = 1$ *for all* $y > 0$.

As described in Section IV.5 in Asmussen (2000), the simplest diffusion approximation is to let $\mu(y)$ and $\sigma^2(y)$ be two positive constants fitted to the first two moments of the compound Poisson process and the desired premium rate p. In this model the ruin probability is given as

$$\psi(y) = \exp\left(-2\frac{\mu}{\sigma^2}y\right).$$

Let γ be the adjustment coefficient as before. As explained in Section IV.6 the simplest diffusion approximation can be refined via so-called correction terms. This leads to the following approximation:

$$\psi(y) \approx \exp\left(-\gamma\left[y + \frac{\mathcal{L}_F{'''}(\gamma)}{3\mathcal{L}_F{''}(\gamma)}\right]\right). \tag{1.1.15}$$

Here $\mathcal{L}_F{''}(\gamma)$ and $\mathcal{L}_F{'''}(\gamma)$ are, respectively, the second and the third derivative of the Laplace transform of the claim size distribution evaluated at the point γ. We will return to the corresponding finite time approximation in Section 1.3.

1.2 Ruin probability in an economic environment

In the classical Cramér-Lundberg model premium income is modeled as a constant rate that does not earn any interest. Neither the claims nor the premium income is subject to inflation.

One of the earlier papers to feature an interest rate is Harrison (1977). In this model it is assumed that the risk reserve is invested in a risk free bank savings account, continuously earning interest at a constant rate r. Further, let the sum $\sum_{n=1}^{N_t} S_n$ be the compound Poisson process from the Cramér-Lundberg model and let $P_t = pt - \sum_{n=1}^{N_t} S_n$. In what follows we will refer to $P^y = y + P_t$ as the *surplus generating process.* With this notation the content of the account at time t is written

$$Y_t = e^{rt}y + \int_0^t e^{r(t-s)}dP_s,$$

or, equivalently

$$Y_t = e^{rt}\left[y + Z_t\right], \quad t \geq 0,$$

where

$$Z_t = \int_0^t e^{-rs}dP_s, \quad t \geq 0.$$

It is shown in Harrison (1977) that $Z_\infty = \lim_{t\to\infty} Z_t$ exists and is finite almost surely. A formula for the characteristic function of Z_∞ (i.e. Ee^{iuZ_∞}) is given. Furthermore it is shown in Theorem 2.3 in Harrison (1977) that

$$\psi(y) = \frac{H(-y)}{E\left[-H\left(Y(\tau)\right)\middle|\tau < \infty\right]}, \tag{1.2.1}$$

where H is the distribution function of Z_∞. For general distributions (1.2.1) may look more like a reformulation of the problem than a solution. However, in Harrison (1977) (1.2.1) is used to derive more explicit ruin formulas for a few specific claim size distributions, including the exponential distribution. The ruin probability is then given as

$$\psi(y) = \frac{\int_y^\infty e^{-\beta x}\left(1+\frac{rx}{p}\right)^{\left(\frac{\lambda p}{r}-1\right)}dx}{\frac{p}{\lambda}+\int_0^\infty e^{-\beta x}\left(1+\frac{rx}{p}\right)^{\left(\frac{\lambda p}{r}-1\right)}dx}.$$

This classical result is also found in Segerdahl (1942).

Another kind of model that is also considered in Harrison (1977) assumes that the surplus generating process is a diffusion process,

$$P_t^y = y + \mu t + \sigma W_t,$$

where μ and σ are positive constants, W_t is a standard Brownian motion and y is the initial value. For this it is shown that

$$\psi(y) = \frac{1-\Phi\left((ay+b)\right)}{1-\Phi(b)},$$

where $a = \sqrt{\left(\frac{2\imath}{\sigma^2}\right)}$, $b = a\frac{\mu}{\imath}$ and Φ is the standardized normal distribution function.

Taylor (1979) is one of the earliest papers to consider the effect inflation may have on premium income and claims size distribution. This paper is notable for its conclusion that probability of ruin is nondecreasing with increasing inflation. Some bounds for ruin in finite horizon are also given in that paper. We will return to those bounds in Section 1.3.

In Delbaen and Haezendonck (1987) the authors incorporate both interest and inflation in their models. Both the interest force and the inflation force (which we denote by r and $\imath$, respectively) are assumed to be constant. In this model the n'th claim size is of size $e^{\imath\tau_n}S_n$ and the premium density at time t is $pe^{\imath t}$. More formally this model can be written as the stochastic equation

$$dY_t = pe^{rt}dt + \imath Y_t dt - e^{rt}S_{N_t}dN_t.$$

The present value $\tilde{Y}_t$ of Y_t can be written as

$$\tilde{Y}_t = y + p \int_0^t e^{-(r-\bar{\imath})u} du - \sum_{n=1}^{N_t} e^{-(r-\bar{\imath})\tau_n} S_n. \tag{1.2.2}$$

As before, y in the above is the initial reserve. The most relevant result in this paper for this thesis is that the probability for eventual ruin can be written as the solution of the integro-differential equation

$$\psi'(y) = \frac{\lambda}{p + (r - \bar{\imath})\, y} \psi(y) - \frac{\lambda}{p + (r - \bar{\imath})\, y} E\left[\psi\left(y - S_1\right)\right], \tag{1.2.3}$$

where $\psi(x) = 1$ for $x < 0$. As before, λ is the intensity of the Poisson process.

Two other papers which take inflation into account are Waters (1983) and Paulsen (1993). In Waters (1983) the risk process is considered in discrete time. At a time $t_n, n \in 1, 2, \ldots$, the reserve is given as

$$Y_{t_n} = y + \sum_{k=1}^{n} c^n X_k.$$

Here c is a constant greater than 1 and the X_k's are i.i.d. variables with finite expectation and a continuous common distribution function such that

$$P\left(X_1 < 0\right) > 0.$$

In this model it is implied that premiums and claims are affected by the same inflation factor c. With no more than the conditions given above it is shown that in this model ultimate ruin is certain, i.e.

$$\psi\left(y\right) = 1,$$

for every $y > 0$. This might seem to imply that avoiding ruin requires increasing premiums by an amount greater than the the increase in claim size. However the paper also shows that eventual ruin is not certain if the risk reserve Y earns interest.

The model in Paulsen (1993) is preferably explained in 5 steps. The first step is that the surplus generating process $P_t^y = y + P_t$ is assumed to be a semimartingale with initial value y. The second step is that the inflation generating process I is assumed to be a semimartingale with $I_0 = 0$, while the level of inflation $\bar{I}$ is given as the solution of

$$d\bar{I}_t = \bar{I}_{t-} dI_t.$$

Here $\bar{I}_0 = 1$. As explained in Paulsen (1993) it then follows that at time t

$$\bar{I}_t = \exp\left\{ I_t - \frac{1}{2}\langle I^c, I^c \rangle_t \right\} \Pi_{0 \leq s \leq t} \left(1 + \Delta I_s\right) e^{-\Delta I_s}.$$

Here $\langle I^c, I^c \rangle$ is the predictable quadratic variation of the continuous martingale part I^c of the semimartingale I. If I is discontinuous at time t then ΔI_t is the jump $I_t^+ - I_t^-$. Otherwise $\Delta I_t = 0$. The third step is that the inflated surplus process $\bar{P}^y$ at time t is given as

$$\bar{P}_t^y = y + \int_0^t \bar{I}_{s-} dP_s^y.$$

The fourth step is that the surplus is assumed to be continuously invested in stochastic assets. The return on investment process R is assumed to be a semimartingale with $R_0 = 0$. In terms of nominal units the total risk process $\bar{Y}_t$ is given as the solution of

$$d\bar{Y}_t = d\bar{P}_t^y + \bar{Y}_{t-} dR_t,$$

where $\bar{Y}_0 = y$. The last step is that the risk process in terms of real units at time t is given as

$$Y_t = \bar{I}_t^{-1} \bar{Y}_t,$$

where $Y_{0-} = y$. At time t let $\bar{R}_t = \exp\left\{R_t - \frac{1}{2}\langle R^c, R^c \rangle_t\right\}$, where $\langle R^c, R^c \rangle$ is the predictable quadratic variation of the continuous martingale part R^c of the semimartingale R. It is shown in Paulsen (1993) that Y can also be written as

$$Y_t = U_t^{-1} \left(y + \int_0^t U_{s-} dP_s \right), \tag{1.2.4}$$

where $U = \bar{I}\bar{R}^{-1}$.

Inspection of (1.2.2) and (1.2.3) above shows that the important quantity is not so much the interest rate or the inflation rate, as the difference between the two. This is often called the real interest rate. This is a consequence of the rate of inflation and the rate of interest being constant in those equations, which are taken from Delbaen and Haezendonck (1987). In Paulsen (1993) it is shown that the so-called real interest rate retains its importance as long as either $R - I$ or I is a continuous deterministic process. Thus, in these cases inflation can be accounted for by focusing on the *inflation-adjusted* rate of return, rather than the nominal rate of return.

In most of Paulsen (1993) it is also assumed that the vector process $\bar{\mathbf{X}} = (P, I, R)$ is a stochastic process with stationary independent increments, with a finite number of jumps on each finite interval. Furthermore, it is assumed that the first component (the surplus generating process P) is independent of the two other components. Under these assumptions it follows (see Krylov (2002)) that $\bar{\mathbf{X}}$ has the representation

$$\bar{\mathbf{X}}_t = \bar{\mathbf{a}}t + \bar{C}\bar{\mathbf{W}}_t + \bar{\mathbf{V}}_t.$$

Here $\bar{\mathbf{a}}$ is a constant vector, $\bar{W}$ is a three-dimensional Brownian motion process, $\bar{\mathbf{V}}$ is a three-dimensional compound Poisson process, independent of $\bar{W}$ and $\bar{C}$ is a 3×3 matrix with the property

$$\bar{C}\bar{C}^{\text{tr}} = \begin{bmatrix} \sigma_P^2 & 0 & 0 \\ 0 & \sigma_I^2 & c\sigma_I\sigma_R \\ 0 & c\sigma_I\sigma_R & \sigma_R^2 \end{bmatrix}.$$

Here $|c| \leq 1$ is the correlation between the continuous part of the inflation process and the return on investment process. In addition $\sigma_P, \sigma_I, \sigma_R$ are non-negative constants, "tr" means "transposed" and the Brownian motion vector $\bar{\mathbf{W}}$. Furthermore, it is assumed that the first component P of $\bar{\mathbf{V}}$ is independent of the other two components, I and R. In terms of the components of $\bar{\mathbf{X}}$ this gives

$$\begin{aligned} P_t &= pt + W_{P,t} - \sum_{n=1}^{N_{P,t}} S_{P,n}, \\ I_t &= \text{ı}t + W_{I,t} + \sum_{n=1}^{N_{I,t}} \tilde{S}_{I,n} \text{ and} \\ R_t &= rt + W_{R,t} + \sum_{n=1}^{N_{R,t}} \tilde{S}_{R,n}. \end{aligned} \tag{1.2.5}$$

Here $(W_P, W_I, W_R)^{\text{tr}} = \bar{C}\bar{\mathbf{W}}$, N_P, N_I and N_R are three Poisson processes with intensities λ, λ_I and λ_R respectively, and N_P is independent of (N_I, N_R). Also the summands in each sum are i.i.d., and $\{S_{P,n}\}_{n=1}^{N_{P,t}}$ and the jumps $\left(\left\{\tilde{S}_{I,n}\right\}_{n=1}^{N_{I,t}}, \left\{\tilde{S}_{R,n}\right\}_{n=1}^{N_{R,t}}\right)$ are independent. Moreover it is assumed that

$$P\left(\tilde{S}_{I,1} \leq -1\right) = P\left(\tilde{S}_{R,1} \leq -1\right) = 0.$$

As explained in Paulsen (1993), this leads to that at time t

$$\bar{I}_t = \exp\left\{\left(\text{ı} - \frac{1}{2}\sigma_I^2\right)t + W_{I,t}\right\}\Pi_{n=1}^{N_{I,t}}\left(1 + \tilde{S}_{I,i}\right) \text{ and}$$

$$\bar{R}_t = \exp\left\{\left(r - \frac{1}{2}\sigma_R^2\right)t + W_{R,t}\right\}\Pi_{n=1}^{N_{R,t}}\left(1+\tilde{S}_{R,i}\right).$$

Also given is the unified process for inflation and return on investment,

$$U_t = \exp\left\{-\alpha_U t + \sigma_U W_{U,t}\right\}\Pi_{n=1}^{N_{I,t}}\left(1+\tilde{S}_{I,i}\right)\Pi_{j=1}^{N_{R,t}}\frac{1}{\left(1+\tilde{S}_{R,j}\right)}, \qquad (1.2.6)$$

where $\alpha_U = r - \bar{\imath} + \frac{1}{2}\left(\sigma_I^2 - \sigma_R^2\right)$, $\sigma_U^2 = \sigma_I^2 - 2c\sigma_I\sigma_R + \sigma_R^2$, and W_U is a Brownian motion. Here we follow the convention that $\Pi_{i=1}^{0} = 1$.

Most of the implications for ruin probability discussed in Paulsen (1993) are easier to grasp if $\Pi_{n=1}^{N_{I,t}}\left(1+\tilde{S}_{I,n}\right)$ and $\Pi_{n=1}^{N_{R,t}}\left(1+\tilde{S}_{R,n}\right)$ are assumed to be independent, which we do for the rest of our discussion of that paper. With this assumption the only dependence between P, I and R is by means of the correlation c between the Brownian motion processes W_I and W_R. It also follows from Lemma 2.1 in Paulsen (1993) that $\Pi_{n=1}^{N_{I,t}}\left(1+\tilde{S}_{I,i}\right)\Pi_{n=1}^{N_{R,t}}\frac{1}{1+\tilde{S}_{R,i}}$ can be written as $V = \Pi_{n=1}^{N_{U,t}} S_{U,n}$, where N_U is a Poisson process with intensity $\lambda_U = \lambda_I + \lambda_R$. Furthermore, the $S_{U,n}$'s are i.i.d. and independent of N_U, and the $S_{U,n}$'s have the common distribution

$$F_U(s) = \frac{\lambda_I}{\lambda_U}F_I(s) + \frac{\lambda_R}{\lambda_U}\left(1 - F_R\left(\frac{1}{s-}\right)\right). \qquad (1.2.7)$$

A key result in Paulsen (1993) is Theorem 3.1, which gives that, with the assumptions made above, the process $Z_t = \int_0^t U_{s-}dP_s$ is a semimartingale. Continuing our assumption that $\Pi_{n=1}^{N_{I,t}}\left(1+\tilde{S}_{I,i}\right)$ and $\Pi_{n=1}^{N_{R,t}}\left(1+\tilde{S}_{R,i}\right)$ are independent, and also assuming that

$$r - \bar{\imath} + c\sigma_I\sigma_R - \sigma_R^2 + \lambda_U\left(1 - ES_{U,1}\right) > 0,$$

then $\lim_{t\to\infty} Z_t = Z_\infty$ exists and converges almost surely in L^1. In Theorem 3.2 in Paulsen (1993) it is shown that the probability of eventual ruin is given by

$$\psi(y) = \frac{H(-y)}{E\left[H\left(-Y_\tau\right)\middle|\tau < \infty\right]}, \qquad (1.2.8)$$

where H is the distribution function of Z_∞ and τ is the ruin time. This formula is similar to the formula (1.2.1) discussed earlier.

As pointed out in Bankovsky et al. (2011), some additional conditions are needed for the results in Paulsen (1993) to hold. These conditions are given in Remark 2(3) in Bankovsky and Sly (2009). However, it is clear from these conditions that these problems can be avoided by assuming that $p \geq 0$.

It follows from Theorem 3.4 in Paulsen (1993) that the distribution H can be derived from a certain integro-differential equation. We state this result below.

Theorem 1.2.1. *Let* $\Psi(u) = E\left[e^{iuS_{P,1}}\right]$ *be the characteristic function of* $S_{P,1}$. *Assume that*

$$\int_{-\infty}^{\infty} |\Psi'(u)|\, du < \infty,$$

and that

$$E\left[\ln S_{U,1}\right] < \infty.$$

In addition assume that either $\sigma_P = \sigma_U = 0$ *and*

$$\int_{-\infty}^{\infty} |\Psi(u)|\, du < \infty,$$

or that

$$\int_{-\infty}^{\infty} |u\Psi(u)|\, du < \infty.$$

Then the distribution function H *of* Z_∞ *is twice continuously differentiable and is the solution of*

$$\begin{aligned}\frac{1}{2}\left(\sigma_P^2 + \sigma_U^2 z^2\right) H''(z) + \left(-p + \left(\alpha_U + \frac{1}{2}\sigma_U^2\right) z\right) H'(z) - \left(\lambda_U + \lambda\right) H(z) \\ + \lambda_U \int_0^{\infty} H\left(\frac{z}{s}\right) dF_U(s) + \lambda \int_{-\infty}^{\infty} H\left(z+s\right) dF(s) = 0.\end{aligned} \tag{1.2.9}$$

Here α_U *is still* $r - \bar{\imath} + \frac{1}{2}\left(\sigma_I^2 - \sigma_R^2\right)$, λ *is still the intensity of the claims process, and* F *is still the claim size distribution. Asymptotic boundary conditions are* $\lim_{z\to-\infty} H(z) = 0$ *and* $\lim_{z\to\infty} H(z) = 1$. *If* $\sigma_U^2 = \sigma_P^2 = 0$, *then* H *is the continuously differentiable solution of* (1.2.9).

From the identity (1.2.8) it is obvious that

$$\psi(y) \leq \frac{H(-y)}{H(0)}. \tag{1.2.10}$$

In the most basic situation, $\lambda = 0$ (no jumps in the claims process), we get equality in (1.2.10). Here, if $S_{P,1}$ has an increasing failure rate, i.e.

$$P\left(S_{P,1} > u + v \middle| S_{P,1} > u\right) \leq P\left(S_{P,1} > u\right), \text{ for } u, v > 0,$$

then

$$\psi(y) \geq \frac{H(-y)}{E\left[H(S_{P,1})\right]}.$$

On the other hand if $\sigma_P^2 = 0$ and $S_{P,1}$ has a decreasing failure rate then

$$\psi(y) \leq \frac{H(-y)}{E\left[H(S_{P,1})\right]}. \tag{1.2.11}$$

The most basic situation with jumps is when the jumps are exponentially distributed and $\sigma_P = 0$, in which case we get equality in (1.2.11). An asymptotic result for the finite horizon ruin probability is given in Proposition 1.3.1.

Perhaps it is because deterministic inflation can be accounted for by considering the real interest rate that very few papers after Paulsen (1993) have included a separate inflation process. In the rest of the thesis we will tacitly make the assumption that inflation is indeed a continuous deterministic function and that the interest rate is the *real* interest rate. An alternative approach could be to consider the process U given in (1.2.6) as the "real" stochastic return on investment process. This might be a topic for further research. In this thesis our only result regarding stochastic inflation is the asymptotic formula in Proposition 1.3.1 in the next section.

Other than not including (explicit) inflation the assumptions in the later paper, Paulsen and Gjessing (1997), are similar to the assumptions in Paulsen (1993). Since in this model there is no I process to worry about it is more convenient to write the surplus generating process P at time t as

$$P_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_{P,t}} S_{P,i}, \quad t \geq 0. \tag{1.2.12}$$

Similarly, the investment return process R is written as

$$R_t = rt + \sigma_R W_R + \sum_{i=1}^{N_{R,t}} S_{R,i}. \tag{1.2.13}$$

Here W_P and W_R are independent Brownian motion processes that are also independent of the compound Poisson processes. As before all the jumps are i.i.d. and independent of the Poisson processes $N_{P,t}$ and $N_{R,t}$. Lastly, $N_{P,t}$ and $N_{R,t}$ are independent. The risk process is then given as the solution of the stochastic differential equation

$$Y_t = y + P_t + \int_0^t Y_{s-} dR_s, \tag{1.2.14}$$

which for time t has the solution

$$Y_t = \bar{R}_t \left(y + \int_0^t \bar{R}_s^{-1} dP_s \right). \tag{1.2.15}$$

Here $\bar{R}_t$ is given as

$$\bar{R}_t = \exp\left\{ \left(r - \frac{1}{2}\sigma_R^2 \right) t + \sigma_R W_{R,t} \right\} \Pi_{n=1}^{N_{R,t}} \left(1 + S_{R,n}\right), \quad t \geq 0.$$

As well as the assumption that $F_R(0) = P\left(1 + S_{R,1} \leq 0\right) = 0$, it is also assumed that both $S_{P,1}$ and $S_{R,1}$ have finite expectation. Under these assumptions it is shown that the risk process Y has the same distribution as $\tilde{Y}$, where

$$\begin{aligned}\tilde{Y}_t = y &+ \int_0^t \left(p + r\tilde{Y}_s \right) ds + \int_0^t \sqrt{\sigma_P^2 + \sigma_R^2 \tilde{Y}_s^2} dW_s - \sum_{i=1}^{N_{P,t}} S_{P,i} \\ &+ \int_0^t \tilde{Y}_{s-} d\left(\sum_{i=1}^{N_{R,t}} S_{R,i} \right).\end{aligned}$$

It is also shown that the infinitesimal generator for $\tilde{Y}$ is given by

$$\begin{aligned}Ag(y) = &\frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) g''(y) + (p + ry)\, g'(y) \\ &+ \lambda \int_0^\infty \left(g\left(y - x\right) - g(y) \right) dF(x) \\ &+ \lambda_R \int_{-1}^\infty \left(g\left(y\left(1 + x\right)\right) - g(y) \right) dF_R(x).\end{aligned} \tag{1.2.16}$$

The result in Paulsen and Gjessing (1997) that is most relevant for this thesis is Theorem 2.1 part (i), which we state below. The proof of this result is based on the generator A given above.

Theorem 1.2.2. *Assume that $g(y)$ is bounded and twice continuously differentiable on $y \geq 0$, with a bounded first derivative there, where we at $y = 0$ mean the right-hand derivative. If $g(y)$ solves*

$$Ag(y) = -\lambda \bar{F}(y) \qquad \text{on } y > 0, \tag{1.2.17}$$

subject to the asymptotic boundary condition

$$\lim_{y \to \infty} g(y) = 0,$$

and, if $\sigma_P > 0$, the boundary condition

$$g(0) = 1$$

holds, then

$$\psi(y) = g(y)$$

for every $y \geq 0$.

In the paper Yuen et al. (2004) it is shown that a smooth solution of (1.2.17) exists provided the following conditions are satisfied:

(i) $\sigma_P = 0$.

(ii) $S_{P,1}$ and $S_{R,1}$ have finite first two moments, the distribution functions F and F_R are three times continuously differentiable, and the limits $F'(0^+), F''(0^+), F'''(0^+), F_R{}'(-1^+), F_R{}''(-1^+)$ and $F_R{}'''(-1^+)$ all exist.

(iii) $2r - \mathrm{Var} S_{P,1} > 0$, $\lambda + \lambda_R - (2r + \mathrm{Var} S_{P,1}) > 0$ and the net profit condition $p - \lambda E S_{P,1} > 0$ is satisfied.

Some alternative sufficient conditions for the existence of a smooth solution of (1.2.17) are given in Paulsen et al. (2005). Here it is shown that if $\lambda_R = 0$ (i.e. no jumps in the return on investment process R), then a smooth solution exists, provided the distribution function F is four times continuously differentiable on $[0, \infty)$ and, for some $c > 0$, $\bar{F}(x)x^c$ is bounded for every $x > 0$. A third set of sufficient conditions is found in Yuen and Wang (2005).

A few examples are given in Paulsen and Gjessing (1997) where the equation (1.2.17) can be explicitly solved. One of the examples is the case when $\sigma_P = \sigma_R = \lambda_R = 0$ and $S_{P,1}$ is exponentially distributed with parameter β. Another example is when $\sigma_P = \sigma_R = \lambda_R = 0$ and F is a mixture of two exponential distributions. For this case the solution of (1.2.17) is more complex and takes the form of an integral representation.

A generalization of the first example in Paulsen and Gjessing (1997) is to let the risk process Y_t take the form

$$Y_t = y + \int_0^t q(Y_s)\,ds - \sum_{i=1}^{N_{P,t}} S_{P,i}. \tag{1.2.18}$$

Here q is a continuous function and the claim sizes are still exponentially distributed with parameter β. It is shown in Dassios and Embrechts (1989) that in this case

$$\psi(y) = \frac{\int_y^\infty \frac{e^{-\beta x+\lambda Q(s)}}{q(x)}dx}{\frac{1}{\lambda} + \int_0^\infty \frac{e^{-\beta x+\lambda Q(s)}}{q(x)}dx},$$

where $Q(x) = \int_0^x q(s)^{-1}ds$.

In Bankovsky et al. (2011) there is a discussion of more general risk processes of the form

$$Y_t = e^{\xi_t}\left(y + \int_0^t e^{-\xi_s}d\eta_s\right), \qquad t \geq 0. \tag{1.2.19}$$

where $(\xi_t, \eta_t)_{t\geq 0}$ is a bivariate Lévy process. The models defined in (1.2.12)-(1.2.15) are special cases of (1.2.19), with $\eta = P$ and

$$\xi_t = \left(r - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t} + \sum_{n=1}^{N_{R,t}} \ln\left(1 + S_{R,n}\right), \quad t \geq 0.$$

For models of type (1.2.19) Bankovsky et al. (2011) derive the theorem below.

Theorem 1.2.3. *Suppose that the following conditions hold:*

(a) $\psi(y) > 0$ *for every* $y \geq 0$.

(b) *There exists* $w > 0$ *such that* $Ee^{-w\xi_1} = 1$.

(c) *There exists* $\epsilon > 0$ *and* $c, d > 1$ *with* $\frac{1}{c} + \frac{1}{d} = 1$ *such that*

$$E\left[e^{-\max(1,w+\epsilon)c\xi_1}\right] < \infty \qquad \textit{and} \quad E\left[|\eta_1|^{\max(1,w+\epsilon)d}\right] < \infty.$$

(d) *The distribution of* ξ_1 *is spread out, i.e. has a convolution power with an absolutely continuous component.*

Then there exists a constant C *such that asymptotically*

$$\psi(y) \sim Cy^{-w}.$$

The result above tells us that under mild conditions, the eventual ruin probability decays like a power law even if the claim distribution is light-tailed. As an example consider the models defined in (1.2.12)-(1.2.15) with $\lambda_R = 0$. Assume that the claim size distribution has moments of all orders and that $r > \frac{1}{2}\sigma_R^2$. A calculation then shows that the theorem holds with $w = 2\frac{r}{\sigma_R^2} - 1$. Similar conclusions can be drawn from Klüppelberg and Kostadinova (2008), Kalashnikov and Norberg (2002) and Frolova et al. (2002). As we shall see in the next section, the ruin probability in finite time is not quite as gravely affected by (moderately) risky investments as is the case for the eventual ruin probability.

In most of the papers that include a return on investment process the return is assumed to be a constant (real) interest force. With this assumption the risk process is of the type (1.2.18), where $q(x)$ is a linear function. In Klüppelberg and Stadtmüller (1998) it is shown that if the claim size distribution is regularly varying with index $\alpha > 1$, then asymptotically

$$\psi(y) \sim \frac{\lambda}{\alpha r}\bar{F}(y).$$

The most noteworthy with this estimate is that it implies that the ruin probability decays as the tail distribution $\bar{F}(y)$, rather than as the integrated tail distribution $\int_y^\infty \bar{F}(x)dx$. Lastly, we mention that the paper Paulsen (1998) offers a fairly extensive survey of other results for eventual ruin. Newer results for eventual ruin are discussed in Paulsen (2008). That paper also discusses ruin in finite time, which is the topic of the next section.

1.3 Ruin probability in finite time

In this section we discuss the probability that ruin (as defined in Section 1.1) occurs within a finite time T. We will denote this probability by $\psi(y, T)$. Unfortunately the known results for ruin in a finite time horizon are generally even less explicit than the results for eventual ruin. The focus here is either on approximations or on results that can be seen as a basis for numerical computation.

Consider again the classical Cramér-Lundberg compound Poisson model, defined in Section 1.1. For a standardized model with premium rate $p = 1$ and standard exponentially distributed claims, assume that the net profit condition is satisfied, which in this simple model just means that $\lambda < 1$.

From Proposition IV.1.3 in Asmussen (2000), we then have that

$$\phi(y,t) = \lambda e^{-(1-\lambda)y} - \frac{1}{\pi}\int_0^{\pi} \frac{f_1(\theta) f_2(\theta)}{f_3(\theta)} d\theta, \tag{1.3.1}$$

where

$$\begin{aligned} f_1(\theta) &= \lambda \exp\left\{2\sqrt{\lambda}T\cos\theta - (1+\lambda)T + y\left(\sqrt{\lambda}\cos\theta - 1\right)\right\}, \\ f_2(\theta) &= \cos\left(y\sqrt{\lambda}\sin\theta\right) - \cos\left(y\sqrt{\lambda}\sin\theta + 2\theta\right) \text{ and} \\ f_3(\theta) &= 1 + \lambda - 2\sqrt{\lambda}\cos\theta. \end{aligned}$$

The result (1.3.1) can be easily extended to more general Cramér-Lundberg models with exponentially distributed claim sizes. As an intermediate step we first show how the ruin probability for a model with a general premium rate p can be expressed in terms of the ruin probability for a model with a standardized premium rate $p = 1$. The last step is to obtain ruin probabilities for a general claim counting process intensity λ and a general exponential parameter β, as well as a general premium rate p. Now, let $\psi(y, T, p, \lambda, \beta)$ be the probability of ruin as a function of the parameters p, λ and β as well as y and T. Since the ruin time,

$$\inf_{t>0}\left\{y + pt - \sum_{n=1}^{N_t} S_n < 0\right\} = \inf_{t>0}\left\{\frac{y}{p} + t - \sum_{n=1}^{N_t} \frac{S_n}{p} < 0\right\},$$

we see that

$$\psi(y, T, p, \lambda, \beta) = \psi\left(\frac{y}{p}, T, 1, \lambda, p\beta\right). \tag{1.3.2}$$

This standardizes the premium rate. To standardize the parameter of the exponential distribution we can use (1.3.2) and the transformation suggested in Proposition IV.1.3 in Asmussen (2000). This yields

$$\psi(y, T, p, \lambda, \beta) = \psi\left(\beta y, p\beta T, 1, \frac{\lambda}{p\beta}, 1\right).$$

Any general Cramér-Lundberg model with exponentially distributed claim sizes can thus be reduced to the standardized model treated in Proposition IV.1.3 in Asmussen (2000).

For general distributions it is shown in Pervozvansky Jr. (1998) that if the premium income is invested with constant (real) interest force $r \geq 0$ and the claim size distribution has a continuously differentiable density f, then $\phi(y,t) = 1 - \psi(y,t)$ is the solution of the following partial integro-differential equation (PIDE):

$$\frac{\partial \phi\left(y,t\right)}{\partial t} = \left(p + ry\right) \frac{\partial \phi\left(y,t\right)}{\partial y} - \lambda \phi\left(y,t\right) + \lambda \int_0^y \phi\left(y-z,t\right) f(z) dz. \quad (1.3.3)$$

Here the solution is subject to the initial condition

$$\phi\left(y,0\right) = 1 \qquad \text{on } y > 0$$

and the asymptotic condition $\lim_{y,t\to\infty} \phi\left(y,t\right) = 1$.

Let

$$IG\left(x;\zeta;u\right) = 1 - \Phi\left(\frac{u}{\sqrt{x}} - \zeta\right) + e^{2\zeta u}\Phi\left(-\frac{u}{\sqrt{x}} - \zeta\sqrt{x}\right),$$

where Φ is the distribution function of the normal distribution. Let $\mathcal{L}_F''(\gamma)$ and $\mathcal{L}_F'''(\gamma)$ be as in Section 1.1. As discussed in Section 1.1 above, and more thoroughly in Section IV.5 in Asmussen (2000), a diffusion model with correction terms can be used to approximate the Cramér-Lundberg model. In finite time this approach leads to the approximation

$$\phi\left(y,T\right) \approx IG\left(\frac{T\delta_1}{y^2} + \frac{\delta_2}{y}; -\frac{1}{2}\gamma y; 1 + \frac{\delta_2}{y}\right), \quad (1.3.4)$$

where T is the time horizon, γ is the adjustment coefficient, $\delta_1 = \lambda \mathcal{L}_F''(\gamma)$ and $\delta_2 = \frac{\mathcal{L}_F'''(\gamma)}{3\mathcal{L}_F''(\gamma)}$. In Asmussen and Højgaard (1999) it is discussed how the ruin probability for general renewal models can be approximated by the formula (1.3.4).

For regularly varying claim distributions Theorem 4.1 in Hult and Lindskog (2011), in combination with Example 3.5 in Hult and Lindskog (2011), can be used to obtain asymptotic formulas for the ruin probability for a fixed finite time horizon for models with investment. Below we have formulated a simplified form of their Theorem 4.1 to fit with models with a surplus generating process P of the form given in (1.2.12), a continuous investment process R of the form given in (1.2.13), and a risk process Y of the form given in (1.2.14).

Theorem 1.3.1. *Assume that the claim size distribution is regularly varying with index α and denote the fixed finite time horizon by T.*

(a) *In the case of the Cramér-Lundberg model the probability of ruin before time T is asymptotically given by*

$$\psi(y,T) \sim \lambda T \bar{F}(y).$$

(b) *Consider a risk process of the form given in* (1.2.14)*. Make the following extra assumptions:*

(i) *Either $\lambda_R = 0$ or for some $\delta > 0$ $E(1+S_R)^{-(\alpha+\delta)} < \infty$.*

(ii)
$$\theta = \frac{1}{2}\sigma_R^2\left(\alpha^2+\alpha\right) - \alpha r + \lambda_R\left(E(1+S_R)^{-\alpha} - 1\right) \neq 0.$$

Then the probability of ruin before time T is asymptotically given by

$$\psi(y,T) \sim \frac{1}{\theta}\left(e^{\theta T} - 1\right)\lambda \bar{F}(y).$$

Proof. As explained in Theorem 3 in Paper C, this follows from Theorem 4.1 and Example 3.5 in Hult and Lindskog (2011). □

Remark: Assume that $\lambda_R > 0$ and let $\tau_{R,1}, \tau_{R,2}, \ldots,$ denote the jump times of the R process. Let $X_i = \ln\left(R_{\tau_{R,i}+}\right) - \ln\left(R_{\tau_{R,i}-}\right)$ for $i \in 1, 2, \ldots,$. The condition $E(1+S_R)^{-\alpha} < \infty$ corresponds to $Ee^{-\alpha X_1} < \infty$. In other words the results in the theorem above only hold if the jumps of the log-returns of the investment process are light-tailed. Below we give a few well-known examples of such models.

Example 1.3.1. *Assume that $\lambda_R > 0$ and that each X_i is normal distributed with parameters μ and σ^2 (as in the* Merton model*, see Merton (1976)). Then for every $\alpha > 0$*

$$\theta = \frac{1}{2}\sigma_R^2\left(\alpha^2+\alpha\right) - \alpha r + \lambda_R\left(\exp\left(-\alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right) - 1\right).$$

Example 1.3.2. *Assume that $\lambda_R > 0$ and that the jumps of the log-returns are as in the* Kou model *(see Kou (2002)), i.e. obey an asymmetric exponential probability distribution with density*

$$f_X(x) = q\beta_1 1_{x>0} e^{-\beta_1 x} + (1-q)\beta_2 1_{x<0} e^{-\beta_2 |x|}$$

for some $q \in (0,1)$. Assume that $\beta_1, \beta_2 > 0$ and that $\beta_2 > \alpha$. Then

$$\theta = \frac{1}{2}\sigma_R^2\left(\alpha^2+\alpha\right) - \alpha r + \lambda_R\left(\frac{q}{1+\frac{\alpha}{\beta_1}} + \frac{1-q}{1-\frac{\alpha}{\beta_2}} - 1\right).$$

A similar result is also valid for the models in Paulsen (1993) with stochastic inflation as indicated below:

Proposition 1.3.1. *Assume that the claim size distribution is regularly varying with index* α. *Let* $U_t, \alpha_U, \sigma_U, S_{U,n}, \lambda_I, \lambda_R, \lambda_U, \tilde{S}_{I,n}, \tilde{S}_{R,n}, N_{I,t}, N_{R,t}, N_{U,t}, F_I$, F_R *and* F_U *be as in* (1.2.6) *and* (1.2.7). *Assume that* $\Pi_{n=1}^{N_{I,t}}\left(1+\tilde{S}_{I,n}\right)$ *and* $\Pi_{n=1}^{N_{R,t}}\left(1+\tilde{S}_{R,i}\right)$ *are independent. Furthermore, assume that* U_t *is a strictly positive process and that, for some* $\delta > 0$,

$$ES_{U,1}^{(\alpha+\delta)} < \infty.$$

Let

$$\theta = \frac{1}{2}\alpha^2\sigma_U^2 - \alpha\alpha_U + \lambda_U\left(ES_{U,1}^{\alpha} - 1\right).$$

Assume that $\theta \neq 0$. *Then asymptotically*

$$\psi(y,T) \sim \frac{1}{\theta}\left(e^{\theta T} - 1\right)\lambda\bar{F}(y),$$

where the the distribution function $F_U(x)$ *of the common distribution of the* $S_{U,1}$*'s is given by*

$$F_U(s) = \frac{\lambda_I}{\lambda_U}F_I(s) + \frac{\lambda_R}{\lambda_U}\left(1 - F_R\left(\frac{1}{s-}\right)\right). \tag{1.3.5}$$

Proof. This follows from Theorem 4.1 and Example 3.5 in Hult and Lindskog (2011), by considering (1.2.4) and making the appropriate time changes as in Theorem 3 in Paper C. It follows from Lemma 2.1 in Paulsen (1993) that the distribution of $S_{U,1}$ is given by (1.3.5). □

In Wang et al. (2012) the result in Theorem 1.3.1 for the classical case ($\sigma_P = r = \sigma_R = \lambda_R = 0$) is generalized to a general renewal process (defined in Section 1.1 above). Assume that the waiting times between claims are i.i.d. with finite expectation λ^{-1}, $ES_1 < \infty$ and the claims satisfy

$$\lim_{z\to 1}\lim_{x\to\infty}\frac{\bar{F}(xz)}{\bar{F}(x)} = 1.$$

Then asymptotically

$$\psi(y,T) \sim \frac{\lambda}{p - \lambda ES_1}\int_y^{y+T(p-\lambda ES_1)}\bar{F}(u)du.$$

In Wang et al. (2012) it is shown that this asymptotic formula holds even for certain kinds of dependence between claim sizes. Yet another generalization of this result is found in Chen et al. (2011), this time to a model where the risk process is a bivariate renewal process.

In the case when $\sigma_P = \sigma_R = \lambda_R = 0$ and $r > 0$ (constant force of interest model), it is shown in Tang (2005) that the asymptotic formula above holds even if the surplus generating process P is generalized to the form

$$P_t = y + C(t) - \sum_{n=1}^{N_t} S_n,$$

where $C(t)$ can be any nondecreasing and right-continuous stochastic process.

Theorem 1.3.1 and the generalization in Wang et al. (2012) only provide approximate ruin probabilities for "large" values of the initial capital and for a special class of distributions. As a basis for numerical calculations it would be useful to also have a formula for the exact ruin probability in finite time for investment models having more general distributions.

For models with a constant force of interest the ruin probability can be calculated by solving the PIDE (1.3.3). We want to obtain a PIDE for the ruin probability that is valid for stochastic investments. Let the integro-differential operator A be as in Theorem 1.2.2. In Paulsen (2008) it is stated that the ruin probability should be the solution of the following partial integro-differential equation (PIDE):

$$\frac{\partial \psi(y,t)}{\partial t} = A\psi(y,t) + \lambda \bar{F}(y), \tag{1.3.6}$$

subject to the initial condition

$$\psi(y,0) = 0, \qquad y > 0,$$

and the asymptotic boundary condition

$$\lim_{y \to \infty} \psi(y,t) = 0.$$

If the diffusion parameter σ_P is positive, we have the extra boundary condition

$$\psi(0,t) = 1.$$

A sufficient condition for the ruin probability to satisfy the above PIDE is that there exists a classical solution of (1.3.6), i.e. a solution that is bounded and smooth on the interior of the domain. This statement can be proved using arguments similar to those in the proof of Theorem 2.1 in Paulsen and Gjessing (1997).

Now let us look at the PIDE (1.3.6) for the case that $\lambda_R = 0$. Let L be the differential operator defined by

$$Lh(y,t) = \frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2)\frac{\partial^2 h(y,t)}{\partial y^2} + (p + ry)\frac{\partial h(y,t)}{\partial y},$$

and let A_0 be the integro-differential operator

$$A_0 h(y,t) = Lh(y,t) + \lambda \int_0^y h(y-z,t)dF(z) - \lambda h(y,t).$$

Here A_0 is the same as the operator A defined in (1.2.16), but with $\lambda_R = 0$.

In Paper D we consider the equation (1.3.6), under the assumptions that $\lambda_R = 0$ and $\sigma_P > 0$, that either $\sigma_R > 0$ or $r = \sigma_R = 0$, and that the tail of the claim size distribution satisfies the bound

$$\bar{F}(x) \leq C\,(1+x)^{-c}, \quad x \geq 0, \tag{1.3.7}$$

for some positive constants C and c. Under these assumptions it is shown in Paper D that a solution of (1.3.6) exists that is smooth for (y,t) away from the origin. In particular, a smooth solution exists even if the claim size distribution is discrete. Since the coefficients are all smooth and the integro-differential operator A is linear, this might seem like a trivial result. There are, however, a number of reasons, listed below, why this is not the case.

- The domain is unbounded.
  Some literature, in particular on PDEs, discuss problems with unbounded domain. In general, however, these treatises require at least that the coefficients of the space derivatives of second order be bounded. In our case the only coefficient of a second order derivative is

  $$\frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2),$$

  which is obviously not bounded for $y \in (0, \infty)$, when $\sigma_R > 0$.

- Violation of compatibility condition.
  The initial condition dictates that $\lim_{y\downarrow 0} \psi(y,0) = 0$, whereas the boundary condition dictates that $\lim_{s\downarrow 0} \psi(0,s) = 1 \neq 0$. The initial condition and the boundary condition are thus incompatible, and any solution of (1.3.6) must hence be discontinuous at the origin. This violates the requirement that a classical solution must be continuous at the boundary.

- Asymptotic boundary condition.
  In addition to the difficulties mentioned above we need to verify that, for any $s \in (0,t]$, $\lim_{y\to\infty} \psi(y,s) = 0$.

The upshot of this is that standard theory does not immediately ensure existence and uniqueness of a solution of equation (1.3.6). It turns out that by far the biggest problem is that the domain is unbounded and that in the general case, when $\sigma_R > 0$, the coefficients grow to infinity. To get around this problem we first consider solutions $\psi_\kappa(y,t)$ of (1.3.6) on a truncated domain with the more standard boundary condition $\psi(\kappa,t) = 0$, for some $\kappa > 0$.

To get around the problem with the singularity at the origin we consider the truncated solution $\psi_\kappa(y,t)$ as a sum of the three functions $\psi_{1,\kappa}(y,t), \ldots, \psi_{3,\kappa}(y,t)$. Here $\psi_{1,\kappa}(y,t)$ is a solution of

$$\begin{cases} \psi_{1,\kappa}(y,0) & = 0, \quad y \in (0,\kappa)\,, \\ \psi_{1,\kappa}(0,t) & = 1, \quad t \in [0,1]\,, \\ \psi_{1,\kappa}\,(\kappa,t) & = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{1,\kappa}(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_{1,\kappa}(y,t)}{\partial y^2} + p\frac{\partial \psi_{1,\kappa}(y,t)}{\partial y}, \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases}$$

$\psi_{2,\kappa}(y,t)$ is a solution of

$$\begin{cases} \psi_{2,\kappa}(y,0) & = 0, \quad y \in (0,\kappa)\,, \\ \psi_{2,\kappa}(0,t) & = 0, \quad t \in [0,1]\,, \\ \psi_{2,\kappa}\,(\kappa,t) & = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{2,\kappa}(y,t)}{\partial t} - L\psi_{2,\kappa} & = H_{1,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases}$$

Here

$$H_{1,\kappa}\,(y,t) = \frac{1}{2}\sigma_R^2 y^2 \frac{\partial^2 \psi_{1,\kappa}\,(y,t)}{\partial^2 y^2} + ry\frac{\partial \psi_{1,\kappa}\,(y,t)}{\partial y} - \lambda \psi_{1,\kappa}\,(y,t) + \lambda \int_0^y \psi_{1,\kappa}\,(y-z,t)\,dF(z) + \lambda \bar{F}(y).$$

Finally, $\psi_{3,\kappa}(y,t)$ is a solution of

$$\begin{cases} \psi_{3,\kappa}(y,0) & = 0, \quad y \in (0,\kappa)\,, \\ \psi_{3,\kappa}(0,t) & = 0, \quad t \in [0,1]\,, \\ \psi_{3,\kappa}\,(\kappa,t) & = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{3,\kappa} y,t)}{\partial t} - A_0\psi_{3,\kappa}\,(y,t) & = H_{2,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{1.3.8}$$

Here

$$H_{2,\kappa}(y,t) = -\lambda\psi_{2,\kappa}(y,t) + \lambda \int_0^y \psi_{2,\kappa}(y-z,t)\,dF(z).$$

In the general case with $\sigma_R > 0$ the coefficients grow to infinity. Therefore we choose the following change of variables: $x = \ln(1+y)$. This leads to a formulation where the coefficients are bounded independently of the value of κ. This makes it easier to obtain suitable bounds on the partial derivative of $\psi_{3,\kappa}(y,t)$. The last step is to obtain the solution of the equation (1.3.6) as a limit of a sequence of functions $\{\psi_{\kappa_n}(y,t)\}_{n=0}^{\infty}$, where $\kappa_n \to \infty$. In addition to existence of a smooth solution of (1.3.6), Paper D also establishes the following: If the tail distribution $\bar{F}(x)$ satisfies

$$\bar{F}(x) \le C(1+x)^{-c}, \quad x \ge 0,$$

for some positive constants C and c, then for some constant C_c, depending on c,

$$\psi(y,T) \le C_c y^{-c}. \tag{1.3.9}$$

In particular we conclude from this that if the moment generating function of the claim size distribution exists in a neighborhood around 0, then for an arbitrarily large c there exists a constant C_c such that (1.3.9) holds. This is markedly different from the asymptotic behavior of the corresponding eventual ruin probability discussed in Section 1.2.

1.4 Numerical calculation of ruin probability with investment

Very few of the results concerning the ruin probability are easily analyzable closed form solutions. Even when relatively simple asymptotic results are known, such as the approximations given in Theorem 1.3.1, it is still desirable to make numerical computations. This is because those approximations are often not very good for moderate, or even quite large, values of the initial reserve. As a consequence much attention has been paid to numerical calculations of ruin probability, especially for the classical Cramér-Lundberg model. Most of these are based on either matrix computation (see e.g. Asmussen and Rolski (1991)) or recursive formulas (see e.g Vylder (1999), Dufresne and Gerber (1994), Dickson and Waters (1991) and especially Steenackers and Goovaerts (1991) and Dickson and Waters (1999)). It is not, however, clear how these methods can be applied to models that include a stochastic return on investment process of the type defined in (1.2.12)-(1.2.14).

In Paulsen et al. (2005) there is a discussion on using numerical methods to calculate the eventual ruin probability. Instead of using recursive formulas the focus in that paper is solving the integro-differential equation (IDE) (1.2.17), assuming $\lambda_R = 0$.

In Paper C there is a discussion of finite-difference methods for solving the equivalent partial integro-differential equation (PIDE) (1.3.6) for ruin in finite time. Numerical examples are given for models fitted to different investment strategies and different data. Integrals are evaluated using pre-calculated Gaussian quadrature rules, while the numerical differentiation is much like that in Halluin et al. (2005).

The claim process is fitted to a classic dataset of Danish fire insurance claims. The investment strategies considered are investing in U.S. treasury bills of 3 month maturity, U.S. Treasury bonds or in American stocks. For each of these investment strategies a geometric Brownian motion model (GBM) is fitted to historical data. We calculated ruin probabilities for several GBM investment models and for some jump-diffusion investment models. For the jump-diffusion investment models and a few GBM investment models we used parameter values from Ramezani and Zeng (2007).

The main finding is that for data covering the period 2000-2011 the models fitted to investments in stocks lead to a ruin probability that is about twice as high as the ruin probabilities with the Cramér-Lundberg model or with the models fitted to investments in bonds. In contrast to this example we find that, when the models are fitted to returns of the S&P 500 for 1962-2003, the resulting ruin probabilities are slightly lower than for investments in bonds or the Cramér-Lundberg model. The results reflect and quantify the relatively high volatility of stock prices since year 2000.

2 Optimal dividend policy

2.1 De Finetti´s dividend problem, dividend policy and the value of an insurance company

As described in Avanzi (2009), during the first part of the twentieth century actuarial literature focused on minimizing the probability of ruin on an infinite time horizon in the Cramér-Lundberg model discussed in Section 1.1. On an infinite time horizon this assumes that insurance companies let their liquidity reserve grow without limit. Bruno de Finetti couldn't see why an older company should hold more capital than a younger one bearing similar risks, only because it is older. As described in Avanzi (2009) the goal of Bruno de Finetti was to propose an alternative formulation that would be sufficiently realistic and tractable to 'study the practical problems regarding risk and reinsurance'(translation in Avanzi (2009)). This led Bruno de Finetti to formulate in de Finetti (1957) the optimal dividends problem for an insurance company: Maximizing the expected sum of the discounted paid out dividends from time zero until ruin, often referred to as 'de Finetti's dividend problem'.

The rationale behind maximizing the expected sum of the discounted paid out dividends is based on the idea that the value of the company is given by this sum. It is remarked in Gordon (1959) that: 'The hypothesis that the investor buys the dividend when he acquires a stock seems intuitively plausible, because the dividend is literally the payment stream that he expects to receive'. An alternative hypothesis discussed in Gordon (1959) is that the investor buys income per share, referred to as 'The Earnings Hypothesis'. Based on some data on price and earnings for companies in four different industries it is concluded in Gordon (1959) that 'the dividend hypothesis is correct regardless of whether the earnings hypothesis is correct. The only point at issue is whether the dividend hypothesis is unnecessary.'

A seemingly different opinion is expressed in the classical paper Miller and Modigliani (1961). There it is argued that

> ... there are no 'financial illusions' in a rational and perfect economic environment. Values there are determined solely by 'real' considerations, in this case the earning power of the firm's assets and its investment policy, and not by how the fruits of the earning power are 'packaged' for distribution.

More precisely, the analysis in Miller and Modigliani (1961) is done in discrete time and with the following assumptions:

> In 'perfect capital markets', no buyer or seller ... is large enough for his transactions to have an appreciable impact on the then ruling price. All traders have equal and costless access to information about the ruling price and all other relevant characteristics of shares. No ... other transaction costs are incurred ... and there are no tax differentials either between distributed and undistributed profits or between dividends and capital gains.

> 'Rational behavior'means that investors always prefer more wealth to less and are indifferent as to whether a given increment to their wealth takes the form of cash payments or an increase in the market value of their holdings of shares.
>
> 'Perfect certainty' implies complete assurance on the part of every investor as to the future investment program and the future profits of every company . . . .
>
> 'Under these assumptions the valuation of all shares would be governed by the following fundamental principle: The price of each share must be such that the rate of return (dividends plus capital gains per dollar invested) on every share will be the same throughout the market over any given interval of time'.

What is then referred to as the 'dividend policy problem' in Miller and Modigliani (1961) is formulated as: 'Which is the better strategy for the firm in financing the investment: To reduce dividends and rely on retained earnings or to raise dividends, but float more new shares?'

Under the assumptions above it is then shown in Miller and Modigliani (1961) that there are at least four equivalent approaches to valuations of a stock that are all valid. These four approaches are the discounted cash flow approach, the current earnings plus future investments approach, the stream of dividends approach and the stream of earnings approach. Based on the equivalence of these approaches they conclude that dividend policy is irrelevant. In Miller and Modigliani (1961) the 'stream of dividends approach' is defined as the view that the current worth of a share is the discounted value of the stream of dividends to be paid on the share in perpetuity, thus very close to the objective function in de Finetti (1957).

It should be clear from the above that the term 'dividend policy' is used in a much more narrow sense in Miller and Modigliani (1961) than what is implied by de Finetti´s dividend problem. Moreover, it is noted in Miller and Modigliani (1961) that, in the special case of exclusively internal financing, '*dividend policy* is indistinguishable from *investment policy*; and there *is* an optimal investment policy which does in general depend on the rate of return'.

It is perhaps clarifying to consider the approach in Stiglitz (1974). Here the decisions of a company are divided into four groups:

(a) How should the firm finance its investment?

(b) How should the firm distribute its revenue?

(c) How much should the firm invest?

(d) Which projects should the firm undertake?

Decisions falling into group (a) and (b) are called *financial* decisions while decisions falling into group (c) and (d) are called *real* decisions. The discussion in Stiglitz (1974) leads to a theorem which establishes sufficient conditions for financial decisions to be inconsequential for the market value of the company. It is clear from the arguments, though, that this theorem does not hold when there is no external financing. In the case of no external financing (c) and (d) cannot be separated from (b), while (a) is constrained to retained earnings. On the other hand, if external financing is available, and absence of transaction costs and the other conditions in Stiglitz (1974) are all satisfied, then the result in Stiglitz (1974) suggests that there might be multiple solutions of the de Finetti dividend problem.

In Sethi (1996) it is argued that some mathematical errors are made in Miller and Modigliani (1961), and that additional conditions need to be satisfied before it can be concluded that the share price equals the present value of future per share dividends. As in Miller and Modigliani (1961) and de Finetti (1957), the analysis in Sethi (1996) is done in discrete time with a constant discounting factor and leads to the result below.

Theorem 2.1.1. *Let $D(t)$ be the present value of the total dividends paid during period t to stockholders of record at the start of the period t, and let $V(t)$ be the present value of the company at the start of period t. The share price equals the present value of the dividends accruing to it if and only if the sum of the dividend yields is infinite, i.e,*

$$\sum_{i=0}^{\infty} \frac{D(i)}{V(i)} = \infty.$$

According to Sethi (1996) this extra condition is necessary to rule out 'bubbles' and 'Ponzi schemes'.

In Rozeff (1982) it is argued that payment of dividends forces the company more frequently to the external capital markets. There the company must undergo the scrutiny of the investment banking and regulatory communities in order to raise new capital. This process thus eliminates much of the need for monitoring by the individual shareholder. Dividend payments serve as a bonding or monitoring function, and thus reduce what economists call the 'agency costs of equity'. On the other hand, going to the external markets incurs substantial transaction costs. According to Rozeff (1982) the optimal dividend policy is therefore a trade-off between those agency costs and the high transaction costs of external financing. This view is supported by the empirical studies Casey et al. (2007) and Puleo et al. (2009) of actual dividend payments from insurance companies to their shareholders. In that paper it is also suggested that 'The more highly regulated property and casualty insurers do appear to pay out more in dividends.'

A return of investment process is incorporated in almost all the models treated in papers A, B, C and D. In addition to affecting the ruin probability, a possible reason for including such a process is for valuation. In the study Foster (1977) of market valuation of non-life insurance companies it is argued that 'The underwriting+investment+capital gains earnings measure provides the best specification of an econometric valuation model.'

2.2 Optimal dividend strategies

We will refer to a solution of de Finetti´s dividend problem as an 'optimal dividend policy', although many economists probably would prefer to call such a solution an 'optimal investment policy'.

Let $\{Y_t\}$ be a risk process. Let $\{D_t^\pi\}$ be a nondecreasing process representing the sum of the dividends distributed over the time interval $[0, t]$. Denote the modified risk process at time t by Y_t^π. In order to be a tractable problem some rules for permissible (or admissible) dividend strategies are needed. These rules are given below and taken from Albrecher and Thonhauser (2009).

Definition 2.2.1. *With each admissible strategy π the corresponding ruin time is given as*

$$\tau^\pi = \inf \{t \geq 0 : Y_t^\pi < 0\}. \tag{2.2.1}$$

Here we assume the following:

(i) Ruin does not occur due to dividend payments.

(ii) The path of D_t^π is non-decreasing.

(iii) Payments stop after the event of ruin.

(iv) Decisions must be made in a predictable way.

The condition (ii) excludes capital injections or other forms of external financing. Some papers discuss optimal dividend policies in more general economic settings, where external financing is permitted, for example Sethi and Taksar (2002); Løkka and Zervos (2008); Paulsen (2008); Kulenko and Schmidli (2008); Bourlés and Henriet (2009) and Scheer and Schmidli (2011). Some of these models include definitions of ruin (or solvency) which are different from Definition 2.2.1.

Some common solutions (i.e. optimal dividend strategies) of the de Finetti dividend problem in the literature are given in the definitions below. These definitions are taken from Albrecher and Thonhauser (2009). In these definitions it is tacitly assumed that the risk process does not have positive jumps.

Definition 2.2.2. *Threshold strategy: Dividend is paid out continuously at a rate a whenever the current reserve is above level b. The cumulated dividend payments is then given by*

$$D_t = \int_0^{\min(t,\tau^\pi)} a 1_{\{Y_s^\pi \geq b\}} ds.$$

Definition 2.2.3. *Barrier strategy: Let x be the current reserve and let p be the rate of premium income (as in Section 1). For a fixed barrier height $b \geq 0$, the cumulated dividend payments are described by*

$$D_t = (x - b)\, 1_{x>b} + \int_0^{\min(t,\tau^\pi)} p 1_{\left\{Y_{s-}^\pi = b\right\}} ds.$$

Such a strategy pays out all the reserve above b at $t = 0+$. Subsequently all incoming premiums that lead to a surplus above b are immediately distributed as dividends.

Definition 2.2.4. *A band strategy is characterized by three sets, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, which partition the state space of the reserve process. Each set is associated with a certain dividend payment action for the current reserve x, as follows: If the current surplus $x \in \mathcal{A}$, then every incoming premium is paid out. If $x \in \mathcal{B}$, then a lump sum is paid out, moving the current reserve to the closest point in $\mathcal{A}$ that is smaller than x. If $x \in \mathcal{C}$ no dividend is paid.*

Definition 2.2.5. *A lump sum barrier strategy (sometimes called an impulse strategy) is characterized by two levels, b_1 and b_2, with $0 \leq b_1 < b_2$. The following rules are used for dividend payments: If the current surplus x is above or equal to b_2, then pay out the amount $x - b_1$ immediately. If the surplus is below b_2, then do nothing until the reserve reaches the level b_2 again.*

In the original paper, de Finetti (1957), the risk process evolves as a random walk with step sizes ± 1. For this model it is proven that the (unique) optimal dividend policy is to follow a barrier strategy. Since then a number of papers have focused on solving de Finetti's dividend problem for various risk models. In Gerber (1969) it is shown that if the risk process is as in the Cramér-Lundberg model, then the optimal dividend policy is a band strategy, while in the case of exponentially distributed claim sizes the optimal dividend policy is a lump sum barrier strategy. In these models it is assumed that no transaction costs are incurred when dividends are paid out. For general reviews of the literature on de Finetti´s dividend problem, which also include several models with transaction costs, we refer to Albrecher and Thonhauser (2009) and especially Avanzi (2009).

In Paper A the uncontrolled risk process is a fairly general diffusion process that satisfies the stochastic differential equation

$$dY_t = \mu\left(Y_t\right) + \sigma\left(Y_t\right) dW_t, \quad Y_0 = y.$$

Here W_t is a Brownian motion process and $\mu(y), \sigma(y)$ are function satisfying the following requirements:

(a) $|\mu(y)| + |\sigma(y)| \leq C\left(1 + y\right)$ for all $x \geq 0$ and some $C > 0$.

(b) $\mu(y), \sigma(y)$ are continuously differentiable and Lipschitz continuous, and the derivatives $\mu'(y)$ and $\sigma'(y)$ are Lipschitz continuous.

(c) $\left(\sigma(y)\right)^2 > 0$ for all $y \geq 0$.

(d) $\mu'(y) \leq d$, where d is the discounting rate.

Especially relevant with regard to Paper A is the following result in Shreve et al. (1984). For a diffusion process satisfying (a)-(d) the optimal dividend policy is a barrier strategy. In Paulsen (2003) it is shown that this result holds even when the dividend is maximized under solvency constraints. In Paulsen (2007) it is shown that, under the same conditions on the uncontrolled risk process, and when payment of divided is assumed to incur fixed and proportional transaction costs, then the optimal dividend strategy is a lump sum strategy. In Paper A it is shown that this last result holds even when the dividend is maximized under solvency constraints.

In Paper B the uncontrolled risk process follows the stochastic differential equation

$$dY_t = \mu(Y_t) + \sigma(Y_t)\, dW_t - dS_t.$$

Here W is a Brownian motion and S is a compound Poisson process. There are some growth conditions on the functions $\mu(y)$ and $\sigma(y)$ similar to (a)-(d). This class of jump-diffusion processes is very similar to the process defined by the stochastic equation (1.2.14) above, except that no jumps are allowed in the return on investment process. In most of Paper B it is assumed that payment of dividends incurs both fixed and proportional transaction costs.

The main finding in Paper B is that for this model a lump sum barrier strategy is optimal when the claim size distribution belongs to a class of light-tailed distributions, including the exponential distribution. When there are no transaction costs it is found that a simple barrier strategy is optimal. Furthermore, a numerical method is developed that can be used to determine whether, for a given calibrated risk process, a lump sum strategy is an optimal dividend strategy.

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3 Paper A

Optimal dividend policies for a class of growth-restricted diffusion processes under transaction costs and solvency constraints

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In this paper, we consider a company where surplus follows a rather general diffusion process and whose objective is to maximize expected discounted dividend payments. With each dividend payment there are transaction costs and taxes and it is shown in [7] that under some reasonable assumptions, optimality is achieved by using a lump sum dividend barrier strategy, i.e. there is an upper barrier $\bar{u}^*$ and a lower barrier $\underline{u}^*$ so that whenever surplus reaches $\bar{u}^*$, it is reduced to $\underline{u}^*$ through a dividend payment. However, these optimal barriers may be unacceptably low from a solvency point of view. It is argued that in that case one should still we should look for a barrier strategy, but with barriers that satisfy a given constraint. We propose a solvency constraint similar to that in [6]; whenever dividends are paid out the probability of ruin within a fixed time T and with the same strategy in the future, should not exceed a predetermined level ε. It is shown how optimality can be achieved under this constraint, and numerical examples are given.

Keywords: Optimal dividends; general diffusion; solvency constraint; quasi-variational inequalities; lump sum dividend barrier strategy.

AMS subject classification: 49N25, 93E20, 91B28, 60J70, 65M06

1. Introduction

Finding optimal dividend strategies is a classical problem in the financial and actuarial literature. The idea is that the company wants to pay some of its surplus as dividends, and the problem is to find a dividend strategy that maximizes the expected total discounted dividends received by the shareholders. The typical time horizon is until ruin occurs, i.e. until the surplus is negative for the first time.

However, left to their own, financial institutions may make decisions that can jeopardize their solvency, and those with a claim on the company, e.g. account holders of a bank or customers of an insurance company, have an unacceptably high probability of loosing all or parts of their claims. As a consequence, most countries impose some regulation on financial companies, and in addition the companies themselves will usually have their own, albeit sometimes lax, capital requirements.

The task for the management is therefore not to maximize expected discounted dividends as such, but to do it under proper solvency constraints. One such constraint was suggested in [6], and we shall apply the same idea in this paper. We also let the capital of the company

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follow the same diffusion process as in [6], originally presented in [11]. To explain, in [11] it was proved that with their model, and provided there are no costs or taxes associated with dividend payments, if an optimal policy exists it is of barrier type, i.e. there is a barrier u^* so that whenever capital reaches u^*, dividends are paid with infinitesimal amounts so that capital never exceeds u^*. The resulting accumulated dividend process is a singular process, hence the name singular control. With the same setup, in [6] it was proved that when solvency requirements prohibit dividend payments unless capital is at least $u_0 > u^*$, then it is optimal to use a singular control at u_0. Therefore, it is natural to use u_0 as a barrier, and it was suggested that u_0 could be determined as follows: whenever capital is at u_0, ruin within a fixed time T by following the same policy should not exceed a small, predetermined number ε. We denote the corresponding u_0 by u_ε. Thus the problem of optimal dividend payments was linked to the problem of calculating ruin probabilities, the latter being a key concept in risk theory. Clearly, increasing u_0 implies that the ruin probability is decreased, so the problem can be reduced to a one dimensional search problem for u_ε. Although in both [11] and [6] there were no transaction costs or taxes, proportional costs or taxes will not change the problem significantly. However, when each dividend payment carries a fixed cost, the problem changes from a singular control problem to an impulse control problem. It was shown in [7], using the same diffusion model as in [11], that if there is an optimal dividend strategy it will be of a two-barrier type. To explain, there is a lower barrier $\underline{u}^* \geq 0$ and an upper barrier $\bar{u}^*$ so that when capital reaches $\bar{u}^*$, dividends are paid bringing the capital down to $\underline{u}^*$.

In this paper we will make the same assumptions as in [7], but slightly differently formulated. With each dividend payment there is a fixed cost K and a tax rate $1-k$ with $0 < k < 1$. We will argue that if the optimal policy is too risky, look for a lower barrier $\underline{u}_\varepsilon > 0$ and an upper barrier $\bar{u}_\varepsilon$ that maximizes expected discounted dividends and at the same time satisfy the solvency constraint as presented in the above paragraph. This problem is more difficult than that in [6] since we must look for a pair $(\underline{u}_\varepsilon, \bar{u}_\varepsilon)$, not just a number u_ε. One issue is to find a fast method to calculate the ruin probability for a given lower and upper barrier, and we will show how we can adapt the Thomas algorithm for solving tridiagonal systems together with the Crank-Nicolson algorithm to solve the relevant partial differential equations. The paper ends with numerical examples.

2. The model and a general optimality result

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and P-complete. Assume that the uncontrolled surplus process follows the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{2.1}$$

where W is a Brownian motion on the probability space and $\mu(x)$ and $\sigma(x)$ are Lipschitz-continuous. Let the company pay dividends to its shareholders, but at a fixed transaction cost $K > 0$ and a tax rate $0 < 1-k < 1$. This means that if $\xi > 0$ is the amount by which the capital is reduced, then the net amount of money the shareholders receive is $k\xi - K$. Since every dividend payment results in a transaction cost $K > 0$, the company should not pay out dividends continuously but only at discrete time epochs. Therefore, a strategy can be described by

$$\pi = (\tau_1^\pi, \tau_2^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \xi_2^\pi \ldots, \xi_n^\pi, \ldots),$$

where τ_n^π and ξ_n^π denote the times and amounts of dividends. Thus, when applying the strategy π, the resulting surplus process X_t^π is given by

$$X_t^\pi = x + \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - \sum_{n=1}^\infty 1_{\{\tau_n^\pi < t\}} \xi_n^\pi. \tag{2.2}$$

The process X^π is left continuous with right limits, so when applying e.g. Itô's formula, it will be on the right continuous with left limit version $\{X_{t+}\}$. Also, we define $\Delta X_t = X_{t+} - X_t$.

Sufficient conditions for existence and uniqueness of (2.2) are assumptions A1 and A2 below.

Definition 2.1. A strategy π is said to be admissible if

(i) $0 \le \tau_1^\pi$ and for $n \ge 1$, $\tau_{n+1}^\pi > \tau_n^\pi$ on $\{\tau_n^\pi < \infty\}$.

(ii) τ_n^π is a stopping time with respect to $\{\mathcal{F}_t\}_{t\ge 0}$, $n = 1, 2. \ldots$

(iii) ξ_n^π is measurable with respect to $\mathcal{F}_{\tau_n^\pi}$, $n = 1, 2. \ldots$

(iv) $P\left(\lim_{n\to\infty} \tau_n^\pi \le T\right) = 0$, $\forall T \ge 0$.

(v) $0 < \xi_n^\pi \le X_{\tau_n}^\pi$.

We denote the set of all admissible strategies by Π.

With each admissible strategy π we define the corresponding ruin time as

$$\tau^\pi := \inf\{t \ge 0 : \ X_t^\pi < 0\}$$

and the performance function $V_\pi(x)$ as

$$V_\pi(x) = E_x\left[\sum_{n=1}^\infty e^{-\lambda\tau_n^\pi}(k\xi_n^\pi - K)1_{\{\tau_n^\pi \le \tau^\pi\}}\right],$$

where by P_x we mean the probability measure conditioned on $X_0 = x$. $V_\pi(x)$ represents the expected total discounted dividends received by the shareholders until ruin when the initial reserve is x. Since we deal with the optimization problem on the time interval $[0, \tau^\pi]$, we can assume without loss of generality that $X_t^\pi \equiv 0$ for $t > \tau^\pi$.

Define the optimal return function

$$V^*(x) = \sup_{\pi\in\Pi} V_\pi(x)$$

and the optimal strategy, if it exists, by π^*. Then $V_{\pi^*}(x) = V^*(x)$.

Definition 2.2 A lump sum dividend barrier strategy $\pi = \pi_{\bar{u},\underline{u}}$ with the parameters $\bar{u}$ and $\underline{u}$ satisfies for $X_0^\pi < \bar{u}$,

$$\tau_1^\pi = \inf\{t > 0 : X_t^\pi = \bar{u}\}, \qquad \xi_1^\pi = \bar{u} - \underline{u},$$

and for every $n \geq 2$,

$$\tau_n^\pi = \inf\{t > \tau_{n-1}^\pi : X_t^\pi = \bar{u}\}, \qquad \xi_n^\pi = \bar{u} - \underline{u}.$$

When $X_0^\pi \geq \bar{u}$,

$$\tau_1^\pi = 0, \qquad \xi_1^\pi = X_0^\pi - \underline{u},$$

and for every $n \geq 2$, τ_n^π is defined as above.

With a given lump sum dividend barrier strategy $\pi_{\bar{u},\underline{u}}$, the corresponding value function is denoted by $V_{\bar{u},\underline{u}}(x)$.

The importance of the lump sum dividend barrier strategies is exemplified in e.g. Theorem 2.1 below, proved in [7]. In order to present the theorem, we make a list of assumptions.

A1. $\mid \mu(x) \mid + \mid \sigma(x) \mid \leq C(1+x)$ for all $x \geq 0$ and some $C > 0$.

A2. $\mu(x)$ and $\sigma(x)$ are continuously differentiable and Lipschitz continuous, and the derivatives $\mu'(x)$ and $\sigma'(x)$ are Lipschitz continuous for all $x \geq 0$.

A3. $\sigma^2(x) > 0$ for all $x \geq 0$.

A4. $\mu'(x) \leq \lambda$ for all $x \geq 0$, where λ is the discounting rate.

Define the operator $\mathcal{L}$ by

$$\mathcal{L}g(x) = \frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) - \lambda g(x)$$

for $g \in C^2(0,\infty)$. It is well known, see e.g. [7], that under the assumptions A1-A3 any solution of $\mathcal{L}g = 0$ is in $C^2(0,\infty)$. Let $g_1(x)$ and $g_2(x)$ be two independent solutions of $\mathcal{L}g(x) = 0$, chosen so that $g(x) = g_1(0)g_2(x) - g_2(0)g_1(x)$ has $g'(0) > 0$. Any such solution will be called a canonical solution. Then any solution $\mathcal{L}V(x) = 0$ with $V(0) = 0$ and $V'(0) > 0$ is of the form

$$V(x) = cg(x), \qquad c > 0.$$

Consider the following set of problems.

B1:
$$\begin{aligned} &\mathcal{L}V(x) = 0, \ \ 0 < x < \bar{u}^*, \\ &V(0) = 0, \\ &V(x) = V(\bar{u}^*) + k(x - \bar{u}^*), \qquad x > \bar{u}^*. \end{aligned}$$

B2:
$$\begin{aligned} &V(\bar{u}^*) = V(\underline{u}^*) + k(\bar{u}^* - \underline{u}^*) - K, \\ &V'(\bar{u}^*) = k, \\ &V'(\underline{u}^*) = k. \end{aligned}$$

B3:
$$\begin{aligned} &V(\bar{u}^*) = k\bar{u}^* - K, \\ &V'(\bar{u}^*) = k. \end{aligned}$$

Note that k and K are equivalent to $\frac{1}{1+d_1}$ and $\frac{d_0}{1+d_1}$ in [7].

Theorem 2.1. (Theorem 2.1 in [7]) *Assume that $A1 - A4$ hold. Then exactly one of the following three cases will occur.*

(i) B1+B2 have a unique solution for unknown $V(x)$, $\bar{u}^*$ and $\underline{u}^*$ and $V^*(x) = V(x) = V_{\bar{u}^*,\underline{u}^*}(x)$ for all $x \geq 0$. Thus the lump sum dividend barrier strategy $\pi^* = \pi_{\bar{u}^*,\underline{u}^*}$ is an optimal strategy.

(ii) B1+B3 have a unique solution for unknown $V(x)$ and $V^*(x) = V(x) = V_{\bar{u}^*,0}(x)$ for all $x \geq 0$. Thus the lump sum dividend barrier strategy $\pi^* = \pi_{\bar{u}^*,0}$ is an optimal strategy.

(iii) There does not exist an optimal strategy, but

$$V^*(x) = \lim_{\bar{u}\to\infty} V_{\bar{u},\underline{u}(\bar{u})}(x)$$

and this limit exists and is finite for every $x \geq 0$. In terms of a canonical solution,

$$V^*(x) = \frac{kg(x)}{\lim_{\bar{u}\to\infty} g'(\bar{u})}.$$

Here $V_{\bar{u},\underline{u}(\bar{u})}(x) = \sup_{\underline{u}\in[0,\bar{u})} V_{\bar{u},\underline{u}}(x)$.

Remark 2.1. *As pointed out in Remark 2.2e in [7], if $\lim_{x\to\infty} g'(x) = \infty$ then either B1+B2 or B1+B3 apply, hence an optimal solution exists. That $\lim_{x\to\infty} g'(x) = \infty$ is almost a necessary condition for existence of a solution can be shown as in Proposition 2.4 of [8]. Therefore, for simplicity we will typically assume that $\lim_{x\to\infty} g'(x) = \infty$.*

Here is a useful sufficient condition for $\lim_{x\to\infty} g'(x) = \infty$. The proof is given in the appendix.

Proposition 2.1. *Assume A1-A4 and that there exists an $x_0 \geq 0$ and an $\varepsilon > 0$ so that*

$$\mu'(x) \leq \lambda - \varepsilon, \quad x \geq x_0.$$

Then for any canonical solution g of $\mathcal{L}g(x) = 0$,

$$\lim_{x\to\infty} g'(x) = \infty.$$

Remark 2.2. *Arguing as in the end of the proof of Theorem 4.1, it follows that if there exists an $x_0 \geq 0$ so that*

$$\mu'(x) = \lambda, \quad x \geq x_0,$$

then $\lim_{x\to\infty} g'(x) < \infty$. Therefore, Proposition 2.1 is quite sharp.

3. Optimality under payout restrictions

Consider e.g. an insurance company that wants to use the optimal barriers $\bar{u}^*$ and $\underline{u}^*$ for its dividend payments. However, when policyholders pay their premiums in advance, they expect to have their claims covered. It is therefore reasonable that the company should not be allowed to pay dividends if that makes the surplus too small. One natural condition is that the surplus is not allowed to be less than some $\underline{u}_0 > 0$ after a dividend payment. Mathematically, for a policy π such a restriction can be written as

$$\sum_{0\leq\tau_n^\pi\leq\tau^\pi} 1_{\{X_{\tau_n^\pi+}<\underline{u}_0\}} = 0. \tag{3.1}$$

Let Π_0 denote the set of all admissible strategies satisfying (3.1). Define the new optimal return function $V_0^*(x)$ as

$$V_0^*(x) = \sup_{\pi \in \Pi_0} V_\pi(x). \tag{3.2}$$

Our aim is to find the optimal return function $V_0^*(x)$ and the optimal strategy $\pi_0 \in \Pi_0$ such that $V_{\pi_0}(x) = V_0^*(x)$.

Following Remark 2.1 we assume that $\lim_{x\to\infty} g'(x) = \infty$ so that either B1+B2 or B1+B3 have a solution. Trivially, if B1+B2 have a solution $V(x)$ for some c^*, $\bar{u}^*$ and $\underline{u}^* \geq \underline{u}_0$, the optimal strategy in case (i) of Theorem 2.1 is feasible under the constraint (3.1). Then $V_0^*(x) = V(x)$ and the optimal strategy is as in case (i) of Theorem 2.1.

Therefore we consider the cases when B1+B2 have a solution $V(x)$ for some c^*, $\bar{u}^*$ and $\underline{u}^* < \underline{u}_0$, or the case when B1+B3 have a solution $V(x)$ for some c^*, $\bar{u}^*$ and $\underline{u}^* = 0$. In these cases, the optimal strategy given by Theorem 2.1 does not satisfy the constraint (3.1). Consequently, we need to look for the optimal return function and the optimal strategy again. To this end, consider the problem for unknown V and $\bar{u}_0$:

C:
$$\begin{aligned} &\mathcal{L}V(x) = 0, \ \ 0 < x < \bar{u}_0, \\ &V'(\bar{u}_0) = k, \\ &V(0) = 0, \\ &V(x) = V(\underline{u}_0) + k(x - \underline{u}_0) - K, \qquad x \geq \bar{u}_0. \end{aligned}$$

The following result is proved in the appendix.

Theorem 3.1. *Assume that A1-A4 hold and that* $\lim_{x\to\infty} g'(x) = \infty$. *Let* $\underline{u}_0 > \underline{u}^*$, *where* $\underline{u}^*$ *is given in Theorem 2.1. Then Problem C has a unique solution for unknown* V *and* $\bar{u}_0$ *and*

$$V_0^*(x) = V(x) = V_{\bar{u}_0, \underline{u}_0}(x),$$

where $V_0^*(x)$ *is defined in (3.2). Thus the lump sum dividend barrier strategy* $\pi_{\bar{u}_0, \underline{u}_0}$ *is an optimal strategy in* Π_0. *Also, for given* $\underline{u}_1$ *so that* $\underline{u}^* < \underline{u}_0 < \underline{u}_1$, *for the corresponding optimal upper barriers it holds that* $\bar{u}^* < \bar{u}_0 < \bar{u}_1$.

According to Theorems 2.1 and 3.1, for a given lower barrier $\underline{u}_0$ the optimal strategy is the lump sum barrier strategy $\pi_{\tilde{u}, \underline{u}_1}$ where,

$$(\tilde{u}, \underline{u}_1) = \begin{cases} (\bar{u}_0, \underline{u}_0), & \text{if } \underline{u}_0 > \underline{u}^*, \\ (\bar{u}^*, \underline{u}^*), & \text{if } \underline{u}_0 \leq \underline{u}^*. \end{cases} \tag{3.3}$$

Here $(\bar{u}^*, \underline{u}^*)$ is as in Theorem 2.1, while $\bar{u}_0$ is as in Theorem 3.1. This addresses the problem of not being allowed to pay dividends that brings the capital too far down. The next result looks at the other end. What if the company cannot make a dividend payment when it wants, but has to postpone it until capital reaches a higher level? Let $\bar{u}_1 > \tilde{u}$ and let Π_1 be the set of all admissible policies satisfying

$$\sum_{0\le\tau_n^\pi\le\tau^\pi} 1_{\{X_{\tau_n^\pi+}<\underline{u}_1\cup X_{\tau_n^\pi}<\bar{u}_1\}} = 0, \tag{3.4}$$

i.e. all policies so that paying dividends when capital is less than $\bar{u}_1$ as well as reducing it below $\underline{u}_1$ through a dividend payment are ruled out. Define the new optimal return function $V_1^*(x)$ as

$$V_1^*(x) = \sup_{\pi\in\Pi_1} V_\pi(x). \tag{3.5}$$

Consider the problem for unknown V.

D:
$$\mathcal{L}V(x) = 0, \;\; 0 < x < \bar{u}_1,$$
$$V(0) = 0,$$
$$V(x) = V(\underline{u}_1) + k(x - \underline{u}_1), \qquad x > \bar{u}_1.$$

We then have the following theorem. It is proved in the Appendix.

Theorem 3.2. *Assume that A1-A4 hold and that* $\lim_{x\to\infty} g'(x) = \infty$. *Let* $\underline{u}_0$ *and* $\bar{u}_1 > \tilde{u}$ *be given, where* $\tilde{u}$ *is defined in (3.3). Then Problem D has a unique solution for unknown* V *and*

$$V_1^*(x) = V(x) = V_{\bar{u}_1,\underline{u}_1}(x),$$

where $V_1^*(x)$ *is defined in (3.5) and* $\underline{u}_1$ *in (3.3). Thus the lump sum dividend barrier strategy* $\pi_{\bar{u}_1,\underline{u}_1}$ *is an optimal strategy in* Π_1

The messages of Theorems 3.1 and 3.2 is that if the optimal barriers are too small, it is still optimal to use lump sum barrier strategies with the barriers as close to the optimal ones as possible in some sense. Therefore, we should look for barrier strategies, but with barriers sufficiently large to satisfy solvency requirements. This is the topic of Section 4.

4. Optimality under a solvency constraint

Having argued in Section 3 that barrier strategies are optimal also under reasonable constraints, we will in this section show how optimal barriers can be found that satisfy a natural solvency restriction. To describe this restriction, let $T < \infty$ be a fixed time horizon and define the survival probability as

$$\phi_{\bar{u},\underline{u}}(T, x) = P_x(\tau^{\pi_{\bar{u},\underline{u}}} > T),$$

where as before P_x means that $X_0 = x$ and $\pi_{\bar{u},\underline{u}}$ is the lump sum dividend strategy with barriers $\bar{u}$ and $\underline{u}$. For a given ruin tolerance ε we say that the strategy $\pi_{\bar{u},\underline{u}}$ is solvency admissible if

$$\phi_{\bar{u},\underline{u}}(T, \underline{u}) \ge 1 - \varepsilon. \tag{4.1}$$

Note that $\phi_{\bar{u},\underline{u}}(T, \underline{u}) = \phi_{\bar{u},\underline{u}}(T, \bar{u})$. This means that for a solvency admissible strategy $\pi_{\bar{u},\underline{u}}$, at the time of paying a dividend the probability of survival during the next time interval of length T using the same strategy cannot be smaller than $1 - \varepsilon$.

Also note that even when case (iii) of Theorem 2.1 applies, in principle condition (4.1) may not hold for any δ-optimal dividend strategy. The reason for this is that $\underline{u}(\bar{u})$ may be bounded as $\bar{u} \to \infty$. The following result shows that even in case (iii) there will exist a δ-optimal dividend strategy. It is proved in the appendix.

Theorem 4.1. *Assume case (iii) of Theorem 2.1. Then for any $b > 0$ and $\bar{u} > 0$ there exists a $\underline{\tilde{u}}(\bar{u}) < \bar{u}$ satisfying $\underline{\tilde{u}}(\bar{u}) \to \infty$ as $\bar{u} \to \infty$ so that*

$$V_{\bar{u},\underline{\tilde{u}}(\bar{u})}(x) \to V^*(x) \quad \forall x \in [0, b] \quad \text{as} \quad \bar{u} \to \infty.$$

By this result we can choose a $\underline{u}$ so large that for any $\delta > 0$ there is a δ-optimal lump sum dividend barrier that satisfies the constraint (4.1). Consequently, from now on it is assumed that $\lim_{x\to\infty} g'(x) = \infty$ as in Remark 2.1.

As in [6] it can be proved that if there exists a $C^{1,2}((0,T) \times (0,\bar{u}))$ function v that satisfies

$$v_t(t,x) = \frac{1}{2}\sigma^2(x)v_{xx}(t,x) + \mu(x)v_x(t,x), \quad (t,x) \in (0,T) \times (0,\bar{u}) \tag{4.2}$$

with initial value

$$v(0,x) = 1, \qquad 0 \le x \le \bar{u} \tag{4.3}$$

and boundary value for $t > 0$,

$$v(t,0) = 0 \quad \text{and} \quad v(t,\bar{u}) = v(t,\underline{u}), \tag{4.4}$$

then $v(T,x) = \phi_{\bar{u},\underline{u}}(T,x)$ is the survival probability. Here v_t means the partial derivative w.r.t. t and so on. In fact it is well known, see e.g. [5], that under assumptions A1-A3 any solution of (4.2) is $C^{1,2}((0,T) \times (0,\bar{u}))$.

Let us discuss how the optimal solvency admissible strategy can be found. By definition, for $\underline{u} > 0$, clearly

$$\begin{aligned} \phi_{\bar{u}_2,\underline{u}}(T,x) &> \phi_{\bar{u}_1,\underline{u}}(T,x), \quad \bar{u}_2 > \bar{u}_1, \\ \phi_{\bar{u},\underline{u}_2}(T,x) &> \phi_{\bar{u},\underline{u}_1}(T,x), \quad \underline{u}_2 > \underline{u}_1. \end{aligned}$$

Let $\phi(T,x) = P_x(X_t > 0 \ \forall t \in [0,T])$ be the survival probability when there is no control. If $\phi(T,\underline{u}) \le 1 - \varepsilon$ then $\underline{u}$ cannot be the lower barrier of a solvency admissible dividend strategy since paying dividends surely increases the ruin probability. However, if $\phi(T,\underline{u}) > 1 - \varepsilon$ then for sufficiently large $\bar{u}$, $\pi_{\bar{u},\underline{u}}$ will be a solvency admissible strategy. The lower bound $\underline{u}_m$ for the lower barrier in a solvency admissible strategy is therefore of interest, and it is given by

$$\phi(T,\underline{u}_m) = 1 - \varepsilon.$$

It is easy to show that if there exists a $C^{1,2}((0,T) \times (0,\infty))$ function w that satisfies

$$w_t(t,x) = \frac{1}{2}\sigma^2(x)w_{xx}(t,x) + \mu(x)w_x(t,x), \quad (t,x) \in (0,T) \times (0,\infty) \tag{4.5}$$

with initial value

$$w(0,x)=1, \qquad 0\le x\le\infty \tag{4.6}$$

and boundary value for $t>0$,

$$w(t,0)=0 \quad \text{and} \quad \lim_{x\to\infty} w(t,x)=1, \tag{4.7}$$

then $w(T,x)=\phi(T,x)$. Again, by A1-A3 any solution of (4.5) is $C^{1,2}((0,T)\times(0,\infty))$.

We are now ready for the optimality algorithm. It is assumed that $\lim_{x\to\infty} g'(x)=\infty$.

1. Calculate the optimal $V^*(x)$ with corresponding barriers $\bar{u}^*$ and $\underline{u}^*$.
2. Calculate $\phi_{\bar{u}^*,\underline{u}^*}(T,\underline{u}^*)$. If $\phi_{\bar{u}^*,\underline{u}^*}(T,\underline{u}^*)\ge 1-\varepsilon$, the optimal strategy satisfies the solvency constraint and we are done. If not continue to step 3.
3. Find $\underline{u}_m$ as the unique solution of $\phi(T,\underline{u}_m)=1-\varepsilon$. This can be done using a one dimensional search.
4. Let $\delta>0$ be a small number, and set $\underline{u}_i=\underline{u}_m+i\delta$, $i=1,2,\ldots$.
5. For each $\underline{u}_i$, find the corresponding optimal upper barrier by solving Problem C, and call this u_i. Calculate $\phi_{u_i,\underline{u}_i}(T,\underline{u}_i)$ and if $\phi_{u_i,\underline{u}_i}(T,\underline{u}_i)\ge 1-\varepsilon$, set $\bar{u}_i=u_i$. Also let $\bar{c}_i$ be the scaling factor so that the solution is $V_0^*(x)=\bar{c}_i g(x)$ for $x\le\bar{u}_i$. On the other hand, if $\phi_{u_i,\underline{u}_i}(T,\underline{u}_i)<1-\varepsilon$, increase u_i in steps of δ until the solvency constraint is satisfied. Let $\bar{u}_i$ be the corresponding upper barrier and $\bar{c}_i$ the scaling factor found by solving Problem D.
6. Do this until $\bar{c}_i$ falls significantly. Then let c_ε be the highest $\bar{c}_i$ and $\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$ be the corresponding $\bar{u}_i$ and $\underline{u}_i$ respectively. The optimal solvency admissible strategy is then $\pi_{\bar{u}_\varepsilon,\underline{u}_\varepsilon}$ and the corresponding value function is

$$V_\varepsilon(x)=\begin{cases} c_\varepsilon g(x), & 0\le x\le\bar{u}_\varepsilon,\\ V_\varepsilon(\bar{u}_\varepsilon)+k(x-\bar{u}_\varepsilon), & x>\bar{u}_\varepsilon.\end{cases}$$

The equations (4.2) and (4.5) together with their respective initial and boundary conditions are not easily solvable, but taking the Laplace transform brings them into ordinary differential equations. To see how, consider (4.2) and define

$$\tilde{v}(s,x)=L_v(s)=\int_0^\infty e^{-st}v(t,x)dt.$$

Straightforward calculations, using (4.3), gives that $\tilde{v}$ satisfies

$$\frac{1}{2}\sigma^2(x)\tilde{v}_{xx}(s,x)+\mu(x)\tilde{v}_x(s,x)-s\tilde{v}(s,x)=-1. \tag{4.8}$$

A particular solution is given by $\tilde{v}_p(s,x)=s^{-1}$. Let $\tilde{v}_1(s,x)$ and $\tilde{v}_2(s,x)$ be independent solutions of the homogeneous equation in (4.8). Then we have

$$\tilde{v}(s,x)=a_1(s)\tilde{v}_1(s,x)+a_2(s)\tilde{v}_2(s,x)+\frac{1}{s},$$

where a_1 and a_2 are determined from the initial and boundary conditions. Now $v(t,0) = 0$ implies that $\tilde{v}(s,0) = 0$ as well, and $v(t,\bar{u}) = v(t,\underline{u})$ implies that $\tilde{v}(s,\bar{u}) = \tilde{v}(s,\underline{u})$. Therefore, after some straightforward calculations

$$a_1(s) = \frac{1}{s}\frac{\tilde{v}_2(s,\bar{u}) - \tilde{v}_2(s,\underline{u})}{\tilde{v}_2(s,0)(\tilde{v}_1(s,\bar{u}) - \tilde{v}_1(s,\underline{u})) - \tilde{v}_1(s,0)(\tilde{v}_2(s,\bar{u}) - \tilde{v}_2(s,\underline{u}))} \tag{4.9}$$

$$a_2(s) = -\frac{1}{s}\frac{\tilde{v}_1(s,\bar{u}) - \tilde{v}_1(s,\underline{u})}{\tilde{v}_2(s,0)(\tilde{v}_1(s,\bar{u}) - \tilde{v}_1(s,\underline{u})) - \tilde{v}_1(s,0)(\tilde{v}_2(s,\bar{u}) - \tilde{v}_2(s,\underline{u}))} \tag{4.10}$$

Let $L_h^{-1}(t)$ be the inverse Laplace transform. Then $L_{s^{-1}}(t) = 1$ and using the Laplace transform property for integrals, we get that

$$v(T,x) = 1 - \int_0^T L^{-1}_{h_1+h_2}(t)dt,$$

where

$$h_i(s,x) = -sa_i(s)\tilde{v}_i(s,x), \quad i = 1,2.$$

Therefore, $P(\tau^\pi \in dt) = L^{-1}_{h_1+h_2}(t)dt$ when $\pi = \pi_{\bar{u},\underline{u}}$.

Similarly, $\tilde{w}(s,x) = L_w(s)$ also satisfies (4.8) with $\tilde{w}(s,0) = 0$ and $\lim_{x\to\infty}\tilde{w}(s,x) = s^{-1}$. Therefore, if we let $\tilde{w}_1(s,x)$ and $\tilde{w}_2(s,x)$ be two independent solutions of the homogeneous equation, and assume that $\hat{w}_i(s) = \lim_{x\to\infty}\tilde{w}_i(s,x)$, $i = 1,2$ exist, then

$$\tilde{w}(s,x) = b_1(s)\tilde{w}_1(s,x) + b_2(s)\tilde{w}_2(s,x) + \frac{1}{s},$$

where

$$b_1(s) = \frac{1}{s}\frac{1}{\tilde{w}_2(s,0)\frac{\hat{w}_1(s)}{\hat{w}_2(s)} - \tilde{w}_1(s,0)},$$

$$b_2(s) = \frac{1}{s}\frac{1}{\tilde{w}_1(s,0)\frac{\hat{w}_2(s)}{\hat{w}_1(s)} - \tilde{w}_2(s,0)}.$$

Inversion formulas are similar to those above.

Example 4.1 Assume that μ and σ^2 are constants. Then it is easy to see that

$$\tilde{v}_i(s,x) = \tilde{w}_i(s,x) = e^{c_i(s)x}, \quad i = 1,2,$$

where

$$c_1(s) = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2s}{\sigma^2}} > 0 \quad \text{and} \quad c_2(s) = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2s}{\sigma^2}} < 0.$$

Plugging this into (4.9) and (4.10) gives $\tilde{v}(s,x)$. Inverting this Laplace transform is unfortunately not straightforward.

Also $\hat{w}_1(s) = \infty$ and $\hat{w}_2(s) = 0$, hence $b_1(s) = 0$ and $b_2(s) = -s^{-1}$. Therefore,

$$\tilde{w}(s,x) = \frac{1}{s} - \frac{1}{s}e^{c_2(s)x}.$$

This can be inverted using standard tables for the Laplace transform. However, the solution can also be obtained by other methods, see e.g. [3] p.196, and is given by

$$w(T,x) = 1 - \frac{1}{\sqrt{2\pi}}\frac{x}{\sigma}\int_0^T t^{-\frac{3}{2}}e^{-\frac{(x+\mu t)^2}{2\sigma^2 t}}dt.$$

Therefore, $\underline{u}_m$ is given as the unique solution of (in x)

$$x = \frac{\sqrt{2\pi}\sigma\varepsilon}{\int_0^T t^{-\frac{3}{2}} e^{-\frac{(x+\mu t)^2}{2\sigma^2 t}}\, dt}.$$

5. Numerical Solutions

In order to provide a complete numerical solution to the problem, several differential equations, both ordinary and partial, have to be solved.

For problems B, C and D it is necessary to find a canonical solution g, either analytically, or if that is not possible or practical, numerically. In the latter case, the Runge-Kutta method can be used, together with linear interpolation between the grid points, this for g, g' and g''. In case the assumption of Proposition 2.1 does not hold, the numerical solution can be helpful to assess whether $\lim_{x\to\infty} g'(x) = \infty$ or not.

Problems B1+B2 or B1+B3. In [7] it is shown how this can be reduced to a one dimensional search problem, but for completeness and since the notation is somewhat different, we include it here. This method will also reveal whether an optimal solution exists.

1. Find the $x^* \in (0, \infty)$, if it exists, so that $g''(x^*) = 0$. If g is convex, we set $x^* = 0$, and if it is concave we set $x^* = \infty$. In the second case there is no solution, and by Lemma A.2b, $x^* = 0$ is equivalent to $\mu(0) \leq 0$, so this case is easy to establish.

2. Choose $x < x^*$ and let $c = \frac{k}{g'(x)}$ so that $cg'(x) = k$.

3. Find (if possible) a $y > x^*$ so that $g'(y) = \frac{k}{c}$. If this is not possible, try with a larger x until it is satisfied.

4. Calculate $k(y - x) - c(g(y) - g(x))$. If this is larger than K increase x. Otherwise decrease x.

5. Repeat the process until a solution is obtained, or until it is clear that there is no solution. In case there is a solution, upon convergence $\underline{u}^* = x$, $\bar{u}^* = y$ and $V^*(x) = cg(x)$ for $x \leq \bar{u}^*$.

Problem C. Assume it is clear that $\lim_{x\to\infty} g'(x) = \infty$. Then the following easy recipe works:

1. Choose an $x > \underline{u}_0$ and let $c = \frac{k}{g'(x)}$ so that $cg'(x) = k$.

2. Calculate $k(x - \underline{u}_0) - c(g(x) - g(\underline{u}_0))$. If this is larger than K decrease x, otherwise increase x.

3. Repeat the process until convergence is obtained. Upon convergence, $\bar{u}_0 = x$ and $V(x) = cg(x)$ for $x \leq \bar{u}_0$.

Problem D. The unique solution is given in (A.15) in the appendix.

The function $v(t, x)$ of (4.2)-(4.4). This is a standard PDE, but with nonstandard boundary conditions. It turns out that the Crank-Nicolson algorithm together with an adaption of the

Thomas algorithm to solve tridiagonal systems are well suited for this problem. For more details on the Crank-Nicolson and the Thomas algorithms the reader can consult [1].

To explain how this adaption works, let h be the grid length and ih, $i = 0, 1, \ldots, m$ the gridpoints so that $mh = \bar{u}$. Similarly, let k be the grid length and jk, $j = 0, 1, \ldots, n$ the gridpoints so that $nk = T$ Previously k is defined as the tax rate, but there should be no ambiguity so we follow the standard notation. Let $\sigma_i^2 = \sigma^2(ih)$ and $\mu_i = \mu(ih)$, $i = 0, 1, \ldots, m$. With v_i^j an approximation to $v(ih, jk)$, the Crank-Nicolson finite difference scheme is

$$\begin{aligned}\frac{1}{k}(v_i^{j+1} - v_i^j) &= \frac{1}{4h^2}\left[\sigma_i^2\left(v_{i+1}^{j+1} - 2v_i^{j+1} + v_{i-1}^{j+1} + v_{i+1}^j - 2v_i^j + v_{i-1}^j\right)\right. \\ &\qquad \left. +\mu_i h\left(v_{i+1}^{j+1} - v_{i-1}^{j+1} + v_{i+1}^j - v_{i-1}^j\right)\right].\end{aligned}$$

Collecting terms, this can be written as

$$\alpha_i v_{i-1}^{j+1} + \beta_i v_i^{j+1} + \gamma_i v_{i+1}^{j+1} = d_i^j, \tag{5.1}$$

where with $r = \frac{k}{h^2}$,

$$\begin{aligned}\alpha_i &= -\frac{r}{4}\left(\sigma_i^2 - \mu_i h\right), \\ \beta_i &= 1 + \frac{1}{2}r\sigma_i^2, \\ \gamma_i &= -\frac{r}{4}\left(\sigma_i^2 + \mu_i h\right), \\ d_i^j &= -\alpha_i v_{i-1}^j + (1 - \frac{1}{2}r\sigma_i^2)v_i^j - \gamma_i v_{i+1}^j.\end{aligned}$$

To start the iterations we use the intial value $v(0, x) = 1$ giving $v_i^0 = 1$ as well, and so the d_i^0, $i = 0, 1, \ldots, m$ can be calculated.

Now to the Thomas algorithm. To use it, for numerical stability we should have

$$|\alpha_i| + |\gamma_i| < |\beta_i|, \quad i = 0, 1, \ldots, m. \tag{5.2}$$

Let us check this condition:

1. $\sigma_i^2 \geq \mu_i h$. Then $|\alpha_i| + |\gamma_i| = \frac{1}{2}r\sigma_i^2 < \beta_i$, so this case is unproblematic.
2. $\sigma_i^2 < \mu_i h$. Then $|\alpha_i| + |\gamma_i| = \frac{1}{2}r\mu_i h < \beta_i$ if and only if $r < \frac{2}{\mu_i h - \sigma_i^2}$.

In order to have case 1 at all gridpoints, we can let

$$h \leq \max_i \frac{\sigma_i^2}{\mu_i},$$

and then for good convergence, a typical choice of r is $r = \frac{1}{2}$.

Assume that (5.2) is satisfied, and for simplicity write $v_i = v_i^{j+1}$ and $d_i = d_i^j$ in (5.1). The idea of the Thomas algorithm is to write

$$v_i = p_{i+1}v_{i+1} + q_{i+1} \tag{5.3}$$

for unknown p_{i+1} and q_{i+1}. Using this in (5.1) with $i - 1$ instead of i, we get

$$\alpha_i(p_i v_i + q_i) + \beta_i v_i + \gamma_i v_{i+1} = d_i. \tag{5.4}$$

Comparing (5.3) and (5.4) gives

$$p_{i+1} = -\frac{\gamma_i}{\alpha_i p_i + \beta_i} \quad \text{and} \quad q_{i+1} = \frac{d_i - \alpha_i q_i}{\alpha_i p_i + \beta_i}. \tag{5.5}$$

The boundary condition $v(t,0) = 0$ implies that $0 = v_0 = p_1 v_1 + q_1$, which is satisfied if $p_1 = q_1 = 0$. We can now use (5.5) to recursively calculate (p_i, q_i), $i = 2, \ldots, m$. Then using (5.3) backwards yields

$$\begin{aligned} v_m &= \frac{v_{m-1} - q_m}{p_m} \\ &= \frac{1}{p_m}\left(\frac{v_{m-2} - q_{m-1}}{p_{m-1}} - q_m\right) \\ &= \cdots \\ &= \frac{1}{P_{l+1}^m} v_l - \sum_{i=l+1}^{m} \frac{q_i}{P_i^m}, \end{aligned}$$

where

$$P_i^m = \prod_{j=i}^{m} p_j.$$

The boundary condition $v(t,\bar{u}) = v(t,\underline{u})$ implies that $v_m = v_l$ where $hl = \underline{u}$. Therefore,

$$v_m = -\frac{\sum_{i=l+1}^m \frac{q_i}{P_i^m}}{1 - \frac{1}{P_{l+1}^m}} = \frac{q_{l+1} + \sum_{i=l+2}^m P_{l+1}^{i-1} q_i}{1 - P_{l+1}^m}.$$

We can now go backwards using (5.3) again.

Remark 5.1 *Since in the Crank-Nicolson method $k = rh^2$, the space grid is typically much coarser than the time grid. In our problem we are searching for optimal points in the space variable, and therefore a fully implicit scheme with $k = rh$ for some r may be more suitable, since this allows for a finer space grid with the same computation time. The relation (5.1) will still apply, but with different coefficients, and so the Thomas algorithm is again applicable. However, we have not tried this method.*

The function $w(t,x)$ of (4.5)-(4.7). This is basically the same problem as that discussed above, except that instead of the nonstandard boundary condition $v(t,\bar{u}) = v(t,\underline{u})$, we impose the standard boundary condition $w(t,\bar{u}) = 1$ for some large $\bar{u}$. This will result in a slightly overestimate of the survival probability, but if $\bar{u}$ is chosen large enough, it should not be a real problem. Deciding when $\bar{u}$ is large enough is not an obvious task, but one way may be to keep x fixed at a moderate value, and then try with increasing $\bar{u}$ until the solution $w(0,x)$ stabilizes. Given the $\bar{u}$, the Crank-Nicolson algorithm together with the standard Thomas algorithm should work fine. Also, to find w analytically is easier than to find v, as we saw in Example 4.1.

6. Numerical examples

In this section we will give two numerical examples where optimal solutions with and without the solvency constraint are compared. In all plots, solid lines are for the case with the solvency constraint, while dashed lines are without solvency constraints. Each figure is split into three panels, where the first panel shows the optimal upper and lower barriers, both without and with the solvency constraint. The second panel shows the amount of dividends paid each time, i.e. $\bar{u}_\varepsilon - \underline{u}_\varepsilon$ and $\bar{u}^* - \underline{u}^*$. The third panel shows the constants c_ε and c^* so that the value functions equal $V_\varepsilon(x) = c_\varepsilon g(x),\ x \le \bar{u}_\varepsilon$ and $V^*(x) = c^* g(x),\ x \le \bar{u}^*$, where g is a canonical solution to be specified in each example. This means that for $x \le \bar{u}^*$, $1 - \frac{c_\varepsilon}{c^*}$ is the percentage loss of value due to the solvency constraint.

Before we give the examples, a few words on the numerics. All programs were written in R, but with subprograms in C for the number crunching. The simple algorithm described in Section 4 had to be modified. The reason is that the finite difference scheme (5.1) for solving (4.2) is accurate of order 2. However, a perturbation of size h of the boundary condition of a PDE will in general induce a change in the solution of order $O(h)$. Experimentally this seems to be the case also in this case for perturbations of $\bar{u}$ and $\underline{u}$, i.e. the most accurate numerical evaluations of the survival probability $\phi_{\bar{u},\underline{u}}$ for a given lump sum strategy $\pi_{\bar{u},\underline{u}}$ seem to come when $\underline{u}$ and $\bar{u}$ are both nodes on the PDE grid. This is especially true for $\underline{u}$. The general idea behind the program is therefore to minimize the calculations of off-grid $\underline{u}$ and $\bar{u}$ by defining the grids so that $\underline{u}$ is on the grid. To find the smallest solvency admissible $\bar{u}$ for a fixed $\underline{u} > u_m$, the program iterates as follows:

1. Start with a fairly coarse grid and find two adjacent points $\bar{v}_1 < \bar{w}_1$ so that according to the numerical solution $\pi_{\bar{w}_1,\underline{u}}$ is solvency admissible, while $\pi_{\bar{v}_1,\underline{u}}$ is not. Then one iteration of the secant method is used to find a $\bar{u}_1$ between $\bar{v}_1$ and $\bar{w}_1$.

2. Repeat the procedure with a finer grid, and find adjacent points $\bar{v}_2 < \bar{w}_2$ with the same properties as $\bar{v}_1$ and $\bar{w}_1$. Since the grid has changed, so has the numerical solution of the ruin probability, and frequently this resulted in $\bar{v}_2 > \bar{w}_1$.

3. Repeat the process a certain number of times. We repeated it until there was about 100 million nodes, where we used $k = \frac{1}{2}h^2$.

Although a bit circumstantial, this routine was in fact quite efficient in terms of total running time. As is seen from several of the figures below, the upper estimated values of $\bar{u}$ are sometimes quite erratic. However, this does not matter much since the corresponding values of c_ε do not vary much. When comparing different plots it is important to note that the y-axis varies, and when the span on the y-axis is very small the results may look more erratic than they actually are.

Example 6.1 Let $\mu(x) = \mu$ and $\sigma(x) = \sigma$ be constants, so that (2.1) becomes

$$X_t = x + \mu t + \sigma W_t.$$

By Proposition 2.1, $\lim_{x\to\infty} g'(x) = \infty$, hence an optimal strategy always exists.

In Figures 1-5 $\mu = \sigma = 1$ and the canonical solution chosen is

$$g(x) = \alpha e^{-\theta x} \sinh(\beta x)$$

with $\alpha = 0.9636$ (a bit arbitrary, admittedly) and

$$\theta = \frac{\mu}{\sigma^2} \quad \text{and} \quad \beta = \frac{1}{\sigma^2}\sqrt{2\lambda\sigma^2 + \mu^2}.$$

The other parameter values used are

$$\lambda = 0.1, \quad T = 10, \quad k = 0.95, \quad K = 0.05, \quad \varepsilon = 0.01.$$

In the figures 4 of these are kept fixed, while one is varying. In the discussion below, $\bar{u}$ is generic for both the unconstrained upper barrier $\bar{u}^*$ and the constrained $\bar{u}_\varepsilon$, and similar with $\underline{u}$.

In Figure 1, the discounting factor λ is varied. When there is no solvency constraint, we see from the first panel that both upper and lower barriers decrease as λ increases, which reflects the fact that with large values of λ early payments are important, since later payments are heavily discounted. When λ is small, the solvency constraint is not binding due to the long term perspective, and hence the necessity to avoid early ruin provides sufficiently large barriers. As λ increases, the constraint becomes binding, and the lower barrier even increases. The reason for this is that with a given constraint, there is more to gain by decreasing the upper barrier $\bar{u}_\varepsilon$ a lot, even if that means a small increase in the lower barrier $\underline{u}_\varepsilon$. However, it is interesting to see from the middle panel that the actual payout $\bar{u} - \underline{u}$ is not much affected by the solvency constraint. From the right panel, we see that the relative impact of the solvency constraint on the values c^* and c_ε increases quite a lot with λ, but for moderate values of λ it only causes small reductions in the value of the company.

In Figure 2 the time horizon T varies. Without the solvency constraint, the optimal solution is independent of T, which is also seen from the figure. For small T, the optimal solution gives sufficiently high survivial probability, hence the solvency constraint is not binding. As T increases, with the solvency constraint both the lower and upper barriers increase, but it is seen from the middle panel that the actual payout is again not much affected by the constraint. Why the payout first goes down and then increases we cannot explain. The ruggednes of the graph in the middle panel is due to numerical issues as discussed above. However, looking at the scale on the y-axis, we see that the variations are not severe. From the right panel it is seen that although the barriers are much influenced by the solvency constraint, the actual values c_ε are far less so.

In Figure 3 the retention rate k varies. As k increases, the amount received, $k(\bar{u} - \underline{u}) - K$, gets positive for lower amounts $\bar{u} - \underline{u}$ paid, and so both barriers decrease with k, both in the unconstrained and the constrained case. The effect of the solvency constraint is just to increase the barriers, but from the middle panel we see that again the payout $\bar{u} - \underline{u}$ is not much affected. From the right panel it is seen that the actual value of the company is not much affected neither.

In Figure 4 the fixed cost K is varied. Since for K large, the payout $\bar{u} - \underline{u}$ must be large in order for the dividend received, $k(\bar{u} - \underline{u}) - K$, to be positive, the optimal payout must increase with K, which is confirmed in the middle panel. For the rest, the picture is much the same as before, with the solvency constrained barriers lying above those without the solvency constraint, but with the payout $\bar{u} - \underline{u}$ rather unaffected. Also, as seen from the right panel, the solvency constraint does not reduce the value of the company by very much.

Finally, in Figure 5 the ruin tolerance ε varies. For sufficient large values of ε the solvency constraint is not binding, but as soon as the constraint becomes binding (read the x-axis from right to left), the picture is much the same as before with both lower and upper barriers increased due to the solvency constraint, but with payouts $\bar{u} - \underline{u}$ almost the same, and values c_ε moderately lower than the optimal c^*. When the solvency constraint is binding, the somewhat rugged behaviour of the curves in the first two panels is again due to numerical issues, but it is seen from the right panel that the optimal values c_ε is not much influenced, hence these numerical issues are rather unproblematic.

The tentative conclusion we can draw from this example is that the solvency constraint can have a quite large impact on the optimal barriers, but except in rather extreme cases, the impact on the actual payout $\bar{u} - \underline{u}$ as well as on the value c_ε versus c^*, is much more modest. This is good news for the shareholders, since what counts for them is how much smaller c_ε is than c^*, i.e. their "loss" due to the solvency constraint.

Figure 6 shows the values of $V_\varepsilon(x)$ and $V^*(x)$ for the standard parameter choice. This gave $(\bar{u}_\varepsilon, \underline{u}_\varepsilon) = (4.65, 3.13)$ and $(\bar{u}^*, \underline{u}^*) = (3.81, 2.22)$. Not so easy to see from the figure, but $V^*(x)$ is concave up to $x = 2.82$ and then convex. As of $V_\varepsilon(x)$ it is also concave up to $x = 2.82$, and then convex up to $\bar{u}_\varepsilon$. However, $V_\varepsilon'(\bar{u}_\varepsilon-) = 0.978 > V_\varepsilon'(\bar{u}_\varepsilon+) = k = 0.95$, and so V_ε is not convex from $x = 2.82$. That $V_\varepsilon'(\bar{u}_\varepsilon-) \geq V_\varepsilon'(\bar{u}_\varepsilon+)$ is a general fact, proved in Lemma A.6 in the appendix.

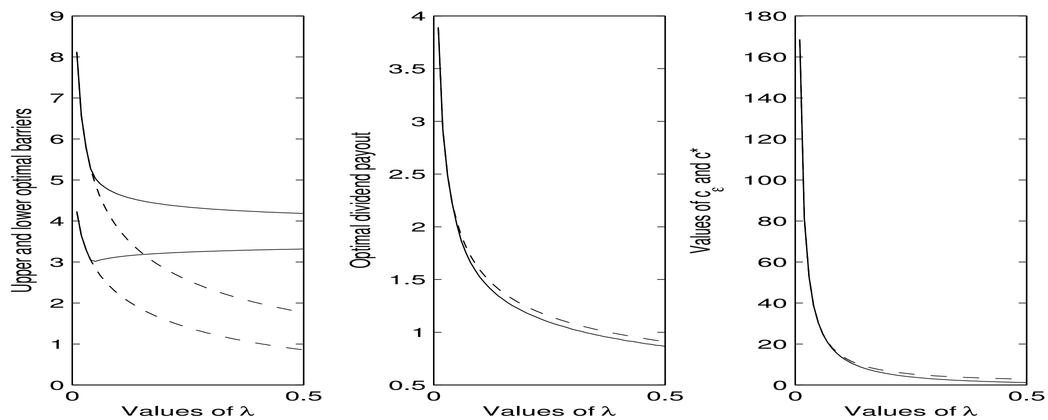

Figure 1: Values for varying λ in Example 6.1. The other values are kept fixed at $T = 10$, $k = 0.95$, $K = 0.05$, $\varepsilon = 0.01$.

Example 6.2 Let the basic income process follow the linear Brownian motion

$$P_t = x + \mu t + \sigma_P W_{P,t},$$

and assume that assets are invested in a risky investment so that the dynamics of the noncontrolled process is

$$dX_t = dP_t + X_t dR_t.$$

We assume that R is a Black-Scholes investment generating process, i.e. $R_t = (\lambda - \alpha)t + \sigma_R W_{R,t}$, and that W_P and W_R are independent. Here λ can be seen as the market rate, also used for discounting, while α is a proportional cost associated with the investment.

We can write X as (same weak solution)

$$dX_t = (\mu + (\lambda - \alpha)X_t)dt + \sqrt{\sigma_P^2 + \sigma_R^2 X_t^2}\, dW_t,$$

where W is a Brownian motion.

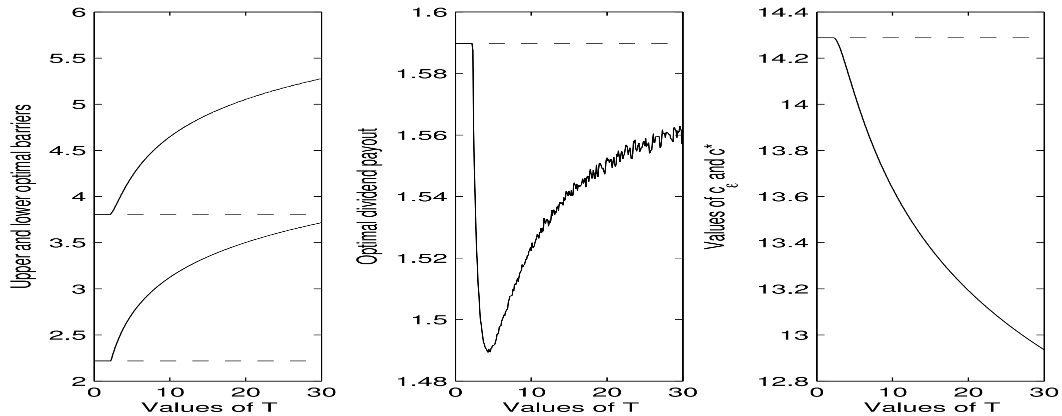

Figure 2: Values for varying T in Example 6.1. The other values are kept fixed at $\lambda = 0.1,\ k = 0.95,\ K = 0.05,\ \varepsilon = 0.01$.

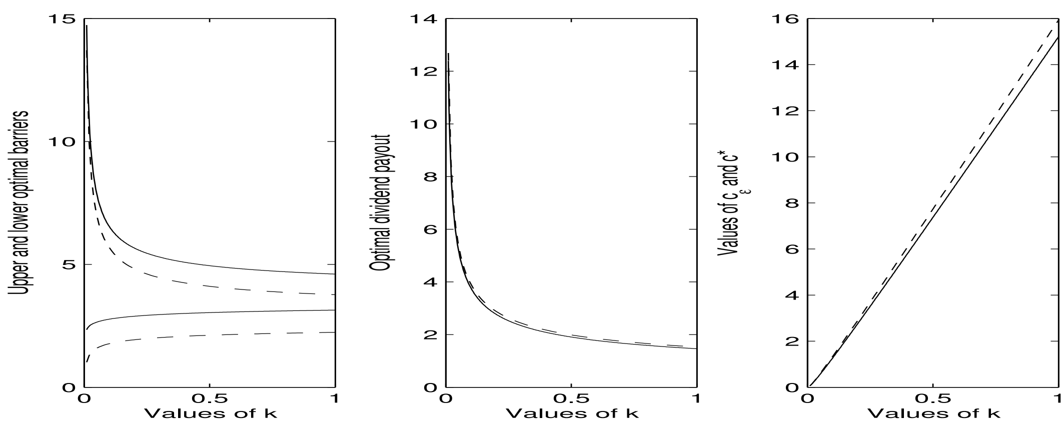

Figure 3: Values for varying k in Example 6.1. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ K = 0.05,\ \varepsilon = 0.01$.

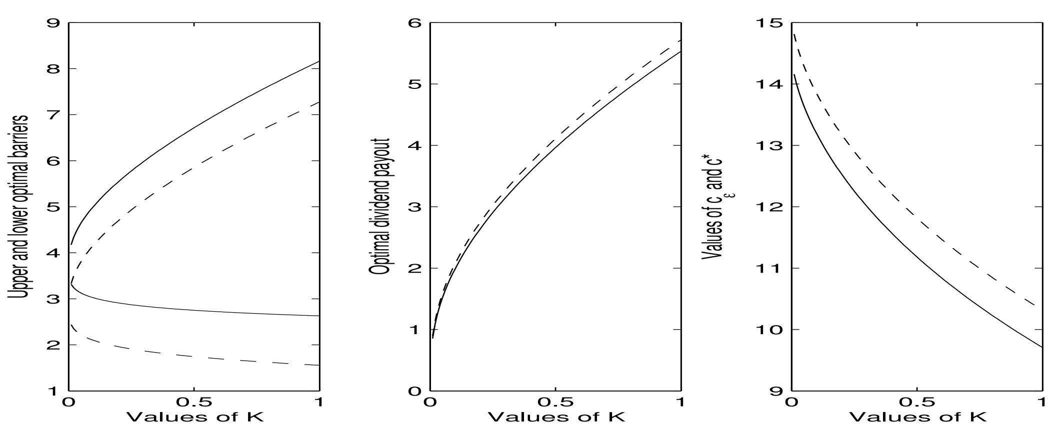

Figure 4: Values for varying K in Example 6.1. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ k = 0.95,\ \varepsilon = 0.01$.

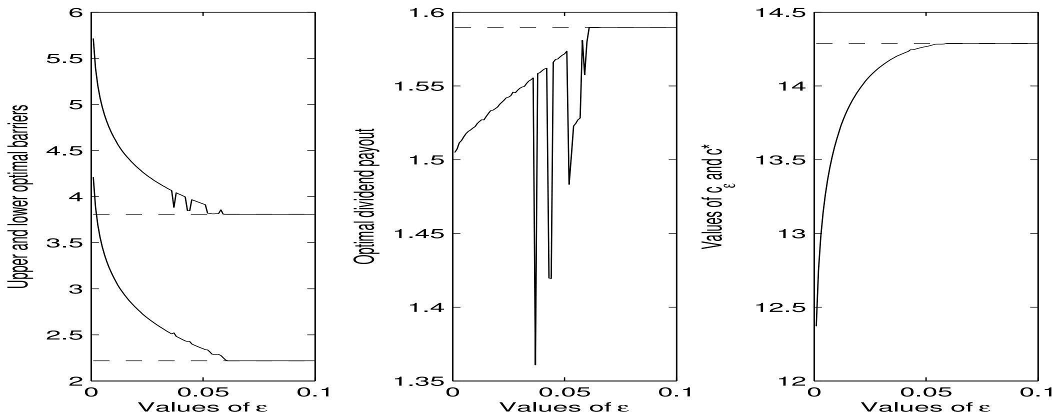

Figure 5: Values for varying ε in Example 6.1. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ k = 0.95,\ K = 0.05$.

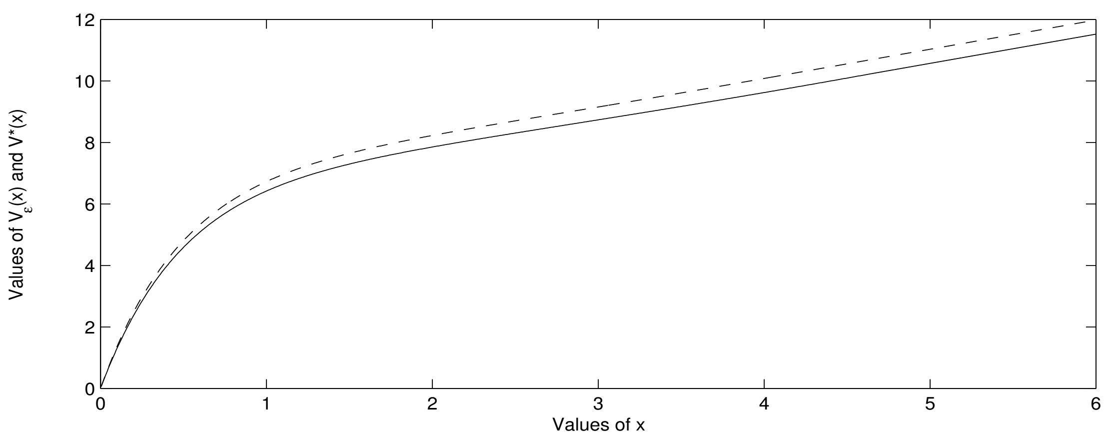

Figure 6: Values of $V_\varepsilon(x)$ and $V^*(x)$ for varying x in Example 6.1. The parameters are $\lambda = 0.1,\ T = 10,\ k = 0.95,\ K = 0.05,\ \varepsilon = 0.01$.

By Proposition 2.1, $\lim_{x\to\infty} g'(x) = \infty$, hence an optimal strategy exists when $\alpha > 0$. Actually, using arguments similar to those in Section 3 in [7] together with the solutions given in the appendix in [9], it can be proved that an optimal strategy exists if and only if $\alpha > 0$. Again using the solutions in that appendix, a canonical solution can be found, but it is complicated so we used the more convenient Runge Kutta method to obtain a numerical solution of $g(x)$, scaled so that $g'(0) = 1$.

In Figures 7-12 $\mu = \sigma_P = 1$, $\sigma_R = 0.25$ and $\alpha = 0.02$. The other parameters used are the same as in Example 6.1, and in the figures 5 of these are kept fixed, while one is varying.

In Figure 7 the discounting factor λ is varied. This is a somewhat different situation from that in Figure 1. Ignoring the random elements, in Example 1 the only income is the linear μ, which is heavily deflated with an increasing λ. In this example there is in addition an investment income $\lambda - \alpha$, which is exponential in nature and therefore partially offsets an increase in λ. When λ is small, the linear income μ dominates, but as λ increases the exponential investment income takes over. This can explain the middle panel in Figure 7, where for small λ the payout decreases with λ as in Figure 1, but as λ increases it starts to increase again. From the left panel we see that the upper barrier starts to increase when λ gets big both in the unconstrained and in the constrained case. However, from the right panel it is seen that the overall effect of increasing λ is somewhat smaller in Figure 7 than in Figure 1, which is to be expected.

Figures 8-11 do not differ very much from Figures 2-5, except that the effect of the solvency constraint seems even less serious here. In Figures 8 and 11 (as well as in Figure 7), the solvency constraint caused some ruggedness due to numerical issues, but again looking at the corresponding right panels shows that this is of no importance.

In Figure 12, the effect of varying the cost factor α is shown. With small α, the investment return $\lambda - \alpha$ is almost as large as the discounting factor λ, and therefore there is no urgency to pay out dividends, hence the barriers can be set high, and the solvency constraint is not binding. As α increases, it is more urgent to pay dividends, and therefore the optimal unconstrained barriers will not satisfy the solvency constraint. Again the payouts $\bar{u} - \underline{u}$ are almost unaffected by the solvency constraint, and from the right panel we see that the reduction in value due to the solvency constraint is not very large.

The conclusion here is much the same as in Example 6.1, the solvency constraint can have a fairly large impact on the optimal policy, but the actual payout as well as the value of the company are only moderately affected.

We also tried with an "investment risk free" version, i.e. with $\sigma_R = 0$ so that

$$dX_t = (\mu + (\lambda - \alpha)X_t)dt + \sigma dW_t.$$

However, this gave much the same results, indicating the the results are quite robust.

Acknowledgments

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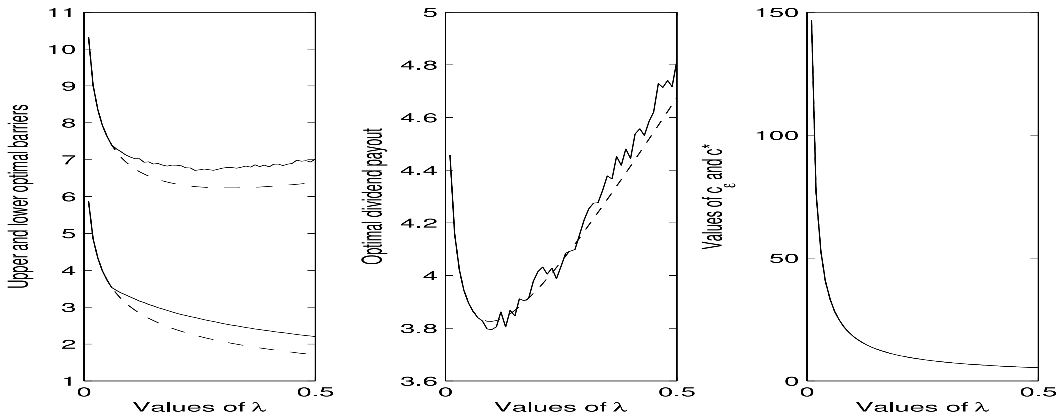

Figure 7: Values for varying λ in Example 6.2. The other values are kept fixed at $T = 10$, $k = 0.95$, $K = 0.05$, $\varepsilon = 0.01$.

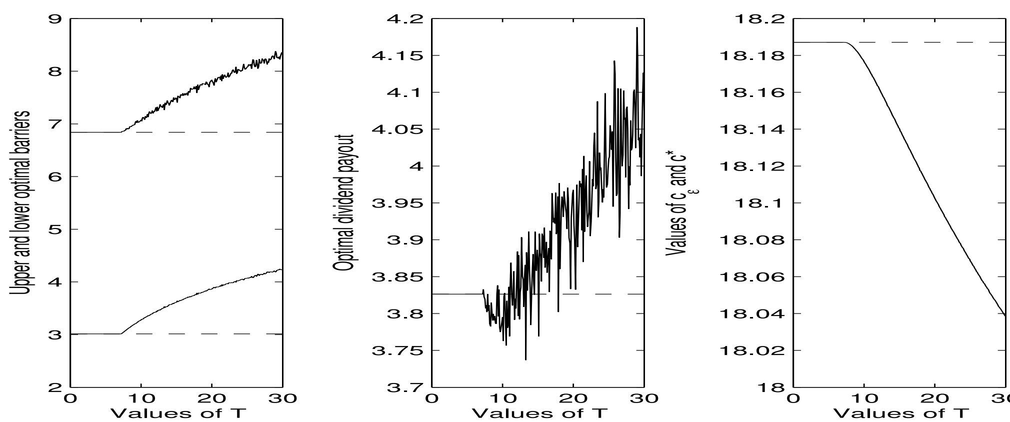

Figure 8: Values for varying T in Example 6.2. The other values are kept fixed at $\lambda = 0.1$, $k = 0.95$, $K = 0.05$, $\varepsilon = 0.01$.

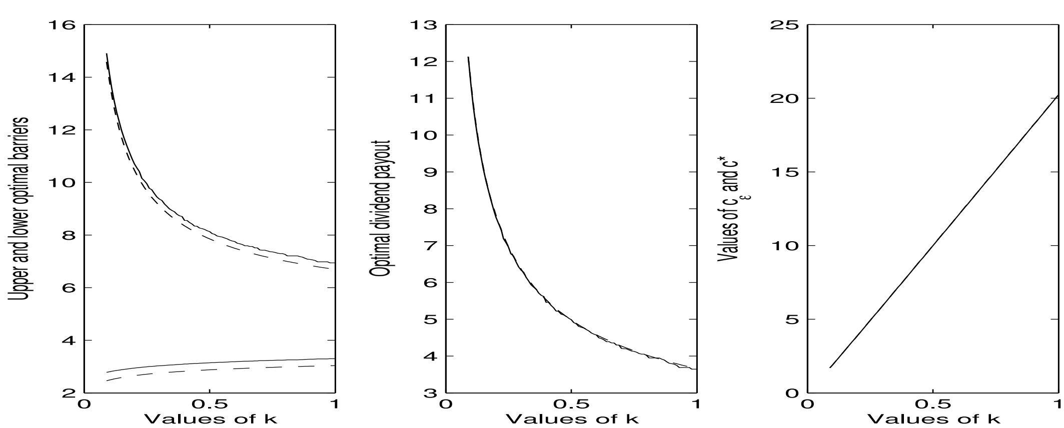

Figure 9: Values for varying k in Example 6.2. The other values are kept fixed at $\lambda = 0.1$, $T = 10$, $K = 0.05$, $\varepsilon = 0.01$.

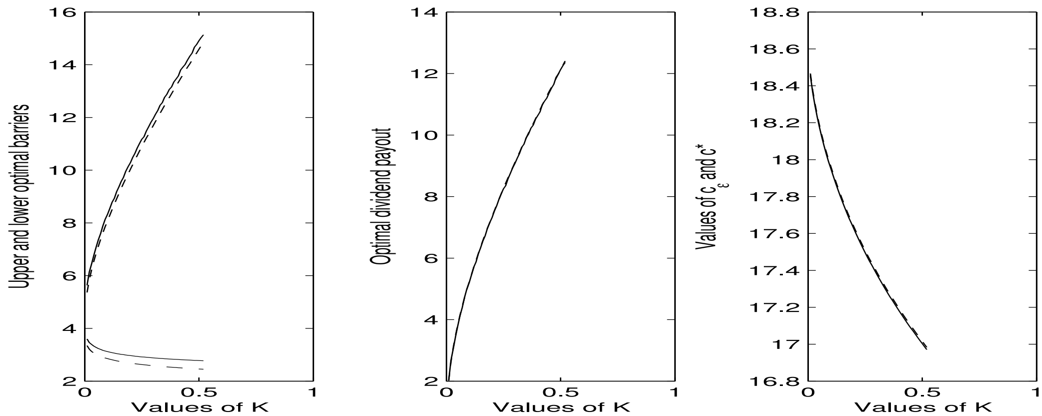

Figure 10: Values for varying K in Example 6.2. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ k = 0.95,\ \varepsilon = 0.01$.

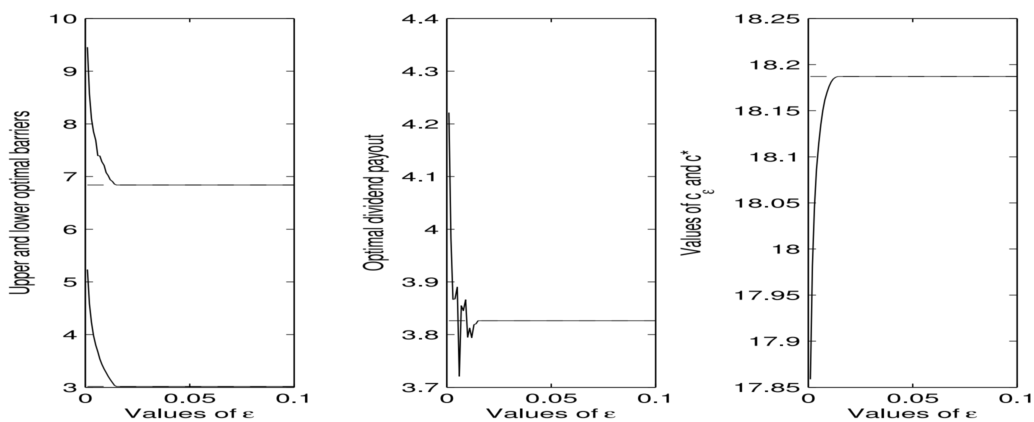

Figure 11: Values for varying ε in Example 6.2. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ k = 0.95,\ K = 0.05$.

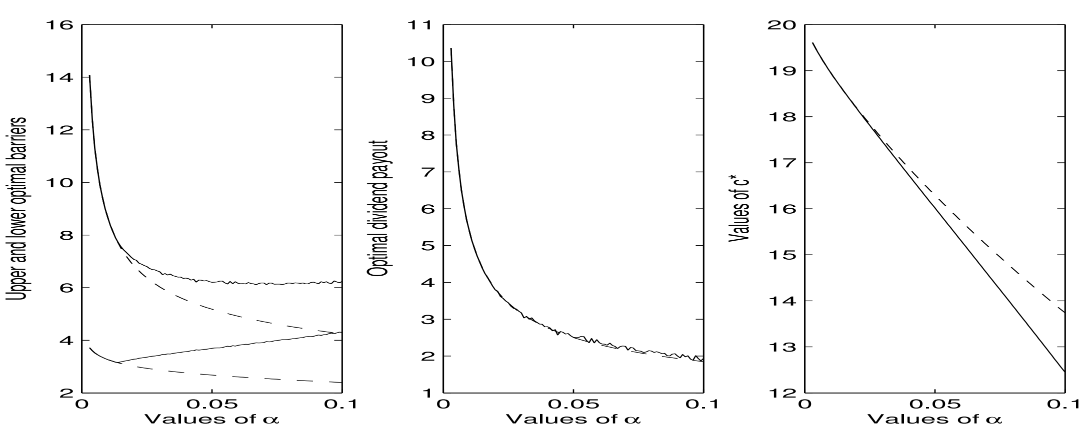

Figure 12: Values for varying α in Example 6.2. The other values are kept fixed at $\lambda = 0.1,\ T = 10,\ k = 0.95,\ K = 0.05$.

Appendix

In this appendix we will prove Proposition 2.1, Theorems 3.1-3.2 and Theorem 4.1. To do so we need the following lemmas, which are the same as Lemmas 2.1 and 2.2 in [7].

Lemma A.1 *Let $\mu(x)$ and $\sigma(x)$ satisfy A2 – A4 and let f be a solution of $\mathcal{L}f(x) = 0$. Consider the interval $[0,\infty)$.*
a) If f has a zero on $[0,\infty)$, then f' has no zero on $[0,\infty)$.
b) If for some $\tilde{x} \in [0,\infty)$, $f'(\tilde{x}) > 0$ and $f''(\tilde{x}) \leq 0$, then f is a concave function on $[0,\tilde{x})$.

Lemma A.2 *Let $\mu(x)$ and $\sigma(x)$ satisfy A2 – A4 and let f satisfy $\mathcal{L}f(x) = 0$, $f(0) = 0$ and $f(\hat{x}) > 0$ for some $\hat{x} > 0$.*
a) f is strongly increasing.
b) There is an $x^* \geq 0$ (possibly taking the value infinity) so that f is concave on $(0,x^*)$ and convex on (x^*,∞). In particular $x^* = 0$ if and only if $\mu(0) \leq 0$ and trivially $f''(x^*) = 0$ when $0 < x^* < \infty$.

Proof of Proposition 2.1. To keep initial conditions fixed, we restrict the definition of a canonical solution to mean that $g(0) = 0$ and $g'(0) = 1$. First note that for any $\delta > 0$,

$$\mu(x) < \mu(0) + \lambda x_0 + \delta + (\lambda - \varepsilon)x,$$

and therefore it follows from Lemma 2.3 in [7] that it is sufficient to prove that for any a, a canonical solution of

$$\frac{1}{2}\sigma^2(x)f''(x) + (a + (\lambda - \varepsilon)x)f'(x) - \lambda f(x) = 0,$$

satisfies $\lim_{x\to\infty} f'(x) = \infty$.

By Lemma A.2b, such a canonical solution f is either ultimately convex or ultimately concave. In either case there exists a $c \leq \infty$ so that

$$\lim_{x\to\infty} f'(x) = c \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{x} = c.$$

Assume that $c < \infty$. Then, since

$$f'(x) = 1 + \int_0^x f''(y)dy,$$

there must exist a sequence $\{x_n\}$ with $x_n \to \infty$ as $n \to \infty$ so that $f''(x_n) = o(x_n^{-1})$. Also

$$\frac{1}{2}\frac{\sigma^2(x)}{x}f''(x) = -\frac{a + (\lambda - \varepsilon)x}{x}f'(x) + \lambda\frac{f(x)}{x} \to \varepsilon c \ \text{ as } \ x \to \infty.$$

Then, considering only the leading terms,

$$\frac{\sigma^2(x_n)}{x_n^2} \sim \frac{2\varepsilon c}{x_n o(x_n^{-1})} \to \infty \ \text{ as } \ n \to \infty.$$

But this contradicts A1, hence $c = \infty$ and we are done. □

The next step is to prove that Problem C really has a solution.

Lemma A.3 *Under the assumptions of Theorem 3.1, Problem C has exactly one solution and $\bar{u}_0 > x^*$, where x^* is given in Lemma A.2.*

Proof. We are looking for a solution $(\bar{c}, \bar{u}_0)$ of

$$\bar{c}g'(\bar{u}_0) = k, \tag{A.1}$$

$$\bar{c}g(\bar{u}_0) = \bar{c}g(\underline{u}_0) + k(\bar{u}_0 - \underline{u}_0) - K. \tag{A.2}$$

Let

$$\hat{c} = \begin{cases} \frac{k}{g'(x^*)}, & \underline{u}_0 \leq x^*, \\ \\ \frac{k}{g'(\underline{u}_0)}, & \underline{u}_0 > x^*, \end{cases}$$

For given $c > 0$, consider the equation

$$cg'(u_c) = k \quad \text{for some} \ \ u_c \geq \max\{\underline{u}_0, x^*\}. \tag{A.3}$$

If $\underline{u}_0 \leq x^*$, since $g'(x)$ is increasing on $[x^*, \infty)$, it is easy to see that (A.3) has a solution if and only if $c \leq \hat{c}$. A similar argument shows that this holds when $\underline{u}_0 > x^*$ as well. We can therefore define the function

$$I(c) = \int_{\underline{u}_0}^{u_c} (k - cg'(y))dy, \quad 0 < c \leq \hat{c}.$$

Then (A.1) and (A.2) are equivalent with the existence of a c so that $I(c) = K$. By the implicit function theorem, u_c is continuously differentiable w.r.t. c and $I'(c) = -\int_{\underline{u}_0}^{u_c} g'(y)dy < 0$, i.e. I is continuous and strictly decreasing in $c \in (0, \hat{c})$. Also $\lim_{c\to 0} u_c = \infty$, hence $\lim_{c\to 0} I(c) = \infty$ as well. Therefore, if we can prove that $I(\hat{c}) \leq 0$, there must exist a unique $\bar{c} \in (0, \hat{c})$ so that $I(\bar{c}) = K$.

To prove that $I(\hat{c}) \leq 0$, assume first that $\underline{u}_0 \leq x^*$. Then since g' has a minimum at x^*,

$$\hat{c}g'(x) = \frac{g'(x)}{g'(x^*)}k \geq k,$$

and consequently $I(\hat{c}) \leq 0$. If $\underline{u}_0 > x^*$, then g' is increasing on $[\underline{u}_0, \infty)$, hence $\hat{c}g'(x) \geq k$ for $x \in [\underline{u}_0, \infty)$, and so $I(\hat{c}) \leq 0$ again.

Denoting the corresponding $u_{\bar{c}}$ by $\bar{u}_0$ so that $\bar{c}g'(\bar{u}_0) = k$ we can thus conclude that

$$V(x) = \begin{cases} \bar{c}g(x), & 0 \leq x \leq \bar{u}_0, \\ \\ V(\bar{u}_0) + k(x - \bar{u}_0), & x > \bar{u}_0. \end{cases}$$

□

Lemma A.4 *Under the assumptions of Theorem 3.1, let V be as in Lemma A.3. Then $V'(x) < k$ for $x \in [\underline{u}_0, \bar{u}_0)$.*

Proof. By Lemma A.2, it is sufficient to prove that $V'(\underline{u}_0) < k$. If $\underline{u}_0 \geq x^*$ the result is trivially true by convexity of g on $[x^*, \infty)$. Assume therefore that $\underline{u}_0 < x^*$ and let $V^*(x) = c^* g(x)$ be the optimal solution from Theorem 2.1. Assume that $\bar{c} \geq c^*$. Then since $c^* g'(\bar{u}^*) = \bar{c} g'(\bar{u}_0)$ it is necessary that $\bar{u}_0 \leq \bar{u}^*$. But then

$$K = \int_{\underline{u}_0}^{\bar{u}_0} (k - \bar{c} g'(x))dx \leq \int_{\underline{u}_0}^{\bar{u}_0} (k - c^* g'(x))dx < \int_{\underline{u}^*}^{\bar{u}^*} (k - c^* g'(x))dx = K,$$

a contradiction. Therefore, $\bar{c} < c^*$ and by concavity of g on $[\underline{u}^*, x^*]$,

$$V'(\underline{u}_0) = \bar{c} g'(\underline{u}_0) < c^* g'(\underline{u}_0) < c^* g'(\underline{u}^*) = k.$$

□

For a function $\phi : [0, \infty) \mapsto [0, \infty)$ define the maximum utility operator M by

$$M\phi(x) := \begin{cases} \sup\{\phi(x - \eta) - K + k\eta : 0 \leq \eta \leq x - \underline{u}_0\}, & \text{if } x \in [\underline{u}_0, \infty), \\ -\infty, & \text{if } x \in [0, \underline{u}_0). \end{cases} \tag{A.4}$$

Lemma A.5. *Let V be as in Lemma A.3. Then V satisfies the quasi-variational inequalities*

$$\mathcal{L}V(x) \leq 0, \tag{A.5}$$
$$V(x) \geq MV(x), \tag{A.6}$$
$$(V(x) - MV(x))(\mathcal{L}V(x)) = 0, \tag{A.7}$$
$$V(0) = 0. \tag{A.8}$$

Furthermore, $MV(x) < V(x)$ when $x \in [0, \bar{u}_0)$ and $MV(x) = V(x)$ when $x \in [\bar{u}_0, \infty)$.

Proof. We first prove (A.5). Since $\mathcal{L}V(x) = 0$ when $x \leq \bar{u}_0$, assume that $x > \bar{u}_0$. Since by Lemma A.3, $\bar{u}_0 > x^*$, $V''(\bar{u}_0-) > 0$ while trivially $V''(\bar{u}_0+) = 0$. Using that $V(x) = V(\bar{u}_0) + k(x - \bar{u}_0)$ we get by Assumption A4,

$$\begin{aligned} \mathcal{L}V(x) &= \mu(x)k - \lambda(V(\bar{u}_0) + k(x - \bar{u}_0)) \\ &= k\int_{\bar{u}_0}^{x} (\mu'(y) - \lambda)dy + k\mu(\bar{u}_0) - \lambda V(\bar{u}_0) \\ &\leq k\mu(\bar{u}_0) - \lambda V(\bar{u}_0) \\ &\leq \frac{1}{2}\sigma^2(\bar{u}_0-)V''(\bar{u}_0-) + \mu(\bar{u}_0-)V'(\bar{u}_0-) - \lambda V(\bar{u}_0-) = 0. \end{aligned}$$

We proceed to prove (A.6). For $x \in [0, \underline{u}_0]$, $MV(x) = -\infty$, hence the inequality is trivially satisfied. When $x > \underline{u}_0$, by Lemma A4 and the definition of $V(x)$, $V'(x) < k$ when $x \in [\underline{u}_0, \bar{u}_0)$, and $V'(x) = k$ when $x \in [\bar{u}_0, \infty)$. Therefore the function $V(x - \eta) + k\eta - K$ is increasing in η for nonnegative η and takes its maximum when $\eta = x - \underline{u}_0$. Hence, for $x \in [\underline{u}_0, \bar{u}_0)$,

$$MV(x) - V(x) = V(\underline{u}_0) + k(x - \underline{u}_0) - K - V(x) = \int_{\underline{u}_0}^{x} (k - V'(y))dy - K < \int_{\underline{u}_0}^{\bar{u}_0} (k - V'(y))dy - K = 0.$$

For $x \geq \bar{u}_0$ we have

$$MV(x) = V(\underline{u}_0) + k(x - \underline{u}_0) - K = V(x).$$

This also proves (A.7) since $\mathcal{L}V(x) = 0$ for $x \in (0, \bar{u}_0)$ and $MV(x) = V(x)$ for $x \in [\bar{u}_0, \infty)$. Finally (A.8) follows by definition of V. □

Proof of Theorem 3.1. Let $\pi \in \Pi_0$ be an arbitrary strategy. By definition, V is continuously differentiable on $(0, \infty)$ and twice continuously differentiable on $(0, \bar{u}_0) \cup (\bar{u}_0, \infty)$. However, for $x = \bar{u}_0$, the continuity of V'' might fail. Since $\{0 \leq t < \tau^\pi : X_t^\pi = \bar{u}_0\}$ has Lebesgue measure zero under each P_x, we can use Itô's formula, see e.g. [2] p.460, together with (A.5) to get

$$\begin{aligned} e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+}) \quad &= \quad V(x) + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\mathcal{L}(X_s^\pi)ds \\ &\quad + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma(X_s^\pi)V'(X_s^\pi)dW_s + \sum_{0\leq\tau_n^\pi\leq t\wedge\tau^\pi} e^{-\lambda\tau_n^\pi}\left(V(X^\pi_{\tau_n^\pi+}) - V(X^\pi_{\tau_n^\pi})\right) \quad \text{(A.9)} \\ &\leq \quad V(x) + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma(X_s^\pi)V'(X_s^\pi)dW_s + \sum_{0\leq\tau_n^\pi\leq t\wedge\tau^\pi} e^{-\lambda\tau_n^\pi}\left(V(X^\pi_{\tau_n^\pi+}) - V(X^\pi_{\tau_n^\pi})\right). \end{aligned}$$

Here we can let $V''(\bar{u}_0) = V''^{-}(\bar{u}_0)$. Another argument for this formula would be to use Lemma A.8 below where now $k = k1$.

Since V' is bounded and the process satisfies Assumptions A1-A4, it is fairly straightforward to show that

$$\int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma(X_s^\pi)V'(X_s^\pi)dW_s$$

is a martingale. Taking expectations on both sides of (A.9) therefore yields

$$E_x\left[e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+})\right] \leq V(x) + E_x\left[\sum_{0\leq\tau_n^\pi\leq t\wedge\tau^\pi} e^{-\lambda\tau_n^\pi}\left(V(X^\pi_{\tau_n^\pi+}) - V(X^\pi_{\tau_n^\pi})\right)\right]. \quad \text{(A.10)}$$

From (A.6) and the fact that $X^\pi_{\tau_n^\pi} > X^\pi_{\tau_n^\pi+} \geq \underline{u}_0$, it follows that

$$e^{-\lambda\tau_n^\pi}\left(V(X^\pi_{\tau_n^\pi+}) - V(X^\pi_{\tau_n^\pi})\right) \leq -e^{-\lambda\tau_n^\pi}(k\xi_n^\pi - K), \quad n = 1, 2, \cdots. \quad \text{(A.11)}$$

on $\{\tau_n^\pi \leq t \wedge \tau^\pi\}$. Then (A.10) and (A.11) together give

$$0 \leq V(x) - E_x\left[\sum_{n=1}^\infty e^{-\lambda\tau_n^\pi}(k\xi_n^\pi - K)1_{\{\tau_n^\pi\leq t\wedge\tau^\pi\}}\right] - E_x\left[e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+})\right]. \quad \text{(A.12)}$$

Letting $t \to \infty$ in (A.12), we have by nonnegativity of V,

$$V(x) \geq E_x\left[\sum_{n=1}^\infty e^{-\lambda\tau_n^\pi}(k\xi_n^\pi - K)1_{\{\tau_n^\pi\leq\tau^\pi\}}\right] = V_\pi(x), \quad \text{(A.13)}$$

which implies that $V(x) \geq V_0^*(x)$.

Now consider the lump sum dividend barrier strategy $\pi_{\bar{u}_0,\underline{u}_0}$ given in Theorem 3.1. Since

$X_s^{\pi_0}$ does not exceed $\bar{u}_0$, $\mathcal{L}(X_s^{\pi_0}) = 0$ a.s. for $0 < s < \tau^{\pi_0}$. Therefore, the inequality in (A.9) becomes an equality with the strategy π_0, i.e.

$$\begin{aligned} e^{-\lambda(t\wedge\tau^{\pi_0})}V(X^{\pi_0}_{t\wedge\tau^{\pi_0}+}) &= V(x) + \int_0^{t\wedge\tau^{\pi_0}} e^{-\lambda s}\sigma(X_s^{\pi_0})V'(X_s^{\pi_0})dW_s \\ &\quad + \sum_{0\le\tau_n^{\pi_0}\le t\wedge\tau^{\pi_0}} e^{-\lambda\tau_n^{\pi_0}}\left(V(X^{\pi_0}_{\tau_n^{\pi_0}+}) - V(X^{\pi_0}_{\tau_n^{\pi_0}})\right). \end{aligned} \tag{A.14}$$

Assume that $x = X_0 \ge \bar{u}_0$. Then

$$V(x) = MV(x) = V(\underline{u}_0) + k(x - \underline{u}_0) - K, \quad x \ge \bar{u}_0,$$

and

$$\xi_1^{\pi_0} = x - \underline{u}_0, \quad \xi_n^{\pi_0} = \bar{u}_0 - \underline{u}_0, \quad n = 2, 3, \cdots.$$

We can conclude that

$$V(X^{\pi_0}_{\tau_1^{\pi_0}+}) - V(X^{\pi_0}_{\tau_1^{\pi_0}}) = V(X^{\pi_0}_{\tau_1^{\pi_0}} - \xi_1^{\pi_0}) - V(X^{\pi_0}_{\tau_1^{\pi_0}}) = -k\xi_1^{\pi_0} + K,$$

and

$$V(X^{\pi_0}_{\tau_n^{\pi_0}+}) - V(X^{\pi_0}_{\tau_n^{\pi_0}}) = -k\xi_n^{\pi_0} + K, \quad n = 2, 3, \cdots.$$

Also by boundedness of $X^{\pi_0}_{t\wedge\tau^{\pi_0}+}$ and the fact that $P(\tau^{\pi_0} < \infty) = 1$ and $X^{\pi_0}_{\tau^{\pi_0}+} = 0$, it follows from the bounded convergence theorem that

$$\lim_{t\to\infty} E_x\left[e^{-\lambda(t\wedge\tau^{\pi_0})}V(X^{\pi_0}_{t\wedge\tau^{\pi_0}+})\right] = 0.$$

Therefore, taking expectations in (A.14) and then letting $t \to \infty$ gives

$$V(x) = V_{\bar{u}_0,\underline{u}_0}(x),$$

which implies that $V(x) \le V_0^*(x)$. In summary, we get $V(x) = V_0^*(x) = V_{\bar{u}_0,\underline{u}_0}(x)$.

When the initial reserve $X_{0-} = x < \bar{u}_0$, the result is proved similarly.

To prove the last part of the theorem, let $\underline{u}^* \le \underline{u}_0 < \underline{u}_1$, and let $V_i(x) = V_{\bar{u}_i,\underline{u}_i}(x)$ be the two value functions. Write $V_i(x) = \bar{c}_i g(x)$ for $x \in [0, \bar{u}_i]$. By what we have just proved, $V_0(x) > V_1(x)$, hence $\bar{c}_0 > \bar{c}_1$. Therefore, for $V_i'(\bar{u}_i) = k$ it is necessary that $\bar{u}_1 > \bar{u}_0$. □

Now to the proof of Theorem 3.2. To prove that there is exactly one solution to the equations in Assumption D, just let $V(x) = \bar{c}g(x)$ so that we get the equation

$$\bar{c}g(\bar{u}_1) = \bar{c}g(\underline{u}_1) + k(\bar{u}_1 - \underline{u}_1) - K.$$

Solving for $\bar{c}$ gives

$$V(x) = \begin{cases} \frac{k(\bar{u}_1-\underline{u}_1)-K}{g(\bar{u}_1)-g(\underline{u}_1)}g(x), & 0 \le x \le \bar{u}_1, \\ V(\bar{u}_1) + k(x - \bar{u}_1), & x > \bar{u}_1. \end{cases} \tag{A.15}$$

Lemma A.6. *Let V be the solution of Problem D. Then there is a $\hat{u} \in [\underline{u}_1, \bar{u}_1]$ so that $V'(x) \le k$ on $[\underline{u}_1, \hat{u}]$ and $V'(x) \ge k$ on $[\hat{u}, \bar{u}_1]$.*

Proof Since $\tilde{u}$ is an upper optimality point, see (3.3) and what follows there, by the previous analysis we know that the corresponding value function is

$$V_0^*(x) = \begin{cases} k\frac{g(x)}{g'(\tilde{u})}, & 0 < x \leq \tilde{u}, \\ \\ V_0^*(\underline{u}_1) + k(x - \underline{u}_1) - K, & x \geq \tilde{u}. \end{cases}$$

Since $V_0^*(\tilde{u}) = V_0^*(\underline{u}_1) + k(\tilde{u} - \underline{u}_1) - K$, we can conclude that

$$k\int_{\underline{u}_1}^{\tilde{u}} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy = K.$$

Define the function G as

$$G(x) = k\int_{\underline{u}_1}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(x)}\right) dy, \quad \tilde{u} \leq x \leq \bar{u}_1.$$

Since $\bar{u}_1 > \tilde{u} > x^*$, $g'(x)$ is increasing on $[\tilde{u}, \bar{u}_1]$. Therefore, G is a continuous and increasing function. Furthermore,

$$\begin{aligned} G(\tilde{u}) &= k\int_{\underline{u}_1}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy \\ &= k\int_{\underline{u}_1}^{\tilde{u}} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy + k\int_{\tilde{u}}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy \\ &= K + k\int_{\tilde{u}}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy \leq K, \end{aligned}$$

and

$$G(\bar{u}_1) = k\int_{\underline{u}_1}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(\bar{u}_1)}\right) dy \geq k\int_{\underline{u}_1}^{\tilde{u}} \left(1 - \frac{g'(y)}{g'(\bar{u}_1)}\right) dy \geq k\int_{\underline{u}_1}^{\tilde{u}} \left(1 - \frac{g'(y)}{g'(\tilde{u})}\right) dy = K,$$

so there must exist a $\hat{u} \in [\tilde{u}, \bar{u}_1]$ such that $G(\hat{u}) = K$, that is

$$k\int_{\underline{u}_1}^{\bar{u}_1} \left(1 - \frac{g'(y)}{g'(\hat{u})}\right) dy = K. \tag{A.16}$$

Let $\hat{V}$ be defined as

$$\hat{V}(x) = \begin{cases} k\frac{g(x)}{g'(\hat{u})}, & 0 < x \leq \bar{u}_1, \\ \hat{V}(\bar{u}_1) + k(x - \bar{u}_1), & x > \bar{u}_1. \end{cases} \tag{A.17}$$

Then $\mathcal{L}\hat{V}(x) = 0$ for $0 < x < \bar{u}_1$ and by (A.16),

$$\hat{V}(\bar{u}_1) = \hat{V}(\underline{u}_1) + k(\bar{u}_1 - \underline{u}_1) - K.$$

Using this together with (A.17) then gives for $x > \bar{u}_1$,

$$\hat{V}(x) = \hat{V}(\underline{u}_1) + k(x - \underline{u}_1) - K.$$

Therefore, $\hat{V}$ also solves Problem D, so by uniqueness $\hat{V} = V$.

To finish the proof, let first $x \in [\underline{u}_1, \tilde{u}]$. It follows from [7] when $(\tilde{u}, \underline{u}_1) = (\bar{u}^*, \underline{u}^*)$, and from Lemma A4 when $(\tilde{u}, \underline{u}_1) = (\bar{u}_0, \underline{u}_0)$, that $V_0^{*\prime}(x) = k\frac{g'(x)}{g'(\tilde{u})} \leq k$. Since $\hat{u} \geq \tilde{u} > x^*$ and $g'(x)$ is increasing on (x^*, ∞), $V'(x) = k\frac{g'(x)}{g'(\hat{u})} \leq k\frac{g'(x)}{g'(\tilde{u})} \leq k$. Finally, let $x \in [\tilde{u}, \tilde{u}_1]$. Since $V'(\tilde{u}) \leq k$, $V'(\hat{u}) = k$ and $V'(x) = k\frac{g'(x)}{g'(\hat{u})}$ is increasing on $[\tilde{u}, \bar{u}_1)$, we can conclude that $V'(x) \leq k$ on $[\tilde{u}, \hat{u}]$ and $V'(x) \geq k$ on $[\hat{u}, \bar{u}_1)$. □

Note that $V'(x)$ and $V''(x)$ exist and are continuous except for when $x = \bar{u}_1$. Let $V^{'-}(\bar{u}_1)$ and $V^{'+}(\bar{u}_1)$ be the left derivative and right derivative of $V(x)$ at $\bar{u}_1$. From Lemma A.6 we can see that $V^{'-}(\bar{u}_1) \geq k = V^{'+}(\bar{u}_1)$. Therefore $V(x)$ may fail to be differentiable at the point $\bar{u}_1$ if $V^{'-}(\bar{u}_1) > k$. Thus, the classical Itô formula can not be applied, but its generalization, the Meyer-Itô formula is applicable. Since we are working with functions of the form $e^{-\lambda t} f(Y_t)$, the standard Meyer-Itô formula needs a slight, but straightforward, modification.

Lemma A.7. *Let f be the difference of two convex functions and $f^{'-}$ be its left derivative. Let*

$$L_t^a = \int_0^t e^{-\lambda t} dL_{t,0}^a,$$

where $L_{t,0}^a$ is the local time of Y at a. Then for a semimartingale Y the following equation holds:

$$\begin{aligned} e^{-\lambda t} f(Y_t) &= f(Y_0) + \int_0^t e^{-\lambda s} f^{'-}(Y_{s-}) dY_s - \int_0^t \lambda e^{-\lambda s} f(Y_{s-}) ds \\ &\quad + \sum_{0<s\leq t} e^{-\lambda s} \Big(f(Y_s) - f(Y_{s-}) - f^{'-}(Y_{s-}) \Delta Y_s \Big) + \frac{1}{2} \int_{-\infty}^{+\infty} L_t^a \mu(da), \end{aligned}$$

where μ is the signed measure (when restricted to compacts) which is the second derivative of f in the generalized function sense. Furthermore, for every bounded Borel measurable function v,

$$\int_{-\infty}^{+\infty} L_t^a v(a) da = \int_0^t e^{-\lambda s} v(Y_s) d[Y,Y]_s^c, \tag{A.18}$$

where $[Y,Y]_s^c$ is the quadratic variation of the continuous martingale part of Y.

Proof. The first part follows from Theorem 70, Chapter IV, in [10] using that $d(e^{-\lambda t} f(Y_t)) = -\lambda e^{-\lambda t} f(Y_t) dt + e^{-\lambda t} df(Y_t)$ and Fubini's theorem on the local time term. Formula (A.18) follows from Corollary 1, Chapter IV, in [10] and an application of Fubini's theorem.

Lemma A.8. *Let V be the solution of Problem D. Then, for $\pi \in \Pi_1$, the following equation holds:*

$$\begin{aligned} e^{-\lambda(t\wedge\tau^\pi)} V(X^\pi_{t\wedge\tau^\pi +}) &= V(X_0^\pi) + \int_0^{t\wedge\tau^\pi} e^{-\lambda s} V^{'-}(X_s^\pi) dX_s^\pi - \int_0^{t\wedge\tau^\pi} \lambda e^{-\lambda s} V(X_s^\pi) ds \\ &\quad + \sum_{0<s\leq t} e^{-\lambda s} \Big(V(X_{s+}^\pi) - V(X_s^\pi) - V^{'-}(X_s^\pi) \Delta X_s^\pi \Big) \\ &\quad - \frac{1}{2} L_{t\wedge\tau^\pi}^{\bar{u}_1} (k_1 - k) + \frac{1}{2} \int_0^{t\wedge\tau^\pi} e^{-\lambda s} \sigma^2(X_s^\pi) V^{''-}(X_s^\pi) ds, \end{aligned}$$

where k_1 is the left derivative of $V(x)$ at $\bar{u}_1$.

Proof Since $V'(x)$ and $V''(x)$ exist and are continuous except for at $x = \bar{u}_1$, and $V^{'\pm}(\bar{u}_1)$, $V^{''\pm}(\bar{u}_1)$ exist and are finite, some fairly straightforward calculations show that $V(x)$ can be written as the difference of the two convex functions

$$\begin{aligned} V_1(x) &= xV^{'+}(0) + \int_0^x \int_0^y (V^{''}(z))^+ dzdy, \\ V_2(x) &= (k_1 - k)(x - \bar{u}_1)^+ + \int_0^x \int_0^y (V^{''}(z))^- dzdy, \end{aligned}$$

where $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. By the property of $V(x)$, we have that

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} L^a_{t\wedge\tau^\pi}\mu(da) &= \frac{1}{2}L^{\bar{u}_1}_{t\wedge\tau^\pi}(V^{'+}(\bar{u}_1) - V^{'-}(\bar{u}_1)) + \frac{1}{2}\int_{-\infty}^{+\infty} L^a_{t\wedge\tau^\pi} V^{''-}(a)da \\ &= \frac{1}{2}L^{\bar{u}_1}_{t\wedge\tau^\pi}(k - k_1) + \frac{1}{2}\int_{-\infty}^{+\infty} L^a_{t\wedge\tau^\pi} V^{''-}(a)da. \end{aligned}$$

The identity (A.18) shows that

$$\frac{1}{2}\int_{-\infty}^{+\infty} L^a_{t\wedge\tau^\pi} V^{''-}(a)da = \frac{1}{2}\int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma^2(X^\pi_s)V^{''-}(X^\pi_s)ds.$$

The result now follows from Lemma A.7. □

Lemma A.9 *Let V be the solution of Problem D and define the operator $\mathcal{L}^-$ by*

$$\mathcal{L}^- V(x) = \frac{1}{2}\sigma^2(x)V^{''-}(x) + \mu(x)V^{'-}(x) - \lambda V(x).$$

Then V satisfies the following quasi-variational inequalities

$$\mathcal{L}^- V(x) = 0, \qquad 0 < x \le \bar{u}_1, \tag{A.19}$$

$$\mathcal{L}^- V(x) \le 0, \qquad x > \bar{u}_1, \tag{A.20}$$

$$V(x) = MV(x), \quad x \ge \bar{u}_1. \tag{A.21}$$

Here the operator M is as in (A.4), but with the lower limit there $\underline{u}_0$ replaced by $\underline{u}_1$.

Proof By the construction of $V(x)$, (A.19) holds. To prove (A.20), let $x > \bar{u}_1$. Then

$$\mathcal{L}^- V(x) = \frac{1}{2}\sigma^2(x)V^{''-}(x) + \mu(x)V^{'-}(x) - \lambda V(x) = \mu(x)k - \lambda V(x).$$

Since by Assumption A4, $\mu'(x) \le \lambda$, and the fact that $V'(x) = k$ on $(\bar{u}_1, \infty)$, the function $\mu(x)k - \lambda V(x)$ is decreasing on $(\bar{u}_1, \infty)$. Therefore,

$$\mathcal{L}^- V(x) = \mu(x)k - \lambda V(x) \le \mu(\bar{u}_1)k - \lambda V(\bar{u}_1).$$

If $\mu(\bar{u}_1) \le 0$, then clearly $\mathcal{L}^- V(x) \le 0$. If $\mu(\bar{u}_1) > 0$, by $\bar{u}_1 > \bar{u} > x^*$, $V^{''-}(\bar{u}_1) = k\frac{g''(\bar{u}_1)}{g(\hat{u})} \ge 0$. Then, since $V^{'-}(\bar{u}_1) \ge k$ and $\mu(\bar{u}_1) > 0$, we have

$$\mu(\bar{u}_1)k - \lambda V(\bar{u}_1) \le \frac{1}{2}\sigma^2(\bar{u}_1)V^{''-}(\bar{u}_1) + \mu(\bar{u}_1)V^{'-}(\bar{u}_1) - \lambda V(\bar{u}_1) = 0, \quad x > \bar{u}_1.$$

Finally, we prove (A.21). By Lemma A.6, for $x \geq \bar{u}_1$,

$$V'(x-\eta) - k \begin{cases} \geq 0, & 0 < \eta < x - \hat{u}, \\ \\ \leq 0, & x - \hat{u} \leq \eta \leq x - \underline{u}_1, \end{cases}$$

so optimality is achieved either by remaining at x or by going down all the way to $\underline{u}_1$. This gives

$$\begin{aligned} MV(x) &= \text{Max}\{V(x) - K, V(\underline{u}_1) - k(x - \underline{u}_1) - K\} \\ &= \text{Max}\{V(\underline{u}_1) - k(x - \underline{u}_1) - 2K, V(\underline{u}_1) - k(x - \underline{u}_1) - K\} \\ &= V(\underline{u}_1) - k(x - \underline{u}_1) - K = V(x). \end{aligned} \tag{A.22}$$

□

Proof of Theorem 3.2. For $\pi \in \Pi_1$ we easily get from Lemma A.8

$$\begin{aligned} e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+}) &= V(x) + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\mathcal{L}^- V(X^\pi_s)ds + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma^2(X^\pi_s)V'^-(X^\pi_s)dW_s \\ &\quad + \sum_{0\leq\tau^\pi_n\leq t\wedge\tau^\pi} e^{-\lambda\tau^\pi_n}\left(V(X^\pi_{\tau^\pi_n+}) - V(X^\pi_{\tau^\pi_n})\right) - \frac{1}{2}L_{t\wedge\tau^\pi}(\bar{u}_1)(k_1 - k). \end{aligned}$$

Since $\pi \in \Pi_1$ it is necessary that that $X^\pi_{\tau^\pi_n-} \geq \bar{u}_1$. Then by Lemma A.9 and the fact that $k_1 \geq k$,

$$e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+}) \leq V(x) + \int_0^{t\wedge\tau^\pi} e^{-\lambda s}\sigma^2(X^\pi_{s-})V'^-(X^\pi_s)dW_s + \sum_{0\leq\tau^\pi_n\leq t\wedge\tau^\pi} e^{-\lambda\tau^\pi_n}(K - k\xi^\pi_n).$$

Taking expectations gives

$$0 \leq V(x) - E_x\left[\sum_{0\leq\tau^\pi_n\leq t\wedge\tau^\pi} e^{-\lambda s}(k\xi^\pi_n - K)\right] - E_x[e^{-\lambda(t\wedge\tau^\pi)}V(X^\pi_{t\wedge\tau^\pi+})].$$

Letting $t \to \infty$, we have by nonnegativity of V,

$$V(x) \geq E_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau^\pi_n}(k\xi^\pi_n - K)\right] = V_\pi(x).$$

Taking the supremum over all strategies in Π_1 gives

$$V(x) \geq V_1^*(x). \tag{A.23}$$

Now consider the lump sum dividend barrier strategy $\pi_1 = \pi_{\bar{u}_1,\underline{u}_1}$. By definition of that strategy, $X^{\pi_1}_s \leq \bar{u}_1$ for all $s > 0$. Therefore, $\mathcal{L}^-(X^{\pi_1}_s) = L^{\bar{u}_1}_s = 0$ for all $s > 0$ and so

$$\begin{aligned} e^{-\lambda(t\wedge\tau^{\pi_1})}V(X^{\pi_1}_{t\wedge\tau^{\pi_1}+}) &= V(x) + \int_0^{t\wedge\tau^{\pi_1}} e^{-\lambda s}\sigma^2(X^{\pi_1}_s)V'^-(X^{\pi_1}_s)dW_s \\ &\quad + \sum_{0\leq\tau^{\pi_1}_n\leq t\wedge\tau^{\pi_1}} e^{-\lambda\tau^{\pi_1}_n}\left(V(X^{\pi_1}_{\tau^{\pi_1}_n+}) - V(X^{\pi_1}_{\tau^{\pi_1}_n})\right). \end{aligned}$$

Furthermore, by (A.22)

$$V(x) = MV(x) = V(\underline{u}_1) + k(x - \underline{u}_1) - K, \quad x \geq \bar{u}_1.$$

Arguing as at the end of the proof of Theorem 3.1 now gives

$$e^{-\lambda(t\wedge\tau^{\pi_1})}V(X^{\pi_1}_{t\wedge\tau^{\pi_1}+}) = V(x)+\int_0^{t\wedge\tau^{\pi_1}} e^{-\lambda s}\sigma^2(X^{\pi_1}_s)V'^{-}(X^{\pi_1}_s)dW_s+\sum_{0\le\tau^{\pi_1}_n\le t\wedge\tau^{\pi_1}} e^{-\lambda\tau^{\pi_1}_n}(K-k\xi^{\pi_1}_n).$$

Taking expectations and then letting $t\to\infty$ results in $V(x)=V_{\bar{u}_1,\underline{u}_1}(x)$ which implies that $V(x)\le V_1^*(x)$. Together with (A.23) we can therefore conclude that $V_1^*(x)=V(x)=V_{\bar{u}_1,\underline{u}_1}(x)$. □

Proof of Theorem 4.1. By Theorem 2.1 and its proof in [7], $V_{\bar{u},\underline{u}(\bar{u})}(x)$ is increasing in $\bar{u}$. If $\underline{u}(\bar{u})\to\infty$ as $\bar{u}\to\infty$, there is nothing to prove, so assume that $\underline{u}(\bar{u})\le m$ for all $\bar{u}$ for some positive m. Given $\delta>0$, choose $\bar{u}>b$ so large that $V_{\bar{u},\underline{u}(\bar{u})}(x)>V^*(x)-\frac{\delta}{2}\ \forall x\in[0,b]$, and also so that $\ln\bar{u}>m$. Consider the two dividend barrier lump sum strategies:

1. The strategy $\pi_0=\pi_{\bar{u},\underline{u}(\bar{u})}$.
2. The strategy $\pi_1=\pi_{\bar{u},\ln\bar{u}}$.

The strategy π_1 clearly satisfies the conditions of the theorem. Let τ be the first time the process hits $\bar{u}$ (with $\tau=\infty$ if it hits 0 before $\bar{u}$). By definition, τ is the same for both strategies when $x\le\bar{u}$. By the strong Markov property we have for $x\in[0,b]$,

$$V_{\pi_i}(x)=E_x[e^{-\lambda\tau}]V_{\pi_i}(\bar{u}),\ \ i=0,1.$$

Now since $\ln\bar{u}>m$,

$$\begin{aligned} V_{\pi_0}(\bar{u}) &\le k\bar{u}+V_{\pi_0}(\ln\bar{u})-K,\\ V_{\pi_1}(\bar{u}) &= k(\bar{u}-\ln\bar{u})+V_{\pi_1}(\ln\bar{u})-K. \end{aligned}$$

Therefore

$$V_{\pi_0}(x)-V_{\pi_1}(x)\le E_x[e^{-\lambda\tau}](k\ln\bar{u}+V_{\pi_0}(\ln\bar{u})-V_{\pi_1}(\ln\bar{u})).$$

Using this equation with $x=\ln\bar{u}$ gives

$$V_{\pi_0}(\ln\bar{u})-V_{\pi_1}(\ln\bar{u})\le k\frac{E_{\ln\bar{u}}\left[e^{-\lambda\tau}\right]}{1-E_{\ln\bar{u}}\left[e^{-\lambda\tau}\right]}\ln\bar{u},$$

and so

$$V_{\pi_0}(x)-V_{\pi_1}(x)\le kE_x[e^{-\lambda\tau}]\frac{1}{1-E_{\ln\bar{u}}\left[e^{-\lambda\tau}\right]}\ln\bar{u}. \tag{A.24}$$

By Assumption A4, $\mu(x)\le\mu(0)+\lambda x$, so by letting τ' be the same as τ, but with the drift $\mu(x)$ replaced by $\mu(0)+\lambda x$, it is clear that $E_x[e^{-\lambda\tau}]\le E_x[e^{-\lambda\tau'}]$. Define $h_{\bar{u}}(x)=E_x[e^{-\lambda\tau'}]$ so that $h_{\bar{u}}(0)=0$ and $h_{\bar{u}}(\bar{u})=1$. Furthermore, by standard results, see e.g. [4] Ch. 15.3, $h_{\bar{u}}$ satisfies

$$\frac{1}{2}\sigma^2(x)h''_{\bar{u}}(x)+(\lambda x+\mu(0))h'_{\bar{u}}(x)-\lambda h_{\bar{u}}(x)=0.$$

One solution of this equation is

$$h_1(x)=\lambda x+\mu(0).$$

Another solution is then given as, see e.g. [12] p.31,

$$\begin{aligned} h_2(x) &= h_1(x)\int_x^\infty \frac{1}{h_1^2(y)} e^{-2\int_0^y \frac{\lambda t+\mu(0)}{\sigma^2(t)}dt}dy \\ &\le h_1(x)\int_x^\infty \frac{1}{h_1^2(y)} e^{-c\int_0^y \frac{1}{1+t}dt}dy \\ &= h_1(x)\int_x^\infty \frac{1}{h_1^2(y)}(1+y)^{-c}dy \to 0 \ \text{ as } \ x\to\infty. \end{aligned}$$

Here we used Assumption A.1 in the first inequality, where c is a suitable positive constant. Fitting the boundary conditions we get

$$h_{\bar{u}}(x) = \frac{1}{\lambda\bar{u}+\mu(0)\left(1-\frac{h_2(\bar{u})}{h_2(0)}\right)}\left(\lambda x+\mu(0)\left(1-\frac{h_2(x)}{h_2(0)}\right)\right).$$

Therefore, $h_{\bar{u}}(x) \sim (\lambda\bar{u})^{-1}$ as $\bar{u}$ gets large and x is fixed, and this proves the result by (A.24), choosing $\bar{u}$ so large that $V_{\pi_0}(x) - V_{\pi_1}(x) \le \frac{\delta}{2}$ for all $x \in [0,b]$. Note that in the proof we may have used $\bar{u}^\gamma$ with $0<\gamma<1$ instead of $\ln\bar{u}$. □

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4 Paper B

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Optimal dividend policies with transaction costs for a class of jump-diffusion processes

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Abstract This paper addresses the problem of finding an optimal dividend policy for a class of jump-diffusion processes. The jump component is a compound Poisson process with negative jumps, and the drift and diffusion components are assumed to satisfy some regularity and growth restrictions. With each dividend payment there is associated a fixed and a proportional cost, meaning that if ξ is paid out by the company, the shareholders receive $k\xi - K$, where k and K are positive. The aim is to maximize expected discounted dividends until ruin. It is proved that when the jumps belong to a certain class of light tailed distributions, the optimal policy is a simple lump sum policy, that is when assets are equal to or larger than an upper barrier $\bar{u}^*$, they are immediately reduced to a lower barrier $\underline{u}^*$ through a dividend payment. The case with $K = 0$ is also investigated briefly, and the optimal policy is shown to be a reflecting barrier policy for the same light tailed class. Methods to numerically verify whether a simple lump sum barrier strategy is optimal for any jump distribution are provided at the end of the paper, and some numerical examples are given.

Keywords Optimal dividends · Jump-diffusion models · Impulse control · Barrier strategy · Singular control · Numerical solution

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and P-complete. Assume that the uncontrolled surplus process follows the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - dY_t, \quad X_0 = x, \tag{1.1}$$

where W is a Brownian motion and Y is a compound Poisson process, i.e.

$$Y_t = \sum_{i=1}^{N_t} S_i,$$

where N is a Poisson process with intensity λ, independent of the i.i.d. positive $\{S_i\}$. We will let S be generic for the S_i, and F be the distribution function of S. A natural interpretation is that X is a model of an insurance business, where Y represents claims and the other terms represent incomes and various business fluctuations. This interpretation is further developed in Example 3.5 below. To avoid lengthy explanations, we will refer to the S_i as claims.

Assume that the company pays dividends to its shareholders, but at a fixed transaction cost $K > 0$ and a tax rate $1 - k < 1$ so that $k > 0$. We will allow $k > 1$, opening up for other interpretations than that $1 - k$ is a tax rate. This means that if $\xi > 0$ is the amount the capital is reduced by due to a dividend payment, the net amount of money the shareholders receive is $k\xi - K$. It can be argued that taxes are paid on dividends after costs, so an alternative would be to use $k(\xi - K) = k\xi - kK$, but clearly this is just a reparametrization. Furthermore, different investors may have different tax rates, so $1 - k$ should be interpreted as an average tax rate.

Since every dividend payment results in a fixed transaction cost, the company should not pay out dividends continuously but only at discrete time epochs. Therefore, a strategy can be described by

$$\pi = (\tau_1^\pi, \tau_2^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \xi_2^\pi, \ldots, \xi_n^\pi, \ldots),$$

where τ_n^π and ξ_n^π denote the times and amounts paid. Thus, when applying the strategy π, the resulting surplus process X_t^π is given by

$$X_t^\pi = x + \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - Y_t - \sum_{n=1}^\infty 1_{\{\tau_n^\pi < t\}}\xi_n^\pi. \tag{1.2}$$

Note that X^π is left continuous at the dividend payments, so that $\xi_n^\pi = X_{\tau_n^\pi}^\pi - X_{\tau_n^\pi+}^\pi$.

Definition 1.1. *A strategy π is said to be admissible if*

(i) $0 \leq \tau_1^\pi$ *and for* $n \geq 1$, $\tau_{n+1}^\pi > \tau_n^\pi$ *on* $\{\tau_n^\pi < \infty\}$.
(ii) τ_n^π *is a stopping time with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$, $n = 1, 2. \ldots$
(iii) ξ_n^π *is measurable with respect to* $\mathcal{F}_{\tau_n^\pi+}$, $n = 1, 2. \ldots$

(iv) $\tau_n^\pi \to \infty$ *a.s. as* $n \to \infty$.
(v) $0 < \xi_n^\pi \le X_{\tau_n}^\pi$.

We denote the set of all admissible strategies by Π.

Another natural admissibility condition is that net money received should be positive, that is $k\xi - K > 0$. However, as we are looking for optimal policies, and a policy that allows $k\xi - K \le 0$ can never be optimal, it can be dropped as a condition.

With each admissible strategy π we define the corresponding ruin time as

$$\tau^\pi = \inf\{t \ge 0 : \ X_t^\pi < 0\}, \tag{1.3}$$

and the performance function $V_\pi(x)$ as

$$V_\pi(x) = E_x\left[\sum_{n=1}^{\infty} e^{-r\tau_n^\pi}(k\xi_n^\pi - K)1_{\{\tau_n^\pi \le \tau^\pi\}}\right], \tag{1.4}$$

where by P_x we mean the probability measure conditioned on $X_0 = x$. $V_\pi(x)$ represents the expected total discounted dividends received by the shareholders until ruin when the initial reserve is x.

The optimal return function is defined as

$$V^*(x) = \sup_{\pi \in \Pi} V_\pi(x) \tag{1.5}$$

and the optimal strategy, if it exists, by π^*. Then $V_{\pi^*}(x) = V^*(x)$. In the control theoretic language, this is an impulse control problem.

Definition 1.1 A lump sum dividend barrier strategy $\pi = \pi_{\bar{u},\underline{u}}$ with parameters $\underline{u} < \bar{u}$, satisfies for $X_0^\pi < \bar{u}$,

$$\tau_1^\pi = \inf\{t > 0 : X_t^\pi = \bar{u}\}, \qquad \xi_1^\pi = \bar{u} - \underline{u},$$

and for every $n \ge 2$,

$$\tau_n^\pi = \inf\{t > \tau_{n-1}^\pi : X_t^\pi = \bar{u}\}, \qquad \xi_n^\pi = \bar{u} - \underline{u}.$$

When $X_0^\pi \ge \bar{u}$,

$$\tau_1^\pi = 0, \qquad \xi_1^\pi = X_0^\pi - \underline{u},$$

and for every $n \ge 2$, τ_n^π is defined as above.

With a given lump sum dividend barrier strategy $\pi_{\bar{u},\underline{u}}$, the corresponding value function is denoted by $V_{\bar{u},\underline{u}}(x)$.

A lump sum dividend strategy $\pi_{\bar{u},\underline{u}}$ is sometimes called a $(\underline{u}, \bar{u})$ strategy.

Since some results in this paper, like Theorem 2.3, can be of interest of their own, we will look for as weak assumptions as possible. The following list of partially inclusive assumptions will therefore be referred to frequently.

A1a. μ and σ are continuous on $[0,\infty)$.
A1b. μ and σ are continuously differentiable on $(0,\infty)$.
A1c. μ and σ are twice continuously differentiable on $(0,\infty)$.
A1d. μ and σ are globally Lipschitz continuous on $[0,\infty)$.
A2a. The distribution function F is continuous.
A2b. The distribution function F has a continuous density f.
A2c. The distribution function F has a continuously differentiable density f.
A2d. The distribution function F has a continuous density f, and there is an $x_f \geq 0$ so that $f(x)$ is decreasing for $x > x_f$.
A3a. μ is continuously differentiable and there is an $\alpha > 0$ so that $\mu'_M \leq r+\lambda-\alpha$, where r is the discounting rate from (1.4) and $\mu'_M = \sup_{x>0}\mu'(x)$.
A3b. μ is continuously differentiable and $\mu'_M \leq r$.
A3c. μ is continuously differentiable and there is an $\alpha > 0$ and an $x_r \geq 0$ so that $\sup_{x\geq x_r}\mu'(x) \leq r-\alpha$.
A3d. μ is continuously differentiable and there is an $\alpha > 0$ so that $\mu'_M \leq r-\alpha$
A4. μ is concave on $[0,\infty)$.
A5. $\sigma^2(x) > 0$ on $[0,\infty)$.
A6. $|\sigma(x)| \leq C(1+x)$ for all $x \geq 0$ and some $C > 0$.
A7. There are nonnegative constants M_1 and M_2 so that

$$\frac{|\mu(x)|+r+\lambda}{\sigma^2(x)} \leq M_1 + M_2 x \quad \text{on } [0,\infty).$$

Note that A6 follows from A1d.

It is argued in [21] that a proper comparison is between $\mu'(x)$, the rate of growth, and r, the discounting factor. It is easy to prove that if for some x_0 and $\delta > 0$, $\mu'(x) > r+\delta$ for all $x > x_0$, then $V^*(x) = \infty$ and there is no optimal policy.

The optimal dividend problem for the classical Lundberg process

$$X_t = x + pt - Y_t, \tag{1.6}$$

where Y is as in (1.1), has a long history when there are no transaction costs. It was proved by Gerber back in 1969 that the optimal strategy can be quite complicated, but for some choices of the claim distribution F, notably the exponential distribution, a simple barrier strategy is optimal [13]. By this is meant that whenever assets hit a barrier u^*, dividends are paid at a rate p until a claim occurs. If initial assets are higher than u^*, they are immediately reduced to u^* through a dividend payment. In general, the optimal dividend strategy is a so-called band strategy, meaning that there are several barriers u_i^*, and whenever assets hit one barrrier, dividends are paid continuously at the rate p until the next claim. If initial assets are higher than the highest barrier, they are reduced to that barrier through a dividend payment.

The methods used by Gerber are somewhat obsolete today, and in their paper Azcue and Muler [4] extended and improved the results from Gerbers paper using very different methods. See also the book [23]. In the same spirit as Azcue and Muler, Albrecher and Thonhauser in [1] allowed for assets to earn interests, and again it was proved that the optimal strategy is a band strategy, but in the case of exponential claims it is a simple barrier strategy as before.

Recently there has been a considerable interest in this problem when X is a Lévy process with spectrally negative jumps, i.e. μ and σ in (1.1) are constants and Y is a nondecreasing pure jump process with stationary, independent increments [3], [18], [16]. In [16] it was proved that if the Lévy measure of Y has a log convex density, then the optimal strategy is a barrier strategy. In particular, when $\sigma > 0$ this means that the dividend process is a singular process, a fact that is well known from the theory of optimal control of ordinary diffusion processes [25]. In [9] a special case of this result was proved when Y is a compound Poisson process with exponential jumps. Extensions and variations of the Lévy problem can be found in [19] and [17].

The introduction of a proportional cost k does not alter any of the above findings in a fundamental way, but if a positive fixed cost K is added, it is a different story. In this case the lump sum barrier strategy corresponds to the simple barrier strategy. Loeffen [19] made use of the results in [16] to prove optimality of a simple lump sum barrier strategy when X is a spectrally negative Lévy process with a log convex jump density. This was also proved in [5] for the simple model (1.6) with exponentially distributed claims, and in [9] where a Brownian motion is added to (1.6), but still with exponentially distributed claims. Loeffen [19] also gives an example where he shows numerically that a simple lump sum dividend strategy cannot be optimal.

Another paper that is related to the present paper is [2], where Y in (1.1) is replaced by the geometric term

$$Y_t = \sum_{i=1}^{N_t} X_{\tau_i -} S_i \quad \text{and} \quad F(1) = 1.$$

Here the τ_i are the times of jump of N. Under assumptions rather different from ours, simple barrier strategies are proved to be optimal in the no-fixed cost case, and simple lump sum dividend strategies in the fixed cost case.

There are several papers that study the fixed cost dividend problem (1.1) when there are no jumps, going back to [15] where X is a linear Brownian motion with drift. The closest to the present paper is [21], where optimality of the simple lump sum barrier strategy is proved. In [5] the basic assumption A3b used in [21] was relaxed, and it was proved that a simple lump sum barrier strategy is no longer always optimal. These exceptional cases are further studied in [7], where it is shown that the optimal strategy sometimes becomes what is called a two-level lump sum dividend strategy.

Further variations of the fixed cost dividend problem for the model (1.1) without jumps can be found in [8] where dividend payouts are subject to certain solvency constraints, and in [22] where reinvestment of capital is allowed after it goes below zero. In both cases, under the same assumptions on the diffusion part of (1.1) as in [21], simple lump sum strategies turned out to be optimal.

Finally we should mention the papers [24] and [11] which are devoted to smoothness properties of the optimal value function for a very general multivariate jump-diffusion process. Their objective, in a setup somewhat different from ours, is to minimize expected discounted costs for some rather general cost functions. In [24] viscosity solution properties are proved, and that is improved to classical solutions in [11].

There is an obvious practical advantage with the lump sum dividend barrier strategy compared to a simple barrier strategy. Paying dividends continuously is rather unfeasable, and one would have to resort to some kind of lump sum payments anyway. So the optimal solution with a fixed positive K is in some sense more attractive.

The aim of this paper is to analyze the dividend problem for the jump-diffusion (1.1) subject to various assumptions. We will be looking for sufficient conditions for a lump sum dividend strategy to be optimal. An, admittedly small, class of distributions, that together with some other rather weak assumptions guarantees that the optimal solution is a lump sum dividend barrier strategy, is found. As could be expected, this class includes the exponential distribution, but not only that. However, in order to belong to the class, it is necessary that the density exists, is decreasing and is light-tailed. For completeness, we have also included the case when $K = 0$. Then, under the same assumptions that yield an optimal solution when $K > 0$, it is proved that the optimal solution is a barrier strategy. At the end of the paper numerical methods to check whether simple lump sum barrier strategies are optimal for any claim distribution, are introduced. Numerical examples showing the usefulness of such methods are provided.

In order to present and prove the optimality results in Section 3 and beyond, it is necessary to make a thorough analysis of a certain boundary value problem associated with the optimality problem. Section 2 is therefore dedicated to this issue.

2 Some results for the associated integro-differential equation

In this section we will study the solution and its properties of the boundary value problem

$$\begin{aligned} Lg(x) &= 0, \quad x > 0, \\ g(0) &= 0, \\ g'(0) &= 1, \end{aligned} \tag{2.1}$$

where L is the integro-differential operator

$$Lg(x) = \frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) - (r+\lambda)g(x) + \lambda \int_0^x g(x-z)dF(z). \tag{2.2}$$

A twice continuously differentiable solution of (2.1) will henceforth be called a canonical solution. We will see in the next section that a canonical solution plays a crucial role in the solution of the optimization problem of this paper.

The results of this section may be of independent interest, for example in generalizing the results of Section 3 and beyond. An example is how the results in [21] are generalized in [6]. We have therefore tried to keep the assumptions at a minimum. All proofs are of technical nature, so they are given in Section 6. Although there exists several proofs for the existence and smoothness of integral-differential equations, we have not found any that covers Theorem 2.3. Lemma 3.1 in [1] covers the case with no diffusion and linear $\mu(x)$. Theorem 5 in [12] is related, but it deals with ruin theory. Another example is Theorem 2.1 in [10], and they refer to Theorem 5 in [14] for a similar proof. As mentioned in the introduction, a very general existence and smoothness result can be found in [11]. It may well be possible to adapt that proof to our setting, but that would only be worthwhile if their assumption A5 can be relaxed, since it excludes much of Example 3.5 which is maybe the most important application of the theory.

Definition 2.1 For given $\beta > 0$ and $\zeta \geq 0$ we will denote by $L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$ the space of Borel measurable functions

$$\mathbf{u} = (u_1, \ldots, u_n) : [0,\infty) \to R^n$$

such that

$$\sup_{x \geq 0} \frac{|\mathbf{u}(x)|}{\exp(\beta x + \zeta x^2)} < \infty.$$

Here

$$|\mathbf{u}(x)| = \max_{1 \leq i \leq n} |u_i(x)|.$$

With $C([0,\infty), R^n)$ the space of continuous functions, we set

$$C_{\beta,\zeta}([0,\infty), R^n) = C([0,\infty), R^n) \cap L^{\infty}_{\beta,\zeta}([0,\infty), R^n).$$

Furthermore, $C^k([0,\infty), R^n)$ is the space of all k-times continuously differentiable functions and $C^k_{\beta,\zeta}([0,\infty), R^n)$ is the subspace so that the k'th derivative belongs to $C_{\beta,\zeta}([0,\infty), R^n)$.

From the definition it is clear that $L^{\infty}_{\beta,\zeta}([0,\infty), R^n) \subset L^{\infty}_{\tilde{\beta},\tilde{\zeta}}([0,\infty), R^n)$ whenever $\tilde{\zeta} > \zeta$ or $\tilde{\zeta} = \zeta$ and $\tilde{\beta} \geq \beta$. The same kind of inclusion obviously holds for $C_{\beta,\zeta}([0,\infty), R^n)$ and $C^k_{\beta,\zeta}([0,\infty), R^n)$.

Lemma 2.2 *The space $L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$ with norm*

$$\|\mathbf{u}\|^{\infty}_{\beta,\zeta} = \sup_{x \geq 0} \frac{|\mathbf{u}(x)|}{\exp(\beta x + \zeta x^2)}$$

is a Banach space. Furthermore, the space $C_{\beta,\zeta}([0,\infty), R^n)$ is closed in $L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$.

Theorem 2.3 *Assume A1a, A5 and A7. Then the boundary value problem (2.1) has a unique solution in $C^2_{\beta,\zeta}([0,\infty), R)$ for some $\beta > 0$ and $\zeta \geq 0$.*

Theorem 2.3 gives sufficient, but not necessary conditions for a canonical solution to exist. The assumption A7 can probably be relaxed, so in order to have results as general as possible, in the following we will just assume that a canonical solution exists.

Theorem 2.4 *Let g be a canonical solution and assume A1a and A5. Then g is strongly increasing on $[0,\infty)$. Moreover, assume there exists positive constants c_1, c_2, c_3 with $c_3 < c_1$ so that for all $x \geq 0$,*

$$(c_1 - c_3)\sigma^2(x) + 2(c_2 + c_3 x)\mu(x) < 2\frac{r}{c_1}(c_2 + c_3 x)^2. \tag{2.3}$$

Then

$$\lim_{x\to\infty} g'(x) = \infty.$$

In particular (2.3) can be satisfied if additionally A3c and A6 are satisfied.

Define

$$x^* = \inf\{x \geq 0 : g''(x) = 0\}.$$

By this definition, g is strictly concave on $(0, x^*)$. Clearly, if $x^* = \infty$ then g is strictly concave.

Theorem 2.5 *Let g be a canonical solution.*

a) If A5 holds then $x^* = 0$ if and only if $\mu(0) \leq 0$.
If in addition A1b and A2a hold and $\mu(0) = 0$ and $\mu'(0) < r + \lambda$, then $g''(0) = 0$ and $g'''(0) > 0$.

b) Assume A5 and that $\mu(0) > 0$. Also assume that there is an $x_0 > 0$ so that

$$\frac{\mu(x_0)}{x_0} = r. \tag{2.4}$$

Then $x^* \leq x_0$.

Clearly $\mu(0) > 0$ and A3c imply (2.4), and in this case since $\mu(x) \leq a + (r - \alpha)x$ for some nonnegative a,

$$x^* \leq \frac{a}{\alpha}.$$

Definition 2.6 A function h defined on $[0,\infty)$ is strictly concave-convex if there is an $x_h \geq 0$ so that h is strictly concave on $x < x_h$ and strictly convex on $x > x_h$.

If $x_h = 0$, h is strictly convex, but for simplicity we include that case in the definition of concave-convex. If h is twice continuously differentiable, a strictly concave-convex function has at most one point x where $h''(x) = 0$. If h is three times continuously differentiable, a concave-convex function has at most one point x where $h''(x) = 0$ and $h'''(x) > 0$.

In [25] it was shown that if $\lambda = 0$, i.e. no jumps, then under conditions similar to those here, the canonical solution is either strictly concave-convex or strictly concave. This was used in [21] to give a solution of the control problem for this case. Inspired by these results, we will look for sufficient conditions to insure concave-convexity for the more general jump-diffusion studied here. Unfortunately, this is not an easy task, and we have only been able to come up with some rather strong conditions. To present the results, define

$$Ag(x) = \int_0^x g(x-z)dF(z). \tag{2.5}$$

If A2a holds, an integration by parts shows that

$$(Ag)'(x) = \int_0^x g'(x-z)dF(z) = -\int_0^x g'(z)dF(x-z), \tag{2.6}$$

and if A2b holds,

$$(Ag)''(x) = f(x) + \int_0^x g''(x-z)f(z)dz. \tag{2.7}$$

Lemma 2.7 *Let g be a canonical solution. Assume A1c, A2b, A3a and A5. Also assume that*

$$\lambda(Ag)''(x) + \mu''(x)g'(x) < 0, \tag{2.8}$$

whenever

$$(Ag)'(x) = \left(\frac{\lambda + r - \mu'(x)}{\lambda}\right) g'(x). \tag{2.9}$$

Then g is strictly concave-convex. Moreover, for every $x > x^*$,

$$(Ag)'(x) < \left(\frac{\lambda + r - \mu'(x)}{\lambda}\right) g'(x), \tag{2.10}$$

i.e. if x_0 satisfies (2.8) and (2.9) then $x_0 \le x^*$.

Unfortunately the assumption (2.8) and (2.9) in Lemma 2.7 is not easy to verify, so something that is more easily verifiable is needed. Assume that the density f is continuously differentiable and consider the condition,

$$-f'(x) > c(x)f(0)f(x), \quad x \ge 0, \tag{2.11}$$

where

$$c(x) = \frac{\lambda}{\lambda + r - \mu'(x)}$$

and it is implicitly assumed that $f(0)$ is finite.

Theorem 2.8 *Let g be a canonical solution of (2.1). Assume A1c, A2c, A3a, A4 and A5. Furthermore, assume that (2.11) holds. Then the canonical solution g is strictly concave-convex.*

Since Theorem 2.8 gives us the result we want, it is of interest to examine a bit closer the class of distribution functions that satisfy (2.11). Clearly, (2.11) and A3a imply that f is strongly decreasing. Furthermore, integrating (2.11) from 0 to infinity and using that $\liminf_{x\to\infty} f(x) = 0$, gives

$$f(0) > f(0)\int_0^\infty c(x)f(x)dx = f(0)E[c(S)].$$

Therefore, it is necessary that $E[c(S)] < 1$.

We can write (2.11) as $\frac{d}{dx}\log f(x) < -c(x)f(0)$, and integrating this yields

$$f(x) < f(0)e^{-f(0)\int_0^x c(y)dy}.$$

By A3a, $c(x) \geq \frac{\lambda}{\alpha} > 0$ for all x and so f must be light tailed.

It is trivial to verify that the exponential distribution satisfies (2.11) provided $c(x) < 1$, i.e. provided A3d holds. The question is whether there are any other distributions that satisfy this inequality. Here are a couple of examples.

Example 2.9 Assume that $\mu'(x) = r - \alpha$ for some $\alpha > 0$ so that $c(x) = c = \frac{\lambda}{\lambda+\alpha}$. Let f be the exponential mixture

$$f(x) = a\beta_1 e^{-\beta_1 x} + (1-a)\beta_2 e^{-\beta_2 x}, \quad x \geq 0,$$

for $0 < a < 1$. Without loss of generality we can assume that $\beta_1 < \beta_2$. Then (2.11) is equivalent to $h(x) > 0$ for all $x \geq 0$, where

$$\begin{aligned} h(x) &= e^{\beta_1 x}(-f'(x) - cf(0)f(x)) \\ &= a\beta_1^2 + (1-a)\beta_2^2 e^{-(\beta_2-\beta_1)x} - c(a\beta_1 + (1-a)\beta_2)\left(a\beta_1 + (1-a)\beta_2 e^{-(\beta_2-\beta_1)x}\right). \end{aligned}$$

Since

$$h'(x) = -(1-a)\beta_2(\beta_2-\beta_1)e^{-(\beta_2-\beta_1)x}(\beta_2 - c(a\beta_1 + (1-a)\beta_2)) < 0,$$

this is satisfied if and only if $\lim_{x\to\infty} h(x) = a\beta_1^2 - c(a\beta_1 + (1-a)\beta_2)a\beta_1 \geq 0$. Easy calculations show that this is equivalent to

$$\frac{\beta_2}{\beta_1} \leq 1 + \frac{1-c}{c}\frac{1}{1-a} = 1 + \frac{\alpha}{(1-a)\lambda}.$$

Example 2.10 Assume again that $\mu'(x) = r - \alpha$ for some $\alpha > 0$ so that $c(x) = c = \frac{\lambda}{\lambda+\alpha}$. Let f be the truncated normal distribution

$$f(x) = \frac{e^{-\frac{1}{2\sigma^2}(x+\gamma)^2}}{\int_0^\infty e^{-\frac{1}{2\sigma^2}(y+\gamma)^2}dy} = \frac{\frac{1}{\sigma}e^{-\frac{1}{2\sigma^2}(x+\gamma)^2}}{H\left(\frac{\gamma}{\sigma}\right)}, \quad x \geq 0, \tag{2.12}$$

for $\gamma > 0$. Here

$$H(u) = \int_u^\infty e^{-\frac{1}{2}y^2}dy.$$

Then (2.11) is equivalent to

$$\frac{1}{\sigma}(x+\gamma) > c\frac{e^{-\frac{1}{2}\left(\frac{\gamma}{\sigma}\right)^2}}{H\left(\frac{\gamma}{\sigma}\right)}.$$

Since the left side is increasing in x, this is equivalent to

$$\frac{\gamma}{\sigma}e^{\frac{1}{2}\left(\frac{\gamma}{\sigma}\right)^2}H\left(\frac{\gamma}{\sigma}\right) > c.$$

Let

$$v(u) = ue^{\frac{1}{2}u^2}H(u).$$

Then $v(0) = 0$, and L'hôpital's rule easily shows that $\lim_{u\to\infty} v(u) = 1 > c$. Therefore, if we can show that v is strongly increasing in u, (2.11) is satisfied if and only if

$$\frac{\gamma}{\sigma} \geq u_0,$$

where u_0 is the unique solution of $v(u) = c$. To show that v is strongly increasing, differentiation gives

$$v'(u) = (1+u^2)e^{\frac{1}{2}u^2}H(u) - u.$$

An integration by parts gives that for $u > 0$,

$$H(u) > u^2\int_u^\infty \frac{1}{y^2}e^{-\frac{1}{2}y^2}dy = u^2\left(\frac{1}{u}e^{-\frac{1}{2}u^2} - H(u)\right),$$

from which we get that $(1+u^2)H(u) > ue^{-\frac{1}{2}u^2}$, and so $v'(u) > 0$. A numerical calculation with $\lambda = 1$ and $\alpha = 0.02$ shows that $u_0 = 6.936$

Remark 2.11 As mentioned in the introduction, in [19] it is shown that for the Lévy model the result of Theorem 2.8 holds if the condition (2.11) is replaced by the condition that $\log f$ is convex. This is a more attractive condition, one reason is that it includes several heavy tailed distributions like the Pareto distribution

$$F(x) = 1 - \frac{\theta^\kappa}{(\theta+x)^\kappa}, \quad x > 0, \tag{2.13}$$

for positive θ and κ. It also includes the heavy tailed Weibull distribution. On the other hand, the log-convexity assumption of f does not include (2.12) since the density in that example is not log-convex.

We conjecture that Theorem 2.8 holds also when f is log-convex. However, the proofs given in [16] and [19] rely on the Lévy strucure, so a different proof is needed.

3 The optimal solution

In this section we will assume that g is the unique canonical solution that satisfies (2.1). Then any function v that satisfies $v(0) = 0$ and $Lv(x) = 0$ is of the form

$$v(x) = cg(x), \tag{3.1}$$

for some constant c. This fact will be utilized in our quest for an optimal solution. Again proofs are of technical nature, and are therefore given in Section 6.

Consider the following set of problems with unknown V, $\bar{u}^*$ and $\underline{u}^*$.

B1:
$$V(0) = 0 \ \text{ and } \ LV(x) = 0, \qquad 0 < x < \bar{u}^*,$$
$$V(x) = V(\bar{u}^*) + k(x - \bar{u}^*), \qquad x > \bar{u}^*.$$

B2:
$$V(\bar{u}^*) = V(\underline{u}^*) + k(\bar{u}^* - \underline{u}^*) - K,$$
$$V'(\bar{u}^*) = k,$$
$$V'(\underline{u}^*) = k.$$

B3:
$$V(\bar{u}^*) = k\bar{u}^* - K,$$
$$V'(\bar{u}^*) = k,$$
$$V'(x) < k, \qquad 0 \le x \le \bar{u}^*.$$

From this and (3.1) we see that $V(x)$ can be written as

$$V(x) = \begin{cases} c^* g(x), & x \le \bar{u}^*, \\ V(\underline{u}^*) + k(x - \underline{u}^*) - K, & x > \bar{u}^*. \end{cases} \tag{3.2}$$

Here

$$c^* = \frac{k}{g'(\bar{u}^*)} = \frac{k(\bar{u}^* - \underline{u}^*) - K}{g(\bar{u}^*) - g(\underline{u}^*)}, \tag{3.3}$$

where in case B3, $\underline{u}^* = 0$. Also, if g is concave-convex then clearly $\underline{u}^* < x^* < \bar{u}^*$.

Theorem 3.1 *Assume that the canonical solution g is strictly concave-convex. Then we have:*

a) If B1+B2 or B1+B3 have a solution, this solution is unique.
b) If in addition $\lim_{x\to\infty} g'(x) = \infty$, then either B1+B2 or B1+B3 will have a solution.

It follows from the proof of Theorem 3.1(b) that it is B1+B2 that have a solution if and only if

$$\int_0^{\bar{u}} \left(1 - \frac{g'(x)}{g'(\bar{u})}\right) dx = \bar{u} - g(\bar{u}) > \frac{K}{k},$$

where $\bar{u}$ is the unique value that satisfies $g'(\bar{u}) = g'(0) = 1$.

If B1+B2 or B1+B3 have a solution, then

$$LV(x) = k\mu(x) - (r+\lambda)(V(\bar{u}^*) + k(x-\bar{u}^*)) + \lambda AV(x), \quad x > \bar{u}^*.$$

Therefore, if the canonical solution g is concave-convex, the fact that $V'(\bar{u}^*) = k$, that V and V' are continuous and that $LV(\bar{u}^*-) = 0$ gives

$$LV(\bar{u}^*+) = -\sigma^2(\bar{u}^*)V''(\bar{u}^*-) \leq 0. \tag{3.4}$$

Theorem 3.2 *Assume that the canonical solution g is a strictly concave-convex. Also assume A1d and A2a. Then we have:*

(i) Assume that either B1+B2 or B1+B3 have a solution, and that

$$LV(x) \leq 0, \quad x > \bar{u}^*. \tag{3.5}$$

Then $V^*(x) = V(x) = V_{\bar{u}^*,\underline{u}^*}(x)$ for all $x \geq 0$, where in case B1+B3, $\underline{u}^* = 0$. Thus the lump sum dividend barrier strategy $\pi^* = \pi_{\bar{u}^*,\underline{u}^*}$ is an optimal strategy. In particular (3.5) is satisfied if

$$(LV)'(x) = \lambda(AV)'(x) - k(r+\lambda-\mu'(x)) \leq 0, \quad x > \bar{u}^*. \tag{3.6}$$

(ii) If neither B1+B2 nor B1+B3 have a solution, then there do not exist an optimal strategy, but

$$V^*(x) = \lim_{\bar{u}\to\infty} V_{\bar{u},0}(x),$$

and this limit exists and is finite for every $x \geq 0$. In terms of the canonical solution,

$$V^*(x) = \frac{k}{g'_\infty} g(x),$$

where $g'_\infty = \lim_{\bar{u}\to\infty} g'(\bar{u})$.

Furthermore, case (i) occurs if $g'_\infty = \infty$. If g is concave, i.e. $x^* = \infty$, then case (ii) occurs.

Assumption A1d was made to guarantee that the stochastic differential equation (1.1) has a unique strong solution. It could be replaced by A1a and any other condition that guarantees a unique strong solution.

Remark 3.3 It was demonstrated in Example 2 in [19] that concave-convexity of g is not a necessary condition for a simple lump sum dividend barrier strategy to be optimal.

The next theorem gives sufficient, verifiable conditions for optimality.

Theorem 3.4 *Assume A1c, A1d, A2c, A3d, A4, A5, A7 and (2.11). Then either B1+B2 or B1+B3 have a solution, and an optimal policy exists. This optimal policy is given in Theorem 3.2(i).*

Example 3.5 Assume that income from the basic insurance business evolves as

$$P_t = pt + \sigma_P W_{P,t} - Y_t,$$

where $\sigma_P^2 > 0$. Also assume that assets earn return according to

$$R_t = (r - \alpha)t + \sigma_R dW_{R,t},$$

where $\alpha > 0$. Here W_P and W_R are standard Brownian motions with correlation ρ. The constant α can be seen as a cost due to inefficient investments, or as an equity premium since $r - (r - \alpha) = \alpha$.

Total assets without dividend payments are then

$$dX_t = dP_t + X_{t-}dR_t, \quad X_0 = x.$$

Combining the two Brownian motions, this can be written as (1.1), where

$$\mu(x) = p + (r - \alpha)x, \quad \sigma^2(x) = \sigma_P^2 + 2\rho\sigma_P\sigma_R x + \sigma_R^2 x^2.$$

In order for assumption A5 to hold it is necessary and sufficient that $\sigma_P^2 > 0$. If in addition A2c is satisfied and (2.11) holds, an optimal solution exists and is given in Theorem 3.4.

4 The case with no fixed transaction costs

In this section results for the case $K = 0$ similar to those in Section 3 will be presented. When $K = 0$ there is the added possibility that dividends may be paid continuously. The controlled process (1.2) therefore becomes

$$\begin{aligned} X_t^\pi = x &+ \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - Y_t \\ &- \sum_{n=1}^\infty 1_{\{\tau_n^\pi < t\}}\xi_n^\pi - D_t^{c,\pi}, \quad t \le \tau^\pi, \end{aligned} \tag{4.1}$$

where $D^{c,\pi}$ is a continuous, nondecreasing and adapted process. The performance function (1.4) becomes

$$V_\pi(x) = E_x\left[\sum_{n=1}^\infty e^{-r\tau_n^\pi}k\xi_n^\pi 1_{\{\tau_n^\pi \le \tau^\pi\}} - \int_0^{\tau^\pi} e^{-rs}kdD_s^{c,\pi}\right]. \tag{4.2}$$

Also, the optimal function V^* is defined as in (1.5).

Definition 4.1 A singular continuous dividend barrier strategy $\pi = \pi_u$ with barrier u satisfies:

- When $X_t^\pi < u$, do nothing.
- When $X_t^\pi > u$, reduce X_t^π to u by paying $X_t^\pi - u$ as a lump sum dividend.
- When $X_t^\pi = u$, pay dividends so that u is a reflecting barrier.

The corresponding value function is denoted by $V_u(x)$.

With the singular continuous dividend barrier strategy a lump sum is only paid at time 0, and only if $x > u$. After that dividends are paid continuously, but if A5 holds it is well known from the theory of singular stochastic control, see e.g. [25], that the dividend process $D^{c,\pi}$ is a singular process. This means that $D^{c,\pi}$ is continuous, nondecreasing and increasing on an uncountable set of Lebesgue measure zero. Therefore, as opposed to the lump-sum dividend strategy of Definition 1.1, from a practical point of view it is impossible to implement a singular continuous dividend policy.

Using the results from Section 2, the following theorem is proved as in [25].

Theorem 4.2 *Assume that a canonical solution exists and is strictly concave-convex. Also assume A1d, A2a and A5. Then we have:*

(i) If $x^* < \infty$ let

$$V(x) = \frac{k}{g'(x^*)} g(x), \qquad x \le x^*,$$
$$V(x) = V(x^*) + k(x - x^*), \qquad x > x^*.$$

If

$$LV(x) \le 0, \quad x \ge x^*, \tag{4.3}$$

then $V^*(x) = V(x) = V_{x^*}(x)$ for all $x \ge 0$, so that the singular continuous dividend barrier strategy $\pi = \pi_{x^*}$ is optimal.

(ii) If $x^* = \infty$ so that g is concave, then there is no optimal strategy, but

$$V^*(x) = \frac{k}{g'_\infty} g(x), \qquad x \ge 0.$$

Note that if $x^* = 0$, assets are immediately reduced to zero and ruin occurs because of A5.

Again, the assumptions of Theorem 3.4 are sufficient for an optimal solution to exist.

5 A numerical approach

Theorem 3.4 gives sufficient conditions for a lump sum barrier strategy to be optimal, but unfortunately the class of distributions that satisfy (2.11) is rather limited. However, if A1d, A2a, A3c, A5 and A6 are satisfied, it follows from Theorems 2.4, 2.5 and 3.2 that all that is needed for a lump sum dividend barrier policy to be optimal is that the canonical solution is strictly concave-convex and that (3.5) is satisfied. In principle, both these conditions can be tested numerically, but such a test will necessarily be on a finite interval, and there is no a priori guarantee that there are points beyond that interval where the assumptions are not satisfied. Therefore, it would be useful to prove theoretically that for some numerically calculable $x_P > x^*$, the conditions hold. In that case it is sufficient to use a numerical check on the interval $(0, x_P)$. Here we will take such an approach. All proofs are again given in Section 6. The first result is concerned with ultimate convexity.

Theorem 5.1 *Let g be a canonical solution.*

a) Assume A1b and A3c. Let

$$x_M = \inf\{x > \max\{x^*, x_r\} : g'(x) \geq g'(0)\}. \tag{5.1}$$

Then $g''(x) > 0$ for all $x > x_M$.

b) Assume A1b, A2b, A3c and A4. Let

$$x_L = \inf\left\{x > \max\{x^*, x_r\} : (r - \mu'(x))g'(x) > \lambda g(x^*) \max_{z \geq x - x^*} f(z)\right\} \tag{5.2}$$

If $x_L < \infty$ and $g''(x) > 0$ for all $x \in (x^*, x_L]$, then g is strictly convex on (x^*, ∞). Furthermore, (2.10) holds for all $x > x_L$.

Remark 5.2 Instead of searching for x_M in (5.1) or x_L in (5.2), an alternative is to take an arbitrary $x_A > \max\{x^*, x_r\}$ and check if the condition in (5.1) or in (5.2) holds. If that is the case, and it is numerically shown that $g''(x) > 0$ on (x^*, x_A), it follows from the definitions of x_M and x_L that $g''(x) > 0$ on (x^*, ∞).

We now turn to condition (3.5). Assume that B1+B2 or B1+B3 have a solution, and let $h(x) = LV(x)$. If we can find a numerically calculable $x_P \geq \bar{u}^*$ so that it is theoretically known that $h(x) \leq 0$ when $x \geq x_P$, then it is enough to numerically test whether $h(x) \leq 0$ on $(\bar{u}^*, x_P)$. By (3.4), this holds if $h'(x) \leq 0$ for $x \in (\bar{u}^*, x_P)$.

Theorem 5.3 *Let g be a canonical solution.*

a) Assume A1b, A2d and A4. Set

$$x_K = \inf\left\{x \geq \bar{u}^* + x_f : \lambda \int_0^{\bar{u}^*} g'(z) f(x - z) dz < (r - \mu'(x)) g'(\bar{u}^*)\right\} \tag{5.3}$$

Then $(LV)'(x) \leq 0$, $x \geq x_K$. Also, $x_K < \infty$ if A3c holds.

b) Assume A1b, A2b and A4. Set

$$x_J = \inf\left\{x > \bar{u}^* : \max_{z \geq x - \bar{u}^*} f(z) < \frac{1}{\lambda}(r - \mu'(x)) \frac{g'(\bar{u}^*)}{g(\bar{u}^*)}\right\}. \tag{5.4}$$

Then $(LV)'(x) \leq 0$, $x \geq x_J$. Also, $x_J < \infty$ if A2d and A3c holds.

Clearly, if $x_K = \bar{u}^*$ or $x_J = \bar{u}^*$, the condition (3.5) holds.

Remark 5.4 As in Remark 5.2 it is not necessary to calculate x_K and x_J. Again it is sufficient to pick an arbitry x_A, with $x_A > \bar{u}^* + x_f$ for x_M and $x_A > \bar{u}^*$ for x_J, and verify that the condition in (5.3) or in (5.4) holds for x_A. If that is the case, (3.5) holds provided it can be shown numerically that $LV(x) \leq 0$ for $x \in (\bar{u}^*, x_A)$.

Example 5.5 In this example we will provide numerical results for the model presented in Example 3.5. We will use two different claimsize distributions, the exponential distribution with expectation β^{-1} and the Pareto distribution (2.13). For the latter, if $\kappa > 2$,

$$E[S] = \frac{\theta}{\kappa - 1} \quad \text{and} \quad E[S^2] = \frac{2\theta^2}{(\kappa-1)(\kappa-2)}.$$

The P process satisfies

$$E[P_t] = (p - \lambda E[S])t \quad \text{and} \quad \text{Var}[P_t] = (\sigma_P^2 + \lambda E[S^2])t.$$

We let p, λ and $E[P_t]$ be the same for the two claimsize distributions. Then $E[S]$ will also be the same, so $\beta = (\kappa - 1)/\theta$. Furthermore, letting $\text{Var}[P_t]$ be the same, and denoting the diffusion parameters by $\sigma_{P,E}^2$ and $\sigma_{P,P}^2$ respectively, gives

$$\sigma_{P,E}^2 = \sigma_{P,P}^2 + \frac{2\lambda\theta^2}{(\kappa-1)^2(\kappa-2)}.$$

For a numerical example we let $p = 1.5$, $\lambda = 1$, $\beta = 1$, $\kappa = 3$, $\theta = 2$, $\sigma_{P,E}^2 = 3$, $\sigma_{P,P}^2 = 1$ and $\rho = 0$, which make $E[P_t]$ and $\text{Var}[P_t]$ the same for the two distributions. Furthermore, let $r = 0.1$, $\alpha = 0.02$, $\sigma_R = 0.2$, $k = 0.9$ and $K = 0.2$. Numerical calculations together with Remarks 5.2 and 5.4 show that the Pareto distribution satisfies the conditions of Theorem 3.2(i), and so the optimal policy is a lump sum dividend policy in both cases. In view of Remark 2.11, this comes as no surprise. The numerical solutions show that in the exponential case $(\bar{u}^*, \underline{u}^*) = (15.96, 6.32)$ so that $\bar{u}^* - \underline{u}^* = 9.65$. In the Pareto case $(\bar{u}^*, \underline{u}^*) = (12.84, 4.11)$ so that $\bar{u}^* - \underline{u}^* = 8.72$. Figure 5.1 shows the value function $V^*(x)$ for increasing x.

It is interesting to note that $\bar{u}^*$, $\underline{u}^*$ and $\bar{u}^* - \underline{u}^*$ are all higher for the exponential distribution than for the Pareto distribution, while the value function $V^*(x)$ is higher for the Pareto distribution. A possible reason for this is that the Pareto distribution yields many small claims and an occasional very large one, while the exponential distribution yields more similar claims. Therefore, not worrying too much about the occasional large claim, the Pareto distribution combined with a lower value of σ_P^2 is less affected with the possibility of ruin, thus allowing a bolder strategy and higher expected payout. If ruin occurs, in the Pareto case it will likely be with a very large deficit, but since the size of the deficit does not matter, this is an advantage for the Pareto distribution and so it can explain the higher value for this distribution.

Figures 5.2-5.9 show optimal barriers $\bar{u}^*$ and $\underline{u}^*$, optimal payout $\bar{u}^* - \underline{u}^*$ and optimal value when $x = 2$, i.e. $V^*(2)$, for the exponential and Pareto distributions. In all figures the parameters are the same as above, except of course

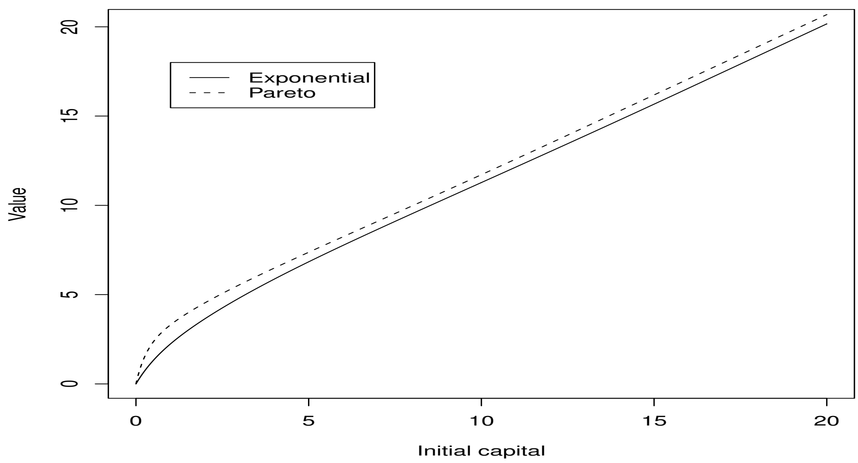

Fig. 5.1 Values of $V^*(x)$ for increasing x, using the exponential and the Pareto distributions for S. The parameters are $p = 1.5$, $\lambda = 1$, $\beta = 1$, $\kappa = 3$, $\theta = 2$, $r = 0.1$, $\alpha = 0.02$, $\sigma_R = 0.2$, $\rho = 0$, $k = 0.9$ and $K = 0.2$. The diffusion parameters are $\sigma_P^2 = 3$ in the exponential case and $\sigma_P^2 = 1$ in the Pareto case.

for the one that varies in that particular figure. Since the Pareto distribution is not covered by Theorem 3.4, a numerical test as described in Remarks 5.2 and 5.4 was used to assure that the optimal policy will always be a lump sum dividend policy. This, not surprisingly, turned out to be the case all the time. We will return to this test in Example 5.6.

Looking at the figures, the first thing to notice is that the Pareto distribution always results in a higher value of $V^*(x)$, thus supporting the argument given above. In most cases, both $\bar{u}^*$ and $\underline{u}^*$ are lower in the Pareto case, as is the payout $\bar{u}^* - \underline{u}^*$.

From Figure 5.2 we see that for $p \leq 0.63$, $\underline{u}^* = 0$ in the exponential case, and $\underline{u}^* = 0$ for $p \leq 0.46$ in the Pareto case. So when the income p is sufficiently small, it is optimal to pay everything in dividends immediately and go bankrupt. The reason is of course that the premium is too small compared to expected claims. The same optimality of immediate bankruptcy is observed in Figure 5.3 when the claim intensity λ is high.

Most plots must be said to be rather reasonable, although not apriori obvious. The main exceptions are Figures 5.2 and 5.3, where $\bar{u}^*$, $\underline{u}^*$ and $\bar{u}^* - \underline{u}^*$ all exhibit some rather unexpected patterns.

Example 5.6 In this example we again study the model of Example 5.5, but with different parameters and distribution function. Let $\sigma_P = \sigma_R = 0$, $p = 21.4$, $\lambda = 10$, $r = 0.1$, $\alpha = 0.08$, $k = 1$ and $K = 0$. Also, let the claimsizes be

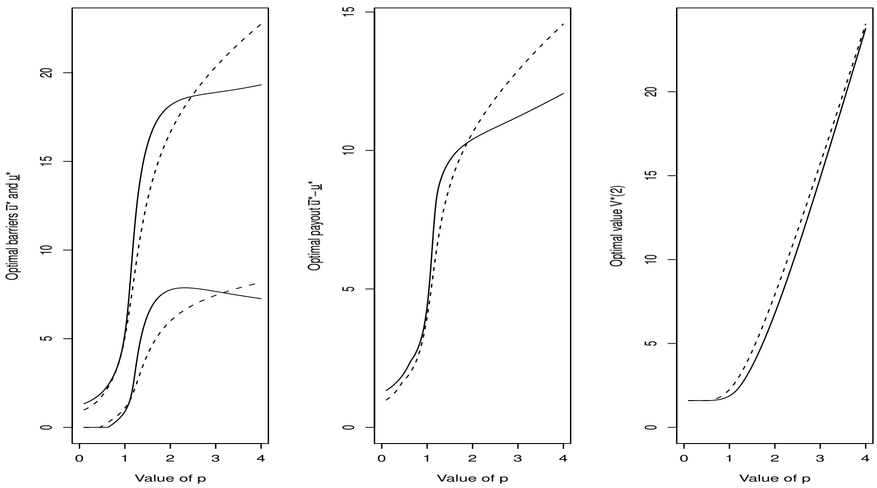

Fig. 5.2 Values for increasing p using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Left panel: Values of the optimal barriers $\bar{u}^*$ and $\underline{u}^*$. Middle panel: Values of the optimal payout $\bar{u}^* - \underline{u}^*$. Right panel: The value function $V^*(2)$. Full line is exponential distribution and broken line is Pareto distribution.

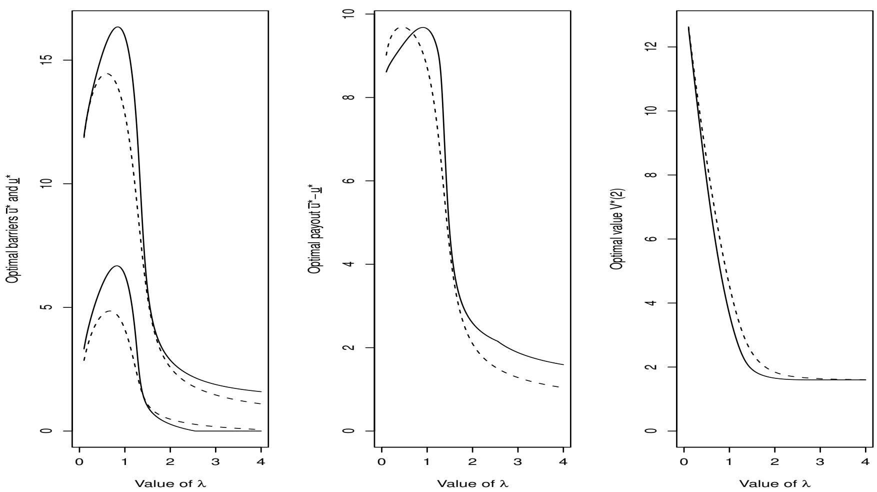

Fig. 5.3 Values for increasing λ using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

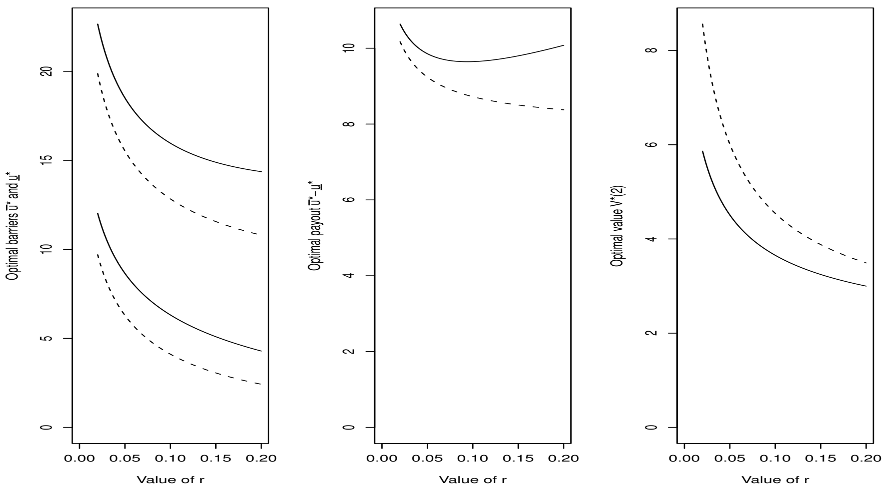

Fig. 5.4 Values for increasing r using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

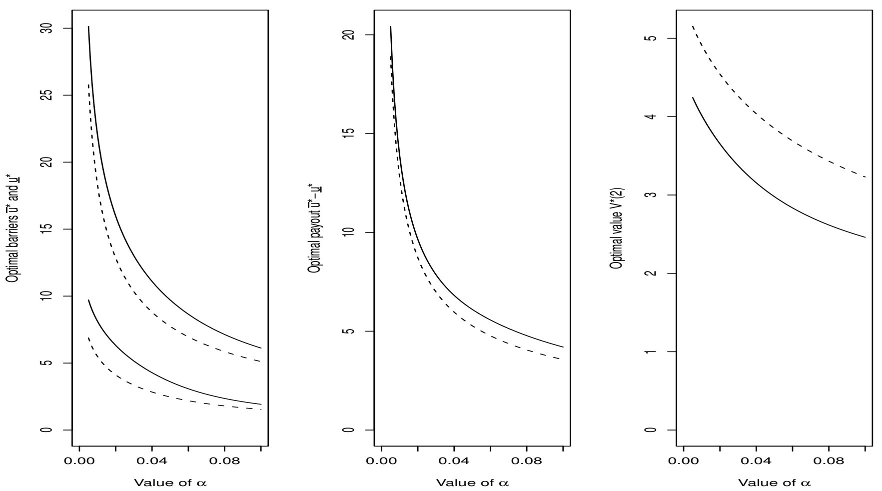

Fig. 5.5 Values for increasing α using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

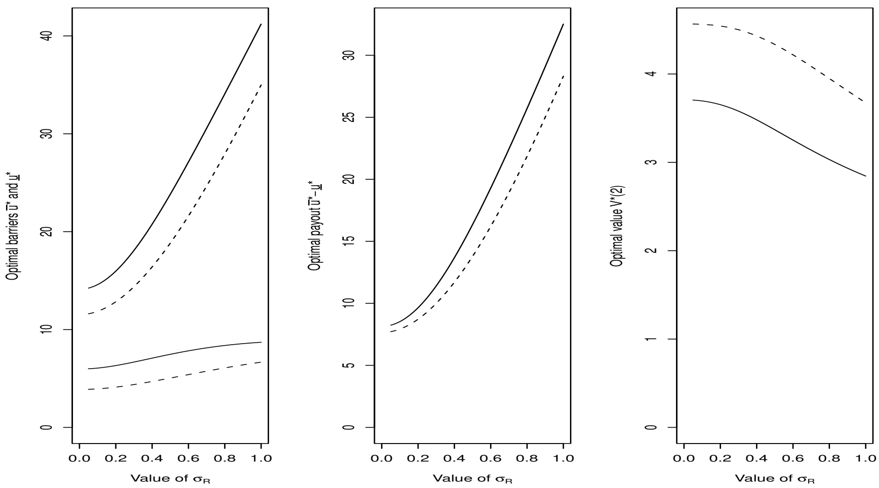

Fig. 5.6 Values for increasing σ_R using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

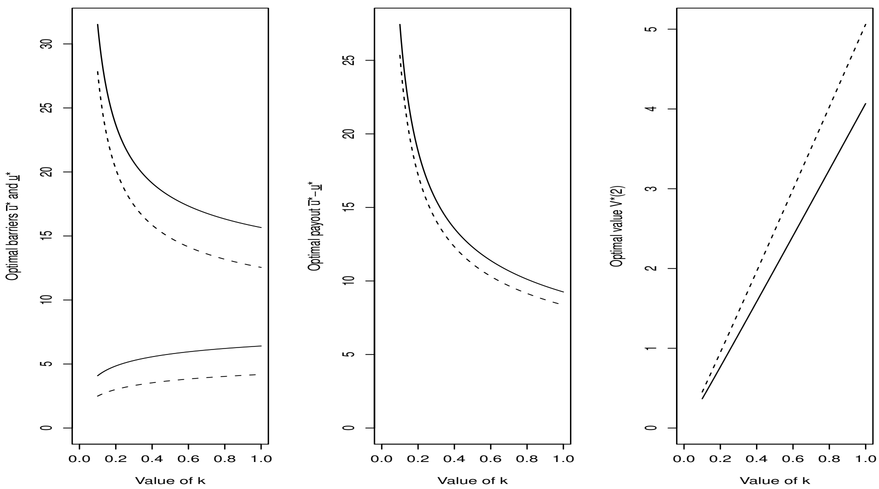

Fig. 5.7 Values for increasing k using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

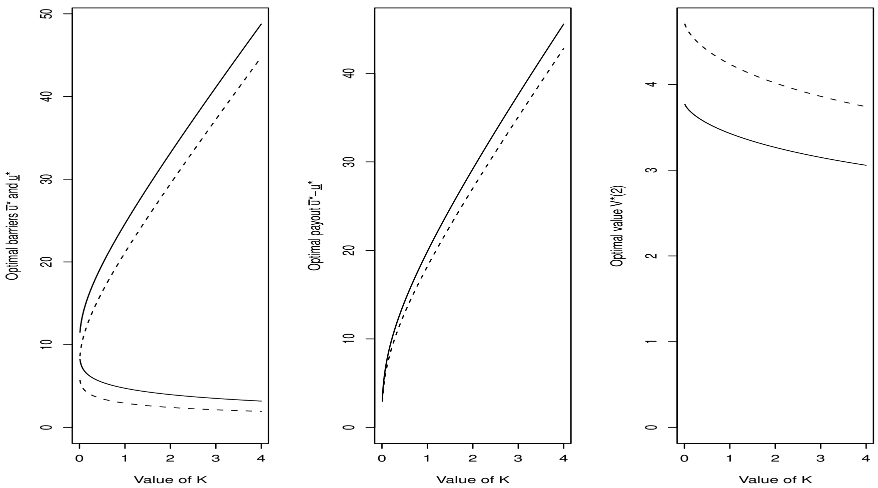

Fig. 5.8 Values for increasing K using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

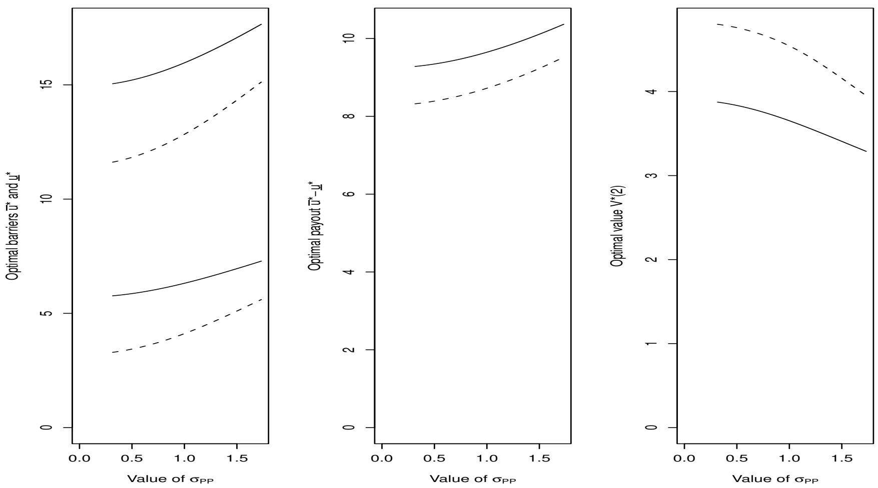

Fig. 5.9 Values for increasing σ_P using the exponential and the Pareto distributions for S. The P-process diffusion parameters are $\sigma_P^2 = \sigma_{P,E}^2 = 2 + \sigma_{P,P}^2$. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

gamma distributed with density

$$f(x) = \beta^2 x e^{-\beta x} 1_{\{x>0\}}, \tag{5.5}$$

with $\beta = 1$. Then it is proved in [1] that for this model a simple barrier strategy cannot be optimal, and the optimal band strategy is identified.

Making a few changes, let $\sigma_P = 0.5$, $\sigma_R = 0.2$ and $\rho = 0$. Although not relevant for the canonical solution, let $k = 0.9$ and $K = 0.2$. The upper left panel in Figure 5.10 shows $g''(x)$ for $x \in (0.064, 50)$. Since there are three roots $x_1 = 0.069$, $x_2 = 1.73$ and $x_3 = 12.66$, we cannot expect a simple lump sum dividend barrier strategy to be optimal, although we cannot rule that out as is shown in [19]. The upper right panel shows $LV(x)$ for $x \in (\bar{u}^*, 50)$, and since the condition in (5.4) turned out to be satisfied for $x = 50$, it follows from Remark 5.4 that (3.5) is satisfied.

Making yet another change, let $\sigma_P = 4$ and as before $\sigma_R = 0.2$. From the lower left panel we have (maybe a bit difficult to see) that there is only one root $x^* = 14.5$. Furthermore, since the condition in (5.2) turned out to be satisfied for $x = 50$, it follows from Remark 5.2 that g is strictly concave-convex. Thus the added diffusion smoothed out the non concave-convexity in the original model. Also, the condition in (5.4) was satisfied for $x = 50$, and so by Remark 5.4 and the lower right panel in Figure 5.10, (3.5) is satisfied. Therefore by Theorem 3.2, the optimal strategy is a simple lump sum dividend strategy.

6 Proofs

Proof of Lemma 2.2 It is straightforward to show that $\|\cdot\|^{\infty}_{\beta,\zeta}$ is a norm on $L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$. To prove completeness, let $\{\mathbf{u}_k\}$ be a Cauchy sequence in $L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$, and for each $x \geq 0$ let

$$\mathbf{u}(x) = \limsup_{k\to\infty} \mathbf{u}_k(x),$$

where the lim sup is componentwise. Choose N_1 large enough so that for $k, l \geq N_1$, $\|\mathbf{u}_k - \mathbf{u}_l\|^{\infty}_{\beta,\zeta} < 1$. Then for every $k \geq N_1$,

$$\|\mathbf{u}_k\|^{\infty}_{\beta,\zeta} \leq \|\mathbf{u}_k - \mathbf{u}_{N_1}\|^{\infty}_{\beta,\zeta} + \|\mathbf{u}_{N_1}\|^{\infty}_{\beta,\zeta} < 1 + \|\mathbf{u}_{N_1}\|^{\infty}_{\beta,\zeta} < \infty,$$

and from this it follows that $\mathbf{u} \in L^{\infty}_{\beta,\zeta}([0,\infty), R^n)$. To show that $\mathbf{u}_k$ converges towards $\mathbf{u}$, for any given $\varepsilon > 0$ choose N_ε so that for any $k, l \geq N_\varepsilon$, $\|\mathbf{u}_k - \mathbf{u}_l\|^{\infty}_{\beta,\zeta} < \frac{\varepsilon}{2}$. Also, for each $x \geq 0$ choose $m_j(x) \geq N_\varepsilon$ large enough so that $|u_{m_j(x),j}(x) - u_j(x)| < \frac{\varepsilon}{2}$. Then for the j'th component,

$$\frac{|u_{k,j}(x) - u_j(x)|}{\exp(\beta x + \zeta x^2)} \leq \frac{|u_{k,j}(x) - u_{m_j(x),j}(x)|}{\exp(\beta x + \zeta x^2)} + \frac{|u_{m_j(x),j}(x) - u_j(x)|}{\exp(\beta x + \zeta x^2)} < \varepsilon.$$

Taking supremum over x and then maximum over j gives that $\|\mathbf{u}_k - \mathbf{u}\|^{\infty}_{\beta,\zeta} < \varepsilon$, and completeness follows.

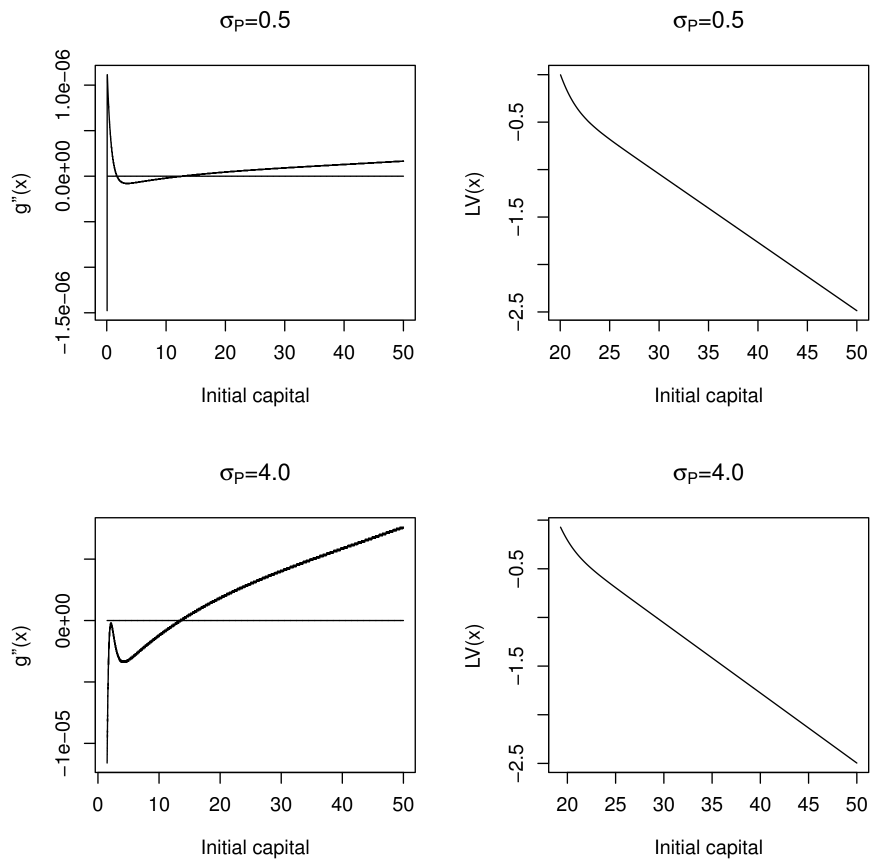

Fig. 5.10 Value of $g''(x)$ (left panels) and $LV(x)$ (right panels). The upper panels are for $\sigma_P = 0.5$, while the lower are $\sigma_P = 4$. The density (5.5) was used for the distribution of S. The other parameters are $p = 21.4$, $\lambda = 10$, $r = 0.1$, $\sigma_R = 0.2$, $\alpha = 0.08$, $k = 1$ and $K = 0$.

It remains to prove that $C_{\beta,\zeta}([0,\infty), R^n)$ is closed in $L^\infty_{\beta,\zeta}([0,\infty), R^n)$. Assume that the $\mathbf{u}_k \in C_{\beta,\zeta}([0,\infty), R^n)$ converge towards $\mathbf{u}$ in the $\|\cdot\|^\infty_{\beta,\zeta}$ norm, but that $\mathbf{u}$ is not continuous at a point x_0. With $b > x_0$ we get,

$$\begin{aligned}\sup_{0\le x\le b} |\mathbf{u}_k(x) - \mathbf{u}(x)| &\le \exp(\beta b + \zeta b^2) \sup_{0\le x\le b} \frac{|\mathbf{u}_k(x) - \mathbf{u}(x)|}{\exp(\beta x + \zeta x^2)} \\ &\le \exp(\beta b + \zeta b^2)\, \|\mathbf{u}_k - \mathbf{u}\|^\infty_{\beta,\zeta}\,.\end{aligned}$$

Hence convergence in the $\|\cdot\|^\infty_{\beta,\zeta}$ norm implies convergence in the standard sup norm on $[0,b]$. But it is well known that $C([0,b], R^n)$ is complete, hence $\mathbf{u}$ must be continuous on $[0,b]$, a contradiction. This proves the lemma.

Let

$$C_{\beta,\zeta,0}\left([0,\infty),R^n\right) = \{\mathbf{u} \in C_{\beta,\zeta}\left([0,\infty),R^n\right) : u_1(0) = 0\},$$

and similarly

$$C^k_{\beta,\zeta,0}\left([0,\infty),R^n\right) = \{\mathbf{u} \in C^k_{\beta,\zeta}\left([0,\infty),R^n\right) : u_1(0) = 0\}.$$

Then it follows trivially from Lemma 2.2 that $C_{\beta,\zeta,0}\left([0,\infty),R^n\right)$ is a Banach space with the $\|\cdot\|^\infty_{\beta,\zeta}$ norm.

Let the operator A be as in (2.5) and define

$$G_1u(x) = \int_0^x u(z)dz.$$

Assume A5 and set

$$G_2\mathbf{u}(x) = \int_0^x \frac{2}{\sigma^2(z)}\left((r+\lambda)u_1(z) - \mu(z)u_2(z) - \lambda Au_1(z)\right)dz, \quad x \geq 0.$$

Finally, let $\mathbf{Gu}(x) = (G_1u_2(x), G_2\mathbf{u}(x))$.

Lemma 6.1 *Let $u \in C_{\beta,\zeta,0}\left([0,\infty),R\right)$. Then $Au \in C_{\beta,\zeta,0}\left([0,\infty),R\right)$ and $G_1u \in C_{\beta,\zeta,0}\left([0,\infty),R\right) \cap C^1_{\beta,\zeta,0}\left([0,\infty),R\right)$. Furthermore,*

$$\|Au\|^\infty_{\beta,\zeta} \leq \|u\|^\infty_{\beta,\zeta} \tag{6.1}$$

and

$$\|G_1u\|^\infty_{\beta,\zeta} \leq \frac{1}{\beta}\|u\|^\infty_{\beta,\zeta}. \tag{6.2}$$

Proof That G_1u is continuously differentiable is obvious. Furthermore,

$$Au(x+h)-Au(x) = \int_0^x (u(x+h-z)-u(x-z))dF(z) + \int_x^{x+h} u(x+h-z)dF(z).$$

The first term goes to zero as h goes to zero because of continuity of u, and the second term goes to zero since $u(0) = 0$. Also by monotonicity of F,

$$\begin{aligned}|Au(x)| &\leq \int_0^x |u(z)||dF(x-z)| \\ &\leq \int_0^x e^{\beta(x-z)+\zeta(x^2-z^2)}|u(z)||dF(x-z)| \\ &\leq e^{\beta x+\zeta x^2}\|u\|^\infty_{\beta,\zeta}.\end{aligned}$$

Therefore,

$$\frac{|Au(x)|}{\exp(\beta x+\zeta x^2)} \leq \|u\|^\infty_{\beta,\zeta}.$$

Taking supremum over x gives (6.1). Next

$$\begin{aligned}|G_1 u(x)| &\le \int_0^x e^{\beta z+\zeta z^2}\frac{|u(z)|}{\exp(\beta z+\zeta z^2)}dz\\ &\le e^{\beta x+\zeta x^2}\|u\|^{\infty}_{\beta,\zeta}\int_0^x e^{-\beta(x-z)}dz\\ &\le \frac{1}{\beta}e^{\beta x+\zeta x^2}\|u\|^{\infty}_{\beta,\zeta}.\end{aligned}$$

The rest of the proof of (6.2) is now the same as above.

Lemma 6.2 *Given the assumptions of Theorem 2.3, let* $\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right)$ *with* $\zeta > 0$ *if* $M_2 > 0$ *in A7. Then* $G_2\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R\right)$ *and*

$$\|G_2\mathbf{u}\|^{\infty}_{\beta,\zeta} < 4\max\left\{\frac{M_1}{\beta},\frac{M_2}{\zeta}\right\}\|\mathbf{u}\|^{\infty}_{\beta,\zeta}, \tag{6.3}$$

with $M_2/\zeta = 0$ *if* $M_2 = \zeta = 0$*. Moreover, for any* $\tilde\beta > \beta$, $G_2\mathbf{u} \in C^1_{\tilde\beta,\zeta,0}\left([0,\infty), R\right)$.

Proof Clearly $G_2\mathbf{u}$ is continuously differentiable with $G_2\mathbf{u}(0) = 0$. Also by assumptions and (6.1),

$$\begin{aligned}|(G_2\mathbf{u})'(x)| &\le \frac{2}{\sigma^2(x)}\left(|\mu(x)||u_2(x)| + (r+\lambda)|u_1(x)| + \lambda|Au_1(x)|\right)\\ &\le 4(M_1+M_2x)e^{\beta x+\zeta x^2}\|\mathbf{u}\|^{\infty}_{\beta,\zeta}.\end{aligned} \tag{6.4}$$

Therefore,

$$\begin{aligned}|G_2\mathbf{u}(x)| &\le 4\|\mathbf{u}\|^{\infty}_{\beta,\zeta}\int_0^x e^{\beta z+\zeta z^2}(M_1+M_2z)dz\\ &\le 4e^{\beta x+\zeta x^2}\|\mathbf{u}\|^{\infty}_{\beta,\zeta}\int_0^x e^{-(x-z)(\beta+\zeta x)}(M_1+M_2z)dz\\ &\le 4e^{\beta x+\zeta x^2}\|\mathbf{u}\|^{\infty}_{\beta,\zeta}\frac{M_1+M_2x}{\beta+\zeta x}\\ &\le 4e^{\beta x+\zeta x^2}\|\mathbf{u}\|^{\infty}_{\beta,\zeta}\max\left\{\frac{M_1}{\beta},\frac{M_2}{\zeta}\right\}.\end{aligned}$$

The result (6.3) now follows as before. From (6.4) we get

$$\frac{|(G_2\mathbf{u})'(x)|}{\exp(\tilde\beta x+\zeta x^2)} \le 4(M_1+M_2x)e^{-(\tilde\beta-\beta)x}\|\mathbf{u}\|^{\infty}_{\beta,\zeta},$$

which shows that $G_2\mathbf{u} \in C^1_{\tilde\beta,\zeta}\left([0,\infty), R\right)$.

Lemmas 6.1 and 6.2 now give:

Lemma 6.3 *Under the assumptions of Theorem 2.3, for* $\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right)$ *and* $\tilde{\beta} > \beta$,

$$\mathbf{Gu} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right) \cap C^1_{\tilde{\beta},\zeta}\left([0,\infty), R^2\right).$$

Furthermore,

$$\|\mathbf{Gu}\|^{\infty}_{\beta,\zeta} \leq c_G \|\mathbf{u}\|^{\infty}_{\beta,\zeta},$$

where

$$c_G = \max\left\{\frac{1}{\beta}, \frac{4M_1}{\beta}, \frac{4M_2}{\zeta}\right\}.$$

Proof of Theorem 2.3 For $\mathbf{u} \in C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ define

$$\mathbf{Hu}(x) = (0,1) + \mathbf{Gu}(x).$$

Choose β_0 and ζ in Lemma 6.3 (with β_0 for β) large enough so that $c_G < 1$. Then $\mathbf{H}$ is a contraction operator on $C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$, and since $C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ is complete, there is a $\mathbf{v} \in C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ so that $\mathbf{Hv} = \mathbf{v}$. Furthermore, by Lemmas 6.1 and 6.2, for $\beta > \beta_0$,

$$\mathbf{v} = \mathbf{Hv} \in C^1_{\beta,\zeta,0}\left([0,\infty), R^2\right).$$

Let $g = v_1$. Then $g' = v_1' = (G_1 v_2)' = v_2$ and so since $g' = v_2$ is continuously differentiable,

$$\begin{aligned} g''(x) &= v_2'(x) \\ &= (G_2\mathbf{v})'(x) \\ &= \frac{2}{\sigma^2(x)}(-\mu(x)v_2(x) + (r+\lambda)v_1(x) - \lambda A v_1(x)) \\ &= \frac{2}{\sigma^2(x)}(-\mu(x)g'(x) + (r+\lambda)g(x) - \lambda A g(x)). \end{aligned}$$

Rearranging this last equation yields $Lg(x) = 0$. Also $g'(0) = v_2(0) = 1 + (G\mathbf{v})_2(0) = 1$, hence g solves (2.1).

Conversely, it can be shown that if $h \in C^2_{\beta,\zeta,0}([0,\infty), R)$ for some $\beta > 0$ and $\zeta \geq 0$ and h solves (2.1), then $\mathbf{w} = (h, h') \in C^1_{\beta,\zeta,0}([0,\infty), R^2)$ and satisfies $\mathbf{Hw} = \mathbf{w}$. Since $\mathbf{H}$ has a unique fixed point it follows that $h = w_1 = v_1 = g$.

In the remaining proofs we shall use the more convenient notation $A_g(x) = Ag(x)$, and similarly $A_g'(x) = (Ag)'(x)$.

Proof of Theorem 2.4 We start by proving that g is strongly increasing. The equation $Lg = 0$ gives

$$g''(x) = \frac{2}{\sigma^2(x)}(-\mu(x)g'(x) + (r+\lambda)g(x) - \lambda A_g(x)). \tag{6.5}$$

Let $x_0 = \inf\{x > 0 : g'(x) = 0\}$. Then g is strongly increasing on $(0, x_0)$, and so

$$\begin{aligned} g(x_0) - A_g(x_0) &= g(x_0) - \int_0^{x_0} g(x_0 - z)dF(z) \\ &\geq g(x_0)(1 - F(x_0)) \geq 0. \end{aligned} \tag{6.6}$$

Therefore, if $x_0 < \infty$,

$$g''(x_0) \geq \frac{2}{\sigma^2(x_0)} rg(x_0) > 0. \tag{6.7}$$

But since $g'(0) = 1 > 0$, it follows from the definition of x_0 that $g''(x_0) \leq 0$, a contradiction. Hence $x_0 = \infty$, and g' is strongly increasing.

Assume that (2.3) holds and let

$$H(x) = g'(x) - \frac{c_1}{c_2 + c_3 x} g(x),$$

so that in particular $H(0) = 1$. Let $x_1 = \inf\{x : H(x) = 0\}$. If $x_1 < \infty$ then $H'(x_1) \leq 0$, and we will show that this leads to a contradiction. So assume $x_1 < \infty$. Then

$$g'(x_1) = \frac{c_1}{c_2 + c_3 x_1} g(x_1)$$

and by (6.5) and (6.6),

$$\begin{aligned} g''(x_1) &\geq \frac{2}{\sigma^2(x_1)} (rg(x_1) - \mu(x_1)g'(x_1)) \\ &= \frac{2}{\sigma^2(x_1)} \left(r - \frac{c_1}{c_2 + c_3 x_1} \mu(x_1) \right) g(x_1). \end{aligned} \tag{6.8}$$

Therefore,

$$\begin{aligned} H'(x_1) &= g''(x_1) - \frac{c_1}{c_2 + c_3 x_1} g'(x_1) + \frac{c_1 c_3}{(c_2 + c_3 x_1)^2} g(x_1) \\ &\geq \left(\frac{2}{\sigma^2(x_1)} \left(r - \frac{c_1}{c_2 + c_3 x_1} \mu(x_1) \right) - \frac{c_1(c_1 - c_3)}{(c_2 + c_3 x_1)^2} \right) g(x_1) > 0, \end{aligned}$$

where the last inequality follows from (2.3). Hence $x_1 = \infty$ and so H is positive. It is easy to verify that the equation

$$g'(x) - \frac{c_1}{c_2 + c_3 x} g(x) = H(x)$$

has the solution

$$g(x) = g(1) \left(1 + \frac{c_3}{c_2} x\right)^{\frac{c_1}{c_3}} + \left(1 + \frac{c_3}{c_2} x\right)^{\frac{c_1}{c_3}} \int_1^x \left(1 + \frac{c_3}{c_2} y\right)^{-\frac{c_1}{c_3}} H(y)dy.$$

Taking the derivative yields that $g'(x) \to \infty$ as $x \to \infty$.

Now assume that A3c and A6 also hold. We will show that (2.3) can be satisfied. Let $\varepsilon < \alpha$ be positive. We will show that for $0 < \varepsilon < \alpha$ we can choose positive c_1, c_2 and c_3 with $c_3 < c_1$ so that

$$\mu(x) < \frac{r-\varepsilon}{c_1}(c_2 + c_3 x), \tag{6.9}$$

and

$$(c_1 - c_3)\sigma^2(x) < \frac{2\varepsilon}{c_1}(c_2 + c_3 x)^2. \tag{6.10}$$

Together, (6.9) and (6.10) prove the claim. To prove (6.9), by A3c there is a constant a so that $\mu(x) < a + (r-\alpha)x$ for all $x \geq 0$. Therefore,

$$\begin{aligned}\frac{r-\varepsilon}{c_1}(c_2 + c_3 x) - \mu(x) &> \frac{r-\varepsilon}{c_1}(c_2 + c_3 x) - (a + (r-\alpha)x) \\ &= \left((r-\varepsilon)\frac{c_2}{c_1} - a\right) + \left((r-\varepsilon)\frac{c_3}{c_1} - (r-\alpha)\right)x.\end{aligned}$$

This is positive for c_2 sufficiently large and c_3 so close to c_1 that $\frac{c_3}{c_1} \geq \frac{r-\alpha}{r-\varepsilon}$. The condition (6.10) is equivalent to

$$\frac{\sigma^2(x)}{2(c_2 + c_3 x)^2} < \frac{\varepsilon}{c_1(c_1 - c_3)}.$$

Using the growth constriction A6 and choosing c_3 sufficiently close to c_1, this can be satisfied and so the theorem is proved.

Proof of Theorem 2.5 By (6.5),

$$g''(0) = -\frac{2}{\sigma^2(0)}\mu(0),$$

hence $\mu(0) \leq 0$ is equivalent to $x^* = 0$. Assume that $\mu(0) = 0$ so that $g''(0) = 0$ as well. Taking the derivative in $Lg(x) = 0$ gives with $\tau(x) = \sigma^2(x)$,

$$\begin{aligned}g'''(x) = \frac{2}{\tau(x)}\Big(&-(\mu(x) + \frac{1}{2}\tau'(x))g''(x) \\ &+(r + \lambda - \mu'(x))g'(x) - \lambda A_g'(x)\Big),\end{aligned} \tag{6.11}$$

since by (2.6), A_g is continuously differentiable. By (2.6), $A_g'(0) = 0$, so therefore

$$g'''(0) = \frac{2}{\sigma^2(0)}(r + \lambda - \mu'(0)) > 0.$$

To prove part b, assume that $x^* = \infty$, meaning that g is strictly concave. Therefore we must have that for $x > 0$, $g(x) > xg'(x)$. This gives

$$rg(x_0) - \mu(x_0)g'(x_0) > \left(r - \frac{\mu(x_0)}{x_0}\right)x_0 g'(x_0) = 0,$$

so by (6.8), $g''(x_0) > 0$, a contradiction. Hence $x^* < \infty$.

Proof of Lemma 2.7 From (6.11) we have

$$g'''(x) = \frac{2}{\tau(x)}\left(-(\mu(x) + \frac{1}{2}\tau'(x))g''(x) + H(x)\right), \tag{6.12}$$

where

$$H(x) = (r + \lambda - \mu'(x))g'(x) - \lambda A_g'(x).$$

Note that (2.9) just says that $H(x) = 0$.

We start by proving that $g''(x) > 0$ for $x \in (x^*, x^* + \delta)$ for some positive δ. Assume the contrary. Then by definition of x^*, $g'''(x^*) = 0$ as well, and so by (6.12), $H(x^*) = 0$. Straightforward differentiation gives

$$g^{(4)}(x) = \frac{2}{\tau(x)}\Big(-(\mu(x) + \tau'(x))g'''(x) + \big(r + \lambda - 2\mu'(x) - \frac{1}{2}\tau''(x)\big)g''(x)$$
$$-\mu''(x)g'(x) - \lambda A_g''(x)\Big).$$

Therefore,

$$g^{(4)}(x^*) = -\frac{2}{\tau(x^*)}(\mu''(x^*)g'(x^*) + \lambda A_g''(x^*)) > 0$$

by assumption. But since $g''(x^*) = g'''(x^*) = 0$, we get

$$g''(x^* + u) = \int_{x^*}^{x^*+u}\int_{x^*}^{y} g^{(4)}(x)dxdy,$$

and the result follows.

We will now show that either $H(x^*) > 0$ or $H(x^*) = 0$ and $H'(x^*) > 0$. If $x^* = 0$ then $H(x^*) = r + \lambda - \mu'(0) > 0$ by assumption. Assume $x^* > 0$. Again by definition of x^*, $g'''(x^*) \geq 0$, and so by (6.12) $H(x^*) \geq 0$. Assume $H(x^*) = 0$. Since

$$H'(x) = (r + \lambda - \mu'(x))g''(x) - (\mu''(x)g'(x) + \lambda A_g''(x)) \tag{6.13}$$

and $g''(x^*) = 0$, it follows from the assumption that $H'(x^*) > 0$.

From the above results we can define

$$x_1 = \min\{x > x^* : g''(x) = 0\},$$
$$x_H = \min\{x > x^* : H(x) = 0\}.$$

Then it follows from the above:

1. $g''(x) > 0$ on (x^*, x_1).
2. $g'''(x_1) \leq 0$.
3. $H(x) > 0$ on (x^*, x_H).
4. $H'(x_H) \leq 0$.

We will prove that $x_1 = \infty$. Consider the three possibilities:

(i) $x_H < x_1$. Then $g''(x_H) \geq 0$ by item 1 and since $H(x_H) = 0$, it follows by assumption that $\mu''(x_H)g'(x_H) + \lambda A''_g(x_H) < 0$. Therefore by (6.13), $H'(x_H) > 0$, which contradicts item 4 above.

(ii) $x_H = x_1 < \infty$. Here we can use the same arguments to arrive at a contradiction.

(iii) $x_H > x_1$. But then by item 3, $H(x_1) > 0$ and so by (6.12), $g'''(x_1) > 0$ as well, which contradicts item 2.

From this it follows that $x_H = x_1 = \infty$ is the only possibility. But the inequality (2.10) just says that $x_H = \infty$, and so that this inequality is proved as well.

Proof of Theorem 2.8 We use the same notation as in the proof of Lemma 2.7. It follows easily from the assumptions that

$$\begin{aligned} A'_g(x) &= f(0)g(x) + \int_0^x g(x-z)f'(z)dz, \\ A''_g(x) &= f(0)g'(x) + \int_0^x g'(x-z)f'(z)dz. \end{aligned}$$

Assume that for some $x_0 > 0$, $\lambda A'_g(x_0) = (\lambda + r - \mu'(x_0))g'(x_0)$. Then

$$\begin{aligned} \lambda A''_g(x_0) + \mu''(x_0)g'(x_0) &\leq \lambda A''_g(x_0) \\ &= \lambda \left(f(0)g'(x_0) + \int_0^{x_0} g'(x_0 - z)f'(z)dz \right) \\ &< \lambda f(0) \left(g'(x_0) - \int_0^{x_0} g'(x_0 - z)c(z)f(z)dz \right) \\ &\leq \lambda f(0) \left(g'(x_0) - c(x_0)A'_g(x_0) \right) = 0. \end{aligned}$$

The result now follows from Lemma 2.7.

Proof of Theorem 3.1 For (a), assume that $(V_i, \bar{u}_i^*, \underline{u}_i^*)$, $i = 1, 2$ are two solutions of B1+B2, and assume without loss of generality that $\bar{u}_1^* < \bar{u}_2^*$. Since $\underline{u}_i^* < x^*$, $i = 1, 2$, and $g'(\underline{u}_i^*) = g'(\bar{u}_i^*)$, it is necessary that $\underline{u}_1^* > \underline{u}_2^*$. But from

$$k(\bar{u}_1^* - \underline{u}_1^*) - (V_1(\bar{u}_1^*) - V_1(\underline{u}_1^*)) = k(\bar{u}_2^* - \underline{u}_2^*) - (V_2(\bar{u}_2^*) - V_2(\underline{u}_2^*))$$

and (3.2) and (3.3), we get

$$\int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_1^*)} \right) dx = \int_{\underline{u}_2^*}^{\bar{u}_2^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)} \right) dx.$$

However, $g'(\bar{u}_1^*) < g'(\bar{u}_2^*)$ and $g'(x) < g'(\bar{u}_i^*)$, $\underline{u}_i^* < x < \bar{u}_i^*$ and so

$$\int_{\underline{u}_2^*}^{\bar{u}_2^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)} \right) dx > \int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)} \right) dx > \int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_1^*)} \right) dx,$$

a contradiction. The proof that B1+B2 and B1+B3 cannot both have a solution is similar, as is the proof that B1+B3 cannot have two different solutions.

For (b), the assumption $\lim_{x\to\infty} g'(x) = \infty$ implies that for each $\underline{u} \in [0, x^*)$ there is a unique $\bar{u} = \bar{u}(\underline{u}) \in (x^*, \infty)$ so that $g'(\bar{u}) = g'(\underline{u})$. By smoothness of g' and strict concave-convexity, this $\bar{u}(\underline{u})$ is continuous in $\underline{u}$. Therefore, if

$$\int_0^{\bar{u}(0)} \left(1 - \frac{g'(x)}{g'(\bar{u}(0))}\right) dx \geq \frac{K}{k}, \tag{6.14}$$

there is a unique pair $(\underline{u}^*, \bar{u}^*)$ so that $g'(\underline{u}^*) = g'(\bar{u}^*)$ and

$$\int_{\underline{u}^*}^{\bar{u}^*} \left(1 - \frac{g'(x)}{g'(\bar{u}^*)}\right) dx = \frac{K}{k}.$$

Then,

$$V(x) = \begin{cases} \dfrac{k}{g'(\bar{u}^*)} g(x), & x \leq \bar{u}^*, \\ V(\bar{u}^*) + k(x - \bar{u}^*), & x > \bar{u}^*, \end{cases}$$

satisfies B1+B2.

If (6.14) does not hold we can find a unique $\bar{u}^*$ so that

$$\int_0^{\bar{u}^*} \left(1 - \frac{g'(x)}{g'(\bar{u}^*)}\right) dx = \frac{K}{k}.$$

Then $V(x)$ defined as above satisfies B1+B3.

Proof of Theorem 3.2 This is proved almost exactly as Theorem 2.1 in [21], and we drop the details.

Proof of Theorem 3.4 By Theorems 2.3, 2.4, 2.5, 2.8 and 3.2 it only remains to verify (3.5), and for this it is sufficient to prove (3.6). Let $h(x) = LV(x)$. Then by (3.6),

$$\begin{aligned} h'(\bar{u}^*) &= \lambda A_V'(\bar{u}^*) - k(r + \lambda - \mu'(\bar{u}^*)) \\ &= k\left(\frac{\lambda}{g'(\bar{u}^*)} A_g'(\bar{u}^*) - k(r + \lambda - \mu'(\bar{u}^*))\right) < 0 \end{aligned}$$

by (2.10). Let $x_0 = \inf\{x > \bar{u}^* : h'(x) > 0\}$. If we can prove that $x_0 = \infty$ we are done. So assume that $x_0 < \infty$ which implies that $\lambda A_V'(x_0) = k(r+\lambda-\mu'(x_0))$. Also by definition of x_0, $h''(x_0) \geq 0$, but a direct calculation gives as in the proof of Theorem 2.8,

$$\begin{aligned} h''(x_0) &= \lambda f(0) V'(x_0) + \lambda \int_0^{x_0} V'(x_0 - z) f'(z) dz \\ &< \lambda k f(0) - \frac{\lambda^2}{r + \lambda - \mu'(x_0)} f(0) \int_0^{x_0} V'(x_0 - z) f(z) dz \\ &= \lambda f(0) \left(k - \frac{\lambda}{r + \lambda - \mu'(x_0)} A_V'(x_0)\right) = 0, \end{aligned}$$

a contradiction.

Proof of Theorem 5.1 By definition of x_M, $g''(x_M) \geq 0$. If $g''(x_M) > 0$ there is a $\delta > 0$ so that

$$g''(x) > 0 \quad \text{on} \quad (x_M, x_M + \delta). \tag{6.15}$$

Assume that $g''(x_M) = 0$. By definition of x_M, $g'(x) \leq g'(x_M)$ for $x \in (0, x_M)$, so that by (2.6), $g'(x_M) \geq A'_g(x_M)$. Therefore, by (6.11),

$$\begin{aligned} g'''(x_M) &= \frac{2}{\sigma^2(x_M)} \big(r + \lambda - \mu'(x_M))g'(x_M) - \lambda A'_g(x_M)\big) \\ &\geq \frac{2\alpha}{\sigma^2(x_M)} g'(x_M) > 0, \end{aligned}$$

and so (6.15) holds in this case as well. Let $x_0 = \inf\{x > x_M : g''(x) = 0\}$. Then if $x_0 < \infty$, $g'''(x_0) \leq 0$. But since $g'(x)$ is increasing on (x_M, x_0), the same calculations as above yield that $g'''(x_0) > 0$, a contradiction. This ends the proof of (a). To prove (b), let $x_0 = \inf\{x > x^* : g''(x) = 0\}$. By assumption, $x_0 > x_L$. Assume that $x_0 < \infty$. Then $g'''(x_0) \leq 0$. Also by assumption, $\max_{z \in [x^*, x_0]} g'(z) = g'(x_0)$, and hence a calculation using (6.11) yields

$$\begin{aligned} g'''(x_0) &= \frac{2}{\sigma^2(x_0)} \big((r + \lambda - \mu'(x_0))g'(x_0) - \lambda A'_g(x_0)\big) \\ &= \frac{2}{\sigma^2(x_0)} \Bigg((r - \mu'(x_0))g'(x_0) - \lambda \int_0^{x^*} g'(z) f(x_0 - z) dz \\ &\quad + \lambda \left(g'(x_0) - \int_{x^*}^{x_0} g'(z) f(x_0 - z) dz\right)\Bigg) \\ &> \frac{2}{\sigma^2(x_0)} \left((r - \mu'(x_0))g'(x_0) - \lambda g(x^*) \max_{z \geq x_0 - x^*} f(z)\right) \\ &\geq \frac{2}{\sigma^2(x_0)} \left((r - \mu'(x_L))g'(x_L) - \lambda g(x^*) \max_{z \geq x_L - x^*} f(z)\right) = 0, \end{aligned}$$

a contradiction. Hence $x_0 = \infty$. From this, using that $Lg(x) = 0$, we also get

$$(r + \lambda - \mu'(x))g'(x) - \lambda A'_g(x) > 0, \quad x \geq x_L. \tag{6.16}$$

Proof of Theorem 5.3 Assume that $x_K < \infty$ and let $h(x) = LV(x)$. Simple calculations using (3.6) and (2.6) yield for $x > \bar{u}^* + x_f$.

$$h'(x) = \lambda \int_0^{\bar{u}^*} V'(z) f(x - z) dz - k\lambda \bar{F}(x - \bar{u}^*) - k(r - \mu'(x)) \tag{6.17}$$

$$\leq \lambda \int_0^{\bar{u}^*} V'(z) f(x - z) dz - k(r - \mu'(x)). \tag{6.18}$$

By definition of x_K, the right side is zero at x_K, and since $f(x-z)$ and $\mu'(x)$ are all decreasing in x when $x > \bar{u}^* + x_f$, the result follows. That $x_K < \infty$ if A3c holds is trivial. Part (b) is proved similarly, since the above gives for $x > \bar{u}^*$,

$$\begin{aligned} h'(x) &= \lambda \int_0^{\bar{u}^*} V'(z) f(x-z) dz - k\lambda \bar{F}(x-\bar{u}^*) - k(r - \mu'(x)) \\ &\le \lambda \int_0^{\bar{u}^*} V'(z) f(x-z) dz - k(r - \mu'(x)) \\ &\le \lambda V(\bar{u}^*) \max_{x-\bar{u}^* \le z \le x} f(z) - k(r-\mu'(x)) \\ &\le \frac{k}{g'(\bar{u}^*)} \left(\lambda g(\bar{u}^*) \max_{z \ge x - \bar{u}^*} f(z) - k(r - \mu'(x)) g'(\bar{u}^*) \right). \end{aligned}$$

By definition, the right side is zero at x_J and is decreasing in x.

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5 Paper C

A numerical approach to ruin probability in finite time for fitted models with investment

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Abstract

In this paper we present a numerical method for solving a partial integro-differential equation (PIDE) associated with ruin probability, when the surplus is continuously invested in stochastic assets. The method uses precalculated Gaussian quadrature rules for the numerical integration. Except for the numerical integration part, the method is based largely on the finite differences method used in Halluin et al. (2005) for a PIDE associated with a more general option pricing problem. In our numerical examples we use historical data for inflation and returns on U.S. Treasury bills, U.S. Treasury bonds and American stocks. The log-returns of the investments are adjusted for an assumed constant force of inflation. We consider four different strategies for continuous investment: (a) U.S. Treasury bills with a constant maturity of 3 months, (b) U.S. Treasury bonds with a constant maturity of 10 years, and (c) the Standard and Poor 500 index and (d) another index of American stocks. For each of these strategies a geometric Brownian motion process is fitted to the aforementioned historical data. The results suggest that the ruin probabilities obtained can vary substantially, depending on whether the models are fitted to data for the last decade or for a longer time period. We also discuss numerical solution of investment models with jumps.

1 Introduction

In the classical Cramér-Lundberg model the risk process of an insurance company at time t is assumed to be of the form

$$Y_t = y + pt - \sum_{n=1}^{N_t} S_n.$$

Here $y > 0$ is the initial capital, pt is the accumulated premium income up to time t, coming at a constant rate p. The sum $\sum_{n=1}^{N_t} S_n$ is a compound Poisson process with only non-negative jumps and whose counting process N has a constant intensity λ. In the following we will follow the convention that $\sum_{n=1}^{0} = 0$ and that $\Pi_{n=1}^{0} = 1$.

In Paulsen and Gjessing (1997) the classical model is generalized to possibly include a scaled Brownian motion $\sigma_P W_P$, where $\sigma_P \geq 0$. In addition it is assumed that the surplus generated from the basic process

$$P_t = pt + \sigma_P W_{P,t} - \sum_{n=1}^{N_t} S_n, \qquad t \geq 0, \tag{1}$$

is continuously invested in risky assets that follow a jump-diffusion process

$$R_t = rt + \sigma_R W_{R,t} - \sum_{n=1}^{N_{R,t}} S_{R,n}, \qquad t \geq 0.$$

In the above $\sigma_R \geq 0$, $r \in \mathbb{R}$, W_R is a Brownian motion, and the sum $\sum_{n=1}^{N_{R,t}} S_{R,n}$ is a compound Poisson process whose counting process $N_{R,t}$ has a constant intensity λ_R and a common jump size distribution F_R. With these assumptions and y as the initial capital, the risk process becomes

$$Y_t = y + P_t + \int_0^t Y_{s-} dR_s, \qquad t \geq 0. \tag{2}$$

It is shown in Paulsen (1998) that the solution of this equation is

$$Y_t = \bar{R}_t \left(y + \int_0^t \bar{R}_s^{-1} dP_s \right), \tag{3}$$

where $\bar{R}_t = \exp\left\{ \left(r - \frac{1}{2}\sigma_R^2 \right) t + \sigma_R W_{R,t} \right\} \Pi_{n=1}^{N_{R,t}} (1 + S_{R,n})$.

In Paulsen (1993) a third process I, representing inflation, is included in the model. In this model inflation is assumed to have the same effect on both the premium income and the insurance claim sizes. It is shown in Paulsen (1993) that if inflation is a deterministic process then the effect on the risk process is the same as if we substituted R with $R - I$. We will assume that there is such an inflation process, with a constant force $\bar{\imath}$, i.e. at time t

$$I_t = \bar{\imath} t.$$

Let the R process be an *inflation-adjusted* return on investment process. This corresponds to replacing the parameter r with $\bar{r} = r - \bar{\imath}$. In this context inflation refers to geometric growth of both insurance claim sizes and premium rates. In the numerical examples we let $\bar{\imath}$ be the geometrical mean of the inflation for the corresponding time periods. The data for annualized inflation are taken from inflationdata.com (2012).

For a risk process like the one defined above, the time of ruin is defined as $\tau := \inf \{ t : Y_t < 0 \}$ and the probability of ruin in finite time is defined as

$$\psi(y, t) := P(\tau \leq t | Y_0 = y). \tag{4}$$

In this paper we will discuss a method for numerical computation of ruin probability in finite time for these models, based on solving an associated partial integro-differential equation (PIDE) using finite differences. In our numerical examples in Section 4 we consider two different claim size distributions. In the first example the claims follow a light-tailed standard exponential distribution, while in the second they follow a mixture of a standard exponential distribution and a heavy-tailed standardized Pareto distribution with expectation 1. For the standardized Pareto distribution part of the mixed distribution we choose a parameter value based on the fitting discussed in chapter 6 in Embrechts et al. (1997) of a Pareto distribution to data for Danish fire insurance claims.

We consider four different strategies for continuous investment: (a) U.S. Treasury bills with a constant maturity of 3 months, (b) U.S. Treasury bonds with a constant maturity of 10 years, (c) the Standard and Poor 500 index and (d) another index of American stocks. We fit a geometric Brownian motion (GBM) to data for annual return of bonds and stocks for the period 1928-2011, taken from Damodaran (2012). In one example we use data for the entire time period. In another example we only use data for 2000-2011. We also calculate ruin probabilities based on data fittings of GBM models, and some jump-diffusion models in Damodaran (2012), to the SP 500 index for the period 1962-2003.

2 Integro-differential equations for the ruin probability

In Paulsen (2008) a partial integro-differential equation (PIDE) is stated for the survival probability $\phi(y,t) = 1 - \psi(y,t)$. First let L be the integro-differential operator

$$\begin{aligned} Lh(y) = {} & \frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2)h''(y) + (p + \bar{r}y)h'(y) \\ & + \lambda \int_0^y (h(y-x) - h(x))\, dF(x) \\ & + \lambda_R \int_{-1}^{\infty} (h(y(1+x)) - h(y))\, dF_R(x), \end{aligned} \tag{5}$$

where L is acting on the variable y, y and t are assumed non-negative, $\bar{r} \in \mathbb{R}$ and $\sigma_P, \sigma_R, p, \lambda$ and λ_R are assumed to be nonnegative. Then the PIDE is given as

$$\frac{\partial}{\partial t}\phi(y,t) = L\phi(y,t). \tag{6}$$

The initial condition is $\phi(y,0) = 1$ for every $y > 0$. Asymptotically the solution must satisfy the condition $\lim_{y\to\infty} \phi(y,0) = 1$. When $\sigma_P > 0$ the infinite variation of the Brownian motion W_P implies that

$$\inf\{t : Y_t < 0\} = \inf\{t : Y_t = 0\}.$$

Hence in this case the survival probability must satisfy $\phi(0,t) = 0$.

2.1 Regularity of solution

Consider the case when $\lambda_R = 0, \sigma_P > 0$, and either $\sigma_R = \bar{r} = 0$ or $\sigma_R > 0$. If an additional weak condition on the probability measure F also holds it is shown in Paper D that the integro-differential equation (6) has a classical solution except at the origin. That is, a solution which is differentiable with respect to

t, twice differentiable with respect to y on the inner domain, and continuous at every point of the boundary except for the origin. It is also known that a classical solution exists when the investment earns an interest with a constant force, i.e., if $\sigma_P = \sigma_R = \lambda_R = 0$ (see Pervozvansky Jr. (1998); Paulsen (2008)). To the author's knowledge there are no known regularity results for other cases, when $\sigma_P^2 + \sigma_R^2 > 0$. However, the behavior of the numerical solution in our experiments suggests that letting $\sigma_P = 0$ or letting $\lambda_R > 0$ (adding the last integral term in (5)) does not negatively affect the smoothness of the solution, at least as long as the distribution functions $F(x)$ and $F_R(x)$ are smooth.

2.2 Localization to a bounded domain and choice of coordinates

The domain of equation (6) is unbounded in the space dimension, which of course is not computationally feasible. Instead we introduce an *artificial* boundary condition (see Section 12.4.1 in Cont and Tankov (2004)), namely that $\phi(y,t) = 1$ for every $y \geq \kappa$. The introduction of an artificial boundary condition leads to an error generally referred to as a *localization error*. Let ϵ_κ be this localization error and let $\bar{\ }(x) = 1 - F(x)$ be the tail distribution. In Paper D it is shown that if $\sigma_P, \sigma_R > 0$, $\lambda_R = 0$, and for some $c > 0$

$$\sup_{x>0} x^c \bar{F}(x) < \infty,$$

then for some constant C

$$|\epsilon_\kappa| < C(1+\kappa)^{-c}$$

for any $\kappa > 0$.

In our numerical experiments we found it more numerically efficient (leading to better accuracy) to make the change of variable $z = \ln(1+y)$. In the following we rewrite the above integro-differential operator L in terms of the new variable z. We also denote the finite time horizon by T. Since $y = e^z - 1$, first let

$$\rho(z,t) := \phi(e^z - 1, t), \quad (z,t) \in [0, \ln(1+\kappa)] \times [0,T].$$

For $z \in [0, \ln(1+\kappa)]$ let

$$\begin{aligned} a_2(z) &:= \frac{1}{2}\left(\sigma_P^2 e^{-2z} + \sigma_R^2\left(1-e^{-z}\right)^2\right), \quad \text{and} \\ a_1(z) &:= pe^{-z} + \bar{r}\left(1-e^{-z}\right) - a_2(z). \end{aligned} \tag{7}$$

Now the operator L becomes

$$\begin{aligned} L_z g(z) = a_2(z)g''(z) &+ a_1(z)g'(z) \\ &+ \lambda \int_0^{e^z-1} \left(g\left(\ln\left(e^z - x\right)\right) - g(z)\right) dF(x) \\ &+ \lambda_R \int_{-1}^{\infty} \left(g\left(\ln\left(1 + \left(e^z-1\right)(1+x)\right)\right) - g(z)\right) dF_R(x). \end{aligned}$$

Making this change of variables and including the artificial boundary condition gives the equation

$$\begin{cases} \rho(z,0) = 1, \quad z \in (0, \ln(1+\kappa)). \\ \rho(\ln(1+\kappa), t) = 1, \quad t \in (0,T]. \\ \frac{\partial \rho(z,t)}{\partial t} = L_z \rho(z,t) \qquad \text{on } (z,t) \in (0, \ln(1+\kappa)) \times (0,T]. \end{cases} \tag{8}$$

Here L_z is acting on the variable z. When $\sigma_P > 0$ we have the extra boundary condition

$$\rho(0,t) = 0, \quad t \in (0,T].$$

In the following we will also define that $\rho(z,t) = 1$ for every $z \geq \ln(1+\kappa)$ and $t \in [0,T]$.

The rest of this paper is a discussion of numerical finite-difference methods for solving (8), with some numerical examples for fitted models with investment in U.S. Treasury bills, U.S. Treasury bonds and American stocks. In all our examples the space grid will be equally spaced on $[0, \ln(1+\kappa)]$. An advantage with this grid, compared with an equally spaced grid in the original coordinate system, is that it gives a more numerically efficient distribution of grid points. This is especially true for the case when $\sigma_P > 0$, since in this case the solution is discontinuous at the origin. Having many grid points near the bottom of the domain seems to give higher accuracy.

3 Numerical algorithm

The finite difference schemes discussed in this paper are adaptations of the schemes developed in Halluin et al. (2005) to fit the problem (8). The basic idea is to solve (8) using Crank-Nicolson time integration on an equally spaced two-dimensional grid. To ensure numerical stability we follow the recommendation in Giles and Carter (2005) and replace the first Crank-Nicolson step with four quarter-timesteps of Backward Euler time integration. After explaining how we do the numerical integration we discuss the difference equations associated with these finite difference schemes.

3.1 Evaluation of the integrals

In the following we assume that both the claim size distribution and the distribution of the jumps of the R-process are smooth. We denote their respective densities as f and f_R. In what follows let m be the grid size and $h = \frac{\ln(1+\kappa)}{m}$ be the step size in the z grid. Thus the nodes in the z grid are $z_i = ih$ for $i \in 0, 1, \ldots m$. Let the nodes in the time grid be $t_0 = 0, t_1, \ldots, t_n$. Since $y = e^x - 1$ let $y_i = e^{z_i} - 1$, for $i \in 0, 1, \ldots, m-1$. Let

$$\begin{aligned} \rho_i^k &= \rho(ih, t_k), \qquad i \in 0, 1, \ldots, m, k \in 0, 1, \ldots, n, \\ I_i^k &= \int_0^{y_i} \rho(\ln(1 + y_i - x), t_k)\, dF(x), \qquad i \in 1, 2, \ldots, m-1, \\ J_i^k &= \int_{-1}^{\infty} \rho(\ln(1 + y_i(1+x)), t_k)\, dF_R(x), \qquad i \in 1, 2, \ldots, m-1, \end{aligned} \tag{9}$$

and

$$\tilde{I}_i^k = \int_{y_{i-1}}^{y_i} \rho\left(\ln\left(1+x\right), t_k\right) f_i(x) dx, \qquad i \in 1, 2, \ldots, m-1.$$

where $f_i(x) = f\left(y_i - x\right)$.

The sequence $\left\{I_i^k\right\}$ defined above is a semi-discretization of the insurance claim integrals

$$I\left(y,t\right) = \int_0^y \left(\rho\left(\ln\left(1+y-x\right), t\right)\right) dF(x) \qquad \text{on } (y,t) \in (0,\kappa) \times (0,T].$$

Similarly, when $\lambda_R > 0$, the sequence $\left\{J_i^k\right\}$ is a discretization of the investment integrals

$$J\left(y,t\right) = \int_{-1}^{\infty} \rho\left(\ln\left(1+y\left(1+x\right)\right), t\right) dF_R(x).$$

In Section 4.5 we discuss some examples with jumps in the investment process. In these examples the J_i^k are calculated as

$$\int_{-\infty}^{\infty} \rho\left(\ln\left(1+y_i e^x\right), t_k\right) f_{\tilde{R}}(x) dx, \tag{10}$$

where

$$f_{\tilde{R}}(x) = e^x f_R\left(e^x - 1\right).$$

As we will see in Section 3.4, for each time step each integral in the sequence $\left\{I_i^k\right\}_{i=1}^{m-1}$ must be computed more than once for every time step, as part of an iteration method. When $\lambda_R > 0$ this also has to be done for each integral in the sequence $\left\{J_i^k\right\}_{i=1}^{m-1}$. Moreover, the integrands in the sequences $\left\{I_i^k\right\}$ and $\left\{J_i^k\right\}$ depend on i. This means that the numerical complexity for numerical integration based on such Newton-Coates quadrature methods as Simpson's rule would be $O(m^2)$ for just one calculation of $\left\{I_i^k\right\}_{i=1}^{m-1}$. Fortunately there are ways of avoiding this, as discussed below.

A popular model is to let the jump sizes be exponential distributed. Below we first show how for this model it is relatively simple to compute the integrals efficiently. We then return to general claim size distributions in 3.1.2.

3.1.1 Exponentially distributed jumps

For $\alpha > 0$ let

$$f_i(x) = \alpha e^{-\alpha(y_i - x)}, \qquad i \in 1, 2, \ldots, m-1.$$

In the special case of exponentially distributed claim sizes with parameter α we observe that

$$\int_0^y \rho\left(\ln\left(1+y-x\right), t\right) f\left(x\right) dx = e^{-\alpha y} \int_0^y \rho\left(\ln\left(1+x\right), t\right) \alpha e^{\alpha x} dx.$$

Thus in this case the insurance claim integrals are dependent on y only through the upper limit and a factor that can be taken outside the integral. Moreover, we have the recursive relation

$$I_{i+1}^k = \exp\left(-\alpha\left(y_{i+1} - y_i\right)\right) I_i^k + \tilde{I}_{i+1}^k. \tag{11}$$

Here we are indebted to the discussion in Toivanen (2008). Due to (11) fast evaluation of the sequence $I_1^k, \ldots, I_{m-1}^k$ is much simpler when the claims are exponentially distributed than in the general case.

As in Toivanen (2008) we approximate the integrand $\rho\left(\frac{x}{1+x}, t_k\right)$ in $\tilde{I}_i^k$ by linear interpolation. This gives the approximation

$$\tilde{I}_i^k \approx \tilde{a}_i^k \int_{y_{i-1}}^{y_i} f_i(x)dx + \tilde{b}_i^k \int_{y_{i-1}}^{y_i} x f_i(x)dx, \tag{12}$$

where

$$\tilde{b}_i^k = \frac{\rho_i^k - \rho_{i-1}^k}{y_i - y_{i-1}}$$

and

$$\tilde{a}_i^k = \rho_i^k - \tilde{b}_i^k y_i.$$

Lastly, we have that

$$\int_{y_{i-1}}^{y_i} f_i(x)dx = 1 - \exp\left(-\alpha\left(y_i - y_{i-1}\right)\right)$$

and that

$$\begin{aligned}\int_{y_{i-1}}^{y_i} x f_i(x)dx = &\left(y_i - \frac{1}{\alpha}\right)\left(1 - \exp\left(-\alpha\left(y_i - y_{i-1}\right)\right)\right) \\ &+ \left(y_i - y_{i-1}\right)\exp\left(-\alpha\left(y_i - y_{i-1}\right)\right).\end{aligned}$$

If the return on investment process R is like that in the Kou model (see Kou (2002)), the jumps of the log-returns follow an asymmetric exponential distribution. That is, for some parameters $\eta_1, \eta_2 > 0$ and a weight $q \in [0,1]$, the probability density $f_{\tilde{R}}(x)$ of the jumps of the log-returns is

$$f_{\tilde{R}}(x) = q 1_{x>0} \eta_1 \exp\left(-\eta_1 x\right) + \left(1-q\right) 1_{x<0} \eta_2 \exp\left(-\eta_2 \left|x\right|\right).$$

In our context this corresponds to letting the investment jump integral in (8) be of the form

$$\int_{-1}^{\infty} \rho\left(\ln\left(1 + y\left(1+x\right)\right), t\right) dF_R(x) = qJ_1 + \left(1-q\right) J_2,$$

where

$$J_1 = \int_0^{\infty} \rho\left(\ln\left(1 + ye^v\right), t\right) \eta_1 \exp\left(-\eta_1 v\right) dv$$

and

$$J_2 = \int_{-\infty}^{0} \rho\left(\ln\left(1 + ye^v\right), t\right) \eta_2 \exp\left(\eta_2 v\right) dv.$$

Making the substitution $w = v + \ln\left(y\right)$ gives

$$J_1 = y^{\eta_1} \int_{\ln y}^{\infty} \rho\left(\ln\left(1 + e^w\right), t\right) \eta_1 \exp\left(-\eta_1 w\right) dw,$$

and

$$J_2 = y^{-\eta_2} \int_{-\infty}^{\ln y} \rho\left(\ln\left(1+e^{w}\right), t\right) \eta_2 \exp\left(\eta_2 w\right) dw.$$

From these formulas one can derive a recursive relation given in Toivanen (2008) and similar to (11). In this model the investment integrals can be evaluated in a way similar to the method described above for the insurance claim integrals.

3.1.2 Computation of Gaussian quadrature rules

Returning to general smooth claim size distributions, we can evaluate the integrals in (9) using Gaussian quadrature methods. The main idea of an l-point Gaussian quadrature rule is to find abscissas $x_1, \ldots, x_l$ and corresponding weights $w_1, \ldots, w_l$ such that, for a known function $\omega(x) : [-1, 1] \to \mathbb{R}$, and given function values of a continuous function $g : [-1, 1] \to \mathbb{R}$,

$$\int_{-1}^{1} g(x)\omega(x)dx \approx \sum_{i=1}^{l} w_i g(x_i). \tag{13}$$

In our numerical method these rules are calculated using the subroutines 'dlancz' and 'dgauss' from the Netlib package 726 'ORTHPOL', developed by Walter Gautschi. The package is an implementation of a Golub-Welsch algorithm. For the integral (13) a Golub-Welsch algorithm (see Golub and Welsch (1969)) involves finding the roots of a sequence of polynomials $p_0(x), \ldots, p_l(x)$. The polynomials in this sequence are required to be orthogonal in the following inner product space, defined by

$$\langle q_1, q_2 \rangle = \int_{-1}^{1} q_1(x) q_2(x) \omega(x) dx,$$

where q_1, q_2 are continuous functions.

Following this procedure it can be shown that the resulting Gaussian quadrature rule is exact for polynomials of degree at most $2l - 1$ (see Theorem 4.7.7 in Cheney (2001)). In order to apply a quadrature rule it is necessary to evaluate the solution at points that are not on the z-grid. We do this by means of linear interpolation.

In our numerical method, $m - 1$ Gaussian quadrature rules are precalculated for each $I_1^k, \ldots, I_{m-1}^k$ before the actual finite differences method begins. The obvious choice of weighting function for these rules is the density $f(x)$. While the weighting function is the same for every I_i^k, the integrals have upper limits that increase with i. This makes it necessary to calculate a separate Gaussian quadrature rule for each I_i^k. However, since the weighting function is the same, we found that the rules were more rapidly and more accurately calculated when a rule calculated for I_k^k is used in the calculation of a rule for the next integral I_{k+1}^k. We also found that when the claim size distribution has a heavy tail it has a positive effect on the accuracy to make the substitution $v = \ln(1 + x)$, and calculate I_i^k as

$$\int_{0}^{\ln(1+y_i)} \rho\left(\ln\left(1 + y_u - e^{v}\right), t\right) e^{v} f\left(e^{v} - 1\right) dv.$$

As is normally the case in numerical problems there is a trade-off between the numerical complexity of the Golub-Welsch algorithm and the accuracy of the results. To control the accuracy of the weights and abscissas, our method first applies the routines 'dlancz' and 'dgauss' with a relatively low complexity. Then the subroutines are called again with increasing resolutions until the differences between succeeding weights and succeeding abscissas are small. In the numerical integration of the $\left\{J_i^k\right\}_{i=1}^{m-1}$ integrals, only one quadrature rule needs to be calculated with the Golub-Welsch algorithm. Denoting the weights of this quadrature rule by $w_{J,1}, \ldots, w_{J,m}$, and denoting the abscissa points by $x_{J,1}, \ldots, x_{J,m_J}$, these integrals are calculated as

$$J_i^k \approx \sum_{j=1}^{m_J} w_{J,j} \rho \left(\ln \left(1 + y_i e^{x_{J,j}}\right), t_k \right).$$

In the special case of the Merton model the required quadrature rule corresponds to Gauss-Hermite quadrature. Calculation of these rules is implemented in the subroutine 'gaussq', also in the 'ORTHPOL' package.

3.2 Backward Euler time integration

As mentioned above we follow a suggestion in Giles and Carter (2005), the numerical differentiation part of our method consists of computing the first four time steps with backward Euler time integration, where each time step is of length Δt. The subsequent time steps are of length $4\Delta t$ and are computed using Crank-Nicolson time integration.

Now let us look at the inner z-grid and time grid points. Here we discretize the time derivative with backward Euler finite differences. In the z variable we discretize both the first and second derivatives by means of central differences. This yields the following set of difference equations, where as before
$\rho_i^k = \rho(ih, t_k)$.

$$\begin{aligned} \frac{\rho_i^{k+1} - \rho_i^k}{\Delta t} = a_2(z_i) \frac{\rho_{i+1}^{k+1} - 2\rho_i^{k+1} + \rho_{i-1}^{k+1}}{h^2} &+ a_1(z_i) \left[\frac{\rho_{i+1}^{k+1} - \rho_{i-1}^{k+1}}{2h} \right] \\ &- \lambda \rho_i^{k+1} + \lambda \sum_{j=0}^{m} c_{i,j} \rho_j^{k+1} \\ &- \lambda_R \rho_i^{k+1} + \lambda_R \sum_{j=0}^{m} d_{i,j} \rho_j^{k+1}. \end{aligned}$$

In the above a_1 and a_2 are defined in (7). The sum $\sum_{j=0}^{m} c_{i,j} \rho_j^{k+1}$ is related to the evaluation of the integral I_i^k, while the sum $\sum_{j=0}^{m} d_{i,j} \rho_j^{k+1}$ is related to the evaluation of the integral J_i^k. Since the $c_{i,j}$'s and $d_{i,j}$'s are integral weights they are non-negative constants.

If we let

$$\begin{aligned} \hat{\lambda} &= \lambda + \lambda_R, \\ \hat{c}_{i,j} &= \frac{\lambda}{\lambda + \lambda_R} c_{i,j} + \frac{\lambda_R}{\lambda + \lambda_R} d_{i,j}, \end{aligned} \tag{14}$$

$$\alpha_i = \frac{a_2(z_i)}{h^2} - \frac{a_1(z_i)}{2h}, \tag{15}$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2} + \frac{a_1(z_i)}{2h}, \tag{16}$$

then the difference equation above can be rearranged as

$$\rho_i^{k+1}\left[1 + \left(\alpha_i + \beta_i + \hat{\lambda}\right)\Delta t\right] - \Delta t \beta_i \rho_{i+1}^{k+1} - \Delta t \alpha_i \rho_{i+1}^{k+1} - \hat{\lambda}\Delta t \sum_{j=0}^{m} \hat{c}_{i,j}\rho_j^{k+1} = \rho_i^k. \tag{17}$$

Now let us see what happens if we change definitions (15) and (16) a little. Let

$$\alpha_i = \frac{a_2(z_i)}{h^2}, \tag{18}$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2} + \frac{a_1(z_i)}{h}. \tag{19}$$

Then (17) corresponds to discretizing the first space derivative using forward differences.

Another alternative is to discretize the first space derivative using backward differences. This gives

$$\alpha_i = \frac{a_2(z_i)}{h^2} - \frac{a_1(z_i)}{h},$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2}.$$

Theorem 1. *Assume that, for every* $i \in 1, \ldots, m-1$, $\alpha_i \geq 0$, $\beta_i \geq 0$ *and*

$$\sum_{j=0}^{m} \hat{c}_{i,j} \leq 1.$$

Then the backward Euler scheme given in (17) *is unconditionally stable in the max norm. Moreover, for any given index* i*, at least one of the options for discretizing* $\frac{\rho(z,t)}{\partial z}$ *given above, i.e, central differences, forward differences and backward differences, gives* $\min(\alpha_i, \beta_i) \geq 0$.

Proof. This follows from Theorem 3.1 in Halluin et al. (2005). □

In the rest of the paper we will assume that $\alpha_i, \beta_i \geq 0$ for every $i \in 1, \ldots, m-1$. Since discretizing the first space derivative with central differences gives a second order convergence rate, whereas forward and backward differences give only first order convergence, we choose central differences for those nodes where this does not lead to negative α_i or β_i.

3.3 Crank-Nicolson time integration

While the fully implicit scheme given in (17) is unconditionally stable, it has the disadvantage of being only first order convergent in the time variable. An alternative, suggested in Giles and Carter (2005) and mentioned above, is to use backward Euler time integration only for the initial four quarter-steps, each with length Δt, and then continue with Crank-Nicolson time integration with time steps of length $\hat{\Delta} t = 4\Delta t$. This approach results in the following set of discrete equations for the Crank-Nicolson part:

$$\begin{aligned}\rho_i^{k+1}&\left[1+\left(\alpha_i+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}\right]-\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^{k+1}-\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^{k+1}\\ &=\rho_i^k\left[1-\left(\alpha_i+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}\right]+\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^{k}+\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^{k}\\ &\quad+\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{i}\hat{c}_{i,j}\rho_j^{k+1}+\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^{k}.\end{aligned}$$

Let

$$\rho^k := \left(\rho_0^k,\rho_1^k,\ldots,\rho_m^k\right)'$$

and define the matrix M such that

$$-\left[M\rho_i^k\right]_i=\rho_i^k\left(\alpha+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}-\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^k-\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^k-\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k. \tag{20}$$

Also let

$$B=[I-M]^{-1}[I+M].$$

Then (20) can be written either as

$$[I-M]\rho^{k+1}=[I+M]\rho^k, \tag{21}$$

or as

$$\rho^k=(B)^k\rho^0.$$

Theorem 2. *Assume that for every $i\in 1,\ldots,m-1$, $\beta_1\geq 0$, $\alpha_i\geq 0$ and*

$$\sum_{j=0}^{m}\hat{c}_{i,j}<1.$$

Then the Crank-Nicolson discretization (20) *is algebraically stable in the sense that there exists a C such that for every n and every grid size*

$$\|(B)^n\|_\infty\leq Cn^{\frac{1}{2}}. \tag{22}$$

The norm used above is the l_∞ norm.

Proof. This follows from Theorem 4.1 in Halluin et al. (2005). □

In contrast to (22), the Lax-Meyer theorem states that strong stability, i.e.

$$\|(B)^n\|_\infty \leq C,$$

for some C independent of n, is a necessary condition for convergence for all initial data. As noted in Halluin et al. (2005), the form of stability given in (22) is clearly weaker than strong stability, and hence yields convergence only for certain initial data. Some caution is thus in order, in particular for the case $\sigma_P > 0$, where the exact solution is discontinuous at the origin. This is why our method uses four quarter-time steps of backward Euler time integration for the first time step, instead of using the Crank-Nicolson method there.

3.4 Fixed-point iteration method

As noted in Halluin et al. (2005), it is computationally very expensive to solve the full linear system of the form (20) or (17), since this means solving a system of linear equations whose numerical complexity grows as $O(m^2)$. Instead we will follow Halluin et al. (2005) and solve the system using the fixed-point iteration method described below. The main advantage with this iteration scheme is that the integrals can be calculated using only the results from the previous time step and the previous iteration. Hence, for a given iteration, the evaluation of the integrals can be considered to be explicit. Thus we define the matrix $\hat{M}$ such that

$$-\left[\hat{M}\rho^k\right]_i = \rho_i^k \left(\alpha_i + \beta_i + r + \hat{\lambda}\right)\hat{\Delta}t - \hat{\Delta}t\beta_i\rho_{i+1}^k - \hat{\Delta}t\alpha_i\rho_{i-1}^k.$$

The only difference between $\hat{M}$ and M is that $\hat{M}$ does not include the integral terms. From the representation (21) it follows that the Crank-Nicolson discretization (20) can be written as follows:

$$\left[I - \frac{1}{2}\hat{M}\right]\rho^{k+1} = \left[I + \frac{1}{2}\hat{M}\right]\rho^k + \frac{1}{2}\hat{\lambda}\Delta t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^{k+1} + \frac{1}{2}\hat{\lambda}\Delta t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k. \quad (23)$$

Using this notation the fixed-point iteration method in Halluin et al. (2005) is described as follows:

Let $\left(\rho^{k+1}\right)^0 = \rho^k$.
Let $\hat{\rho}^j = \left(\rho^{k+1}\right)^j$.
For $j = 0, 1, 2, \ldots$ until convergence
    Solve $\left[I - \frac{1}{2}\hat{M}\right]\hat{\rho}^{j+1} = \left[I + \frac{1}{2}\hat{M}\right]\rho^k$
        $+\frac{1}{2}\hat{\lambda}\hat{\Delta}\sum_{j=0}^{m}\hat{c}_{i,j}\hat{\rho}_j^j + \frac{1}{2}\hat{\lambda}\hat{\Delta}\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k$.
    If $\max_i\left|\hat{\rho}_i^{j+1} - \hat{\rho}_i^j\right| < tolerance$, then quit.
EndFor

In Theorem 5.1 in Halluin et al. (2005) it is proven not only that the iteration scheme above converges, but that the error $e^j = \rho^{k+1} - \hat{\rho}^j$ has an upper bound

$$\| e^{j+1} \|_\infty \leq \| e^j \|_\infty \frac{\frac{1}{2}\hat{\lambda}\hat{\Delta}t}{1 + \frac{1}{2}\hat{\lambda}\hat{\Delta}t}. \quad (24)$$

We used an integration algorithm very similar to the above algorithm for the initial backward Euler timesteps. In our implementation the iteration is set to terminate when the maximal absolute difference between ρ_i^{k+1}-values of consecutive iterations is less than 10^{-8}. We found that for good convergence of this iteration scheme it was advantageous to choose time steps $\hat{\Delta}t$ smaller than $\frac{1}{\lambda+\lambda_R}$.

4 Experimental results

In this section we will discuss numerical examples, where we first fit parameter values for the risk models discussed in the introduction, and then calculate the corresponding ruin probabilities by solving the PIDE (6). We will first consider the case when the claim sizes follow the standard exponential distribution. Then we let the claim sizes follow a mixture of a standard exponential distribution and a Pareto distribution, standardized to have expectation 1. For both claims processes we choose a value for the intensity λ based on data for inflation-adjusted Danish insurance claims. In the examples where the claim distribution is a mixture of a Pareto distribution and an exponential distribution, we let the tail index of the Pareto distribution be the same as the fitted value in Embrechts et al. (1997).

The Danish fire insurance data set consists of 2167 claims over a period of 11 years. We choose a year as the time unit, which gives a maximum likelihood estimate for λ of 197 with a standard error of 4.26. In all our examples we let $\lambda = 197$, let the claims have expectation value 1, and let $p = 216.7$. This corresponds to letting the premium be decided by the expected value principle, with safety loading of 0.1. As already mentioned we adjust the returns of the investments for a constant force of inflation $\bar{\imath}$. We use inflation data from inflationdata.com (2012) to choose an $\bar{\imath}$ for each time period that we consider. These values are given in Table 1.

For the investment return process we consider three different strategies. The first strategy is to continuously invest in Treasury bills with a 3-month rate, the second strategy is to continuously invest in 10-year Treasury bonds that also earn coupons and price appreciation. The last strategy is to invest in American stocks. We use a dataset from Damodaran (2012), which covers annual returns on U.S. Treasury bills, U.S. Treasury bonds and American stocks for the period from 1928 to 2011. For the S&P 500 data for 1962-2003, we use parameter estimates from Ramezani and Zeng (2007) for a geometric Brownian motion model, a Merton model and a Kou model.

4.1 Fitting of geometric Brownian motion to data

In a geometric Brownian motion investment model with drift parameter r and diffusion parameter σ, the log-returns (log-differences) are normally distributed with variance $\sigma^2 t$ and expectation $\left(r - \frac{1}{2}\sigma^2\right) t$. Let $X_0, X_1, \ldots, X_l$ be $l + 1$ observations of the index values at equally spaced times $t_0 = 0, t_1, \ldots, t_l = lt_1$, with one year as the unit of time. Let $Z_1 = \ln\left(\frac{X_1}{X_0}\right), \ldots, Z_l = \ln\left(\frac{X_l}{X_{l-1}}\right)$ be the log-returns.

Period	1928 -2011	1963-2003	2000-2011
Force of inflation	0.03058	0.04406	0.02506

Table 1: The assumed constant force of inflation $\bar{\imath}$ fitted to different time periods.

Let $\bar{Z}$ be the sample mean and let S^2 be the sample variance of the Z_i's. Since the log-returns are i.i.d. normal distributed $\mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)t_1, \sigma^2 t_1\right)$, the method of moment estimator for σ is $\sqrt{\frac{1}{t_1}S^2}$. We thus use $\sqrt{\frac{1}{t_1}S^2}$ as our statistic for σ_R. The method of moment estimator for r is $\frac{1}{t_1}\left(\bar{Z} + \frac{1}{2}S^2\right)$. Since we are adjusting the log-returns for an assumed constant force of inflation $\bar{\imath}$, we use $\frac{1}{t_1}\left(\bar{Z} + \frac{1}{2}S^2\right) - \bar{\imath}$ as the statistic for $\bar{r}$. The resulting estimated parameter values for $\bar{r}$ and σ_R that we use in the geometric Brownian models are given in Table 2. The confidence intervals for σ_R and the standard errors for $\bar{r}$ are based on the fact that $\frac{l-1}{\sigma_R^2}S^2$ is χ^2_{l-1}-distributed and that the sample mean and sample variance of normal random variables are independent. The latter property leads to a standard error for $\bar{r}$ of $\sqrt{S^2\left(\frac{1}{l} + \frac{1}{2}\frac{S^2}{l-1}\right)}$. The standard error for the $\bar{r}$ parameter based on the daily S&P 500 data for 7/1962-12/2003 is the same as the standard error given in Ramezani and Zeng (2007) multiplied with 252. This last multiplication is due to the standardization of the time dimension. Our estimates for U.S. Treasury bills, U.S. bonds and American stocks are based on data for annual returns for 1928-2001 (83 observations for each asset class) from Damodaran (2012). The estimates for S&P 500 are annualized and inflation-adjusted versions of the parameter estimates given in Ramezani and Zeng (2007). These estimates are based on 10446 dividend-adjusted daily observations covering the period 7/1962-12/2003. The estimates for S&P 500 1/2000-11/2011 are based on 3000 observations. Our estimates for U.S. Treasury bills, U.S. bonds and American stocks for the period 2000-2011 are based on just 12 observations. To determine the force of inflation $\bar{\imath}$ we used historical data from inflationdata.com (2012).

An alternative parameterization is to let $\tilde{r} = \bar{r} - \frac{1}{2}\sigma_R^2$. For this parameter the natural statistic (for both method of moments and maximum likelihood) is $\bar{Z} - \bar{\imath}$, where $\bar{Z}$ is the sample mean of the log-returns. The fact that $\frac{\bar{Z} - \bar{\imath} - \tilde{r}}{\sqrt{\frac{S^2}{l}}}$ is t-distributed can be used to construct confidence intervals. Estimates for $\tilde{r}$ as well as 95% confidence intervals are given in the rightmost column in Table 2.

4.2 About the implementation and execution

4.2.1 Software and hardware

We implemented the algorithms described in Section 3 using R software. This was augmented by some Fortran subroutines. In particular the Net lib 'ORTHPOL' package 726 by W. Gautschi was used to calculate the Gaussian quadrature rules.

parameter	$\bar{r}$	σ_R	$\tilde{r}$
U.S. T-bills 1928-2011	0.00534 (0.00316)	0.02900 (0.02518, 0.03419)	0.00492 (−0.00138, 0.01121)
U.S. T-bonds 1928-2011	0.02259 (0.0078)	0.07131 (0.06192, 0.08409)	0.02005 (0.00457, 0.03552)
U.S. Stocks 1928-2011	0.07815 (0.02165)	0.19648 (0.17060, 0.23169)	0.05885 (0.01621, 0.10149)
S&P 500 7/1962-12/2003	0.08194 (0.0252)	0.15081 (0.14879, 0.15288)	0.07057 n.a.
U.S. T-bills 2000-2011	−0.00233 (0.0056)	0.01941 (0.01375, 0.03296)	−0.00252 (−0.01485, 0.00982)
U.S. T-bonds 2000-2011	0.04811 (0.0242)	0.08367 (0.05927, 0.14206)	0.04461 (−0.00855, 0.09777)
U.S. Stocks 2000-2011	0.00129 (0.06001)	0.20551 (0.14558, 0.34894)	−0.01982 . (−0.15040, 0.11075)
S&P 500 1/2000-11/2011	−0.00125 (0.00394)	0.21331 (0.20804, 0.21884)	−0.02789 (−0.03553, −0.02026)

Table 2: Parameter estimates for the geometric Brownian motion investment model with normally distributed inflation-adjusted log-returns. $\bar{r}$ is $r - \bar{\imath}$ (nominal return subtracted with the inflation force $\bar{\imath}$), while $\tilde{r}$ is defined as $\bar{r} - \frac{1}{2}\sigma_R^2$. The drift term for nominal log-returns (r) can be obtained by adding the corresponding inflation forces in Table 1. All the asset returns except the S&P 500 returns for 1/2000-11/2011 include dividends or coupons. 95% confidence intervals for σ_R and $\tilde{r}$, and standard errors for $\bar{r}$ are given in parentheses.

4.2.2 Grid sizes and tolerance values

In the implementations we let $\kappa = 2000$. We let the artificial boundary condition be at $z = \ln(1+\kappa)$. Recall that $\hat{\Delta}t$ is the length of the Crank-Nicolson time steps. The error bound (24) and our experiments suggest that letting $(\lambda + \lambda_R)\,\hat{\Delta}t$ be large may lead to poor convergence in the integral iteration described in section 3.4. Our experience shows that to avoid excessive iterations $\hat{\Delta}t$ should be less than $\frac{1}{\lambda+\lambda_R}$. In the case of exponentially distributed jumps we calculated the integral terms as described in section 3.1.1. In the examples where we illustrate the convergence rate, we use the same spatial grid sizes as in Halluin et al. (2005). Unless denoted otherwise the space grid has $m = 2^K$ for $K \in 7, \ldots, 12$, for the results in the tables. As explained in Section 3.3, in the time domain the first four time-steps have length $\frac{1}{24m}$, while the rest of the steps have length $\frac{1}{6m}$. For the results in the tables the size of the space grid is mostly $m = 4096$, or 2^{12}. For the examples with Pareto-distributed claim sizes, Gaussian quadrature rules of length $2K$ are calculated before the finite-differences method begins. In the example with the Merton model, a Gauss-Hermite rule of length 24 is applied. As convergence criterion for the fixed-point iteration method we required the max norm difference between two iterations to be less than 10^{-8}. In the tables containing the experimental results, the abbreviation 'C.R' refers to the convergence ratio, defined in equation (8.2) in Halluin et al. (2005) as

$$C.R = \left| \frac{\rho_{\text{approx}}(h/2) - \rho_{\text{approx}}(h))}{\rho_{\text{approx}}(h/4) - \rho_{\text{approx}}(h/2)} \right|.$$

Here h is a given step size.

4.3 Exponentially distributed jumps

In the case of exponentially distributed jumps we can calculate the integrals as described in Section 3.1.1. For the Cramér-Lundberg model with exponentially distributed claim sizes, the ruin probability was calculated using an integral formula given in Chapter IV in Asmussen (2000). We used this solution to check the accuracy of the method. Table 3 shows the relative errors of the calculated ruin probabilities using the method described in 3.1.1, with parameter vales $p = 216.7, \lambda = 197$, $\alpha = 1$ (standard exponential) and $t = 1$ (1 year). We also used the case with exponentially distributed jumps to check the accuracy of using Gaussian quadrature rules to evaluate the integrals. The results from this test suggest that the errors from the numerical integration are small in comparison to the errors from the numerical differentiation.

In order to avoid oscillations, α_i and β_i in (20) should be non-negative. For this to be satisfied the derivative terms in the space variable have to be discretized using forward differences. The drawback of forward differences is that it gives only first order convergence (i.e. consistency error $O(h)$) as opposed to the second order convergence (i.e. consistency error $O(h^2)$) we get with using the central differences as in (15). So although the convergence of Crank-Nicolson time integration itself is of second order, the overall convergence is only first order. For models where σ_R^2 is not very small we can use central differences on most of the domain without violating the conditions in Theorem 1 and Theo-

m		128	512	2048	4096
n		105	408	1620	3237
y	Exact	Relative	Relative	Relative	Relative
	solution	error	error	error	error
0	0.90080	0.00018	0.00044	0.00014	0.00007
7	0.43782	0.07168	0.01502	0.00354	0.00175
15	0.18097	0.30361	0.06881	0.01667	0.00829
31	0.02474	1.67517	0.32926	0.07630	0.03766
63	0.00017	35.45299	2.40341	0.39177	0.18208

Table 3: Relative errors for the Cramér-Lundberg model with various values of m. The parameter values are $p = 216.7$, $\lambda = 197$, and the claim sizes are standard exponentially distributed. The units for y and p are the expected value of a claim. λ corresponds to the expected number of claims per year. Hence $y = 63$ corresponds to a starting capital equal to 32% of the expected annual cost of claims.

rem 2. Thus for these models it is important to have an accurate method of integration in order to utilize the accuracy of the differential terms.

For Treasury bonds the variance in the log-returns is not large enough to allow use of central differences at more than a small minority of the grid points. There the convergence rate seems to be very similar to what is was with the Cramér-Lundberg model. For the log-return on stocks the volatility is higher, and for the grids with space grid size $m > 1000$ central differences can be used at a majority of the grid points. As seen in Table 4 this gives improved convergence. Since convergence is slower when the diffusion term (in effect ellipticity) is small, we used an even finer space grid ($m = 8192$) for the Treasury bond data.

Regarding ruin probabilities for models fitted to long term trends, we see in Table 5 that in the case of U.S. Treasury bills the difference between the classical Cramer-Lundberg model and the investment model fitted to annual returns is very small. For the investment model fitted to annual returns of U.S. Treasury bonds and the model fitted to daily returns of the SP 500 index, the ruin probabilities are slightly lower. For the model fitted to annual returns of American stocks the ruin probabilities are slightly higher than in the classical model, especially for the highest intial capital ($y = 63$). Lastly, we note that the increases in ruin probabilities flatten out after 5 years.

For years after 2000 the results are very different. In particular, at the highest intial capital ($y = 63$) the ruin probabilities for models with stocks are $2 - 3$ times higher than for models with bonds or with the classical model. Again the increases in ruin probability flatten out after 5 years.

4.4 Heavy Tail Models

For regularly varying claim size distributions (defined below) we have the following asymptotic result, based on Theorem 2 and Example 1 in Hult and Lindskog

m	128	512	2048	4096
n	105	408	1620	3237
	y=31			
Value	0.06414	0.03222	0.02488	0.02444
C.R	n.a.	2.44	1.36	7.06
	y=45			
Value	0.02261	0.00631	0.00367	0.00359
C.R	n.a.	3.07	2.44	9.63
	y=63			
Value	0.00628	0.00071	0.00026	0.00025
C.R	n.a.	4.54	6.29	8.86

Table 4: For standard exponentially distributed claim sizes, experimental results showing convergence as the number of grid points increases. The model includes a return on investment process, which is a geometric Brownian motion process. The parameter values are $\sigma_R = 0.19648$, $p = 216.7$ (premium rate corresponding to a safety loading of 0.1), $\bar{r} = 0.07815$ (real rate of interest) and $\lambda = 197$. The data used is return on American stocks for the period 1928-2011.

Model	Cramér-Lundberg	Geometric Brownian Motion			
Data Period	n.a. n.a.	U.S. T-bills 1928-2011 (annual)	U.S. T-bonds 1928-2011 (annual)	U.S. Stocks 1928-2011 (annual)	S&P 500 7/1962-12/2003 (daily)
			$T = 1$		
$y = 31$	0.02468	0.02503	0.02458	0.02444	0.02312
$y = 45$	0.00330	0.00342	0.00331	0.00359	0.00313
$y = 63$	0.00017	0.00018	0.00017	0.00025	0.00018
			$T = 2$		
$y = 31$	0.04188	0.04228	0.04110	0.04034	0.03765
$y = 45$	0.00897	0.00920	0.00874	0.00918	0.00787
$y = 63$	0.00103	0.00110	0.00101	0.00132	0.00095
			$T = 5$		
$y = 31$	0.05294	0.05330	0.05116	0.04989	0.04562
$y = 45$	0.01437	0.01465	0.01359	0.01407	0.01164
$y = 63$	0.00260	0.00273	0.00241	0.00301	0.00206
			$T = 10$		
$y = 31$	0.05423	0.05458	0.05220	0.05094	0.04627
$y = 45$	0.01516	0.01544	0.01422	0.01475	0.01202
$y = 63$	0.00294	0.00307	0.00267	0.00336	0.00222

Table 5: Ruin probabilities for the Cramér-Lundberg model and four fitted models, with investment following geometric Brownian motion (GBM) fitted to long term trends. The premium rate p is 216.7, $\lambda = 197$, and claim sizes are assumed to be standard exponentially distributed. Again, y = 63 corresponds to an initial capital equal to 32% of the expected annual claim cost. Note that the increases in ruin probability flatten out after T = 5 years.

Data Period	U.S. T-bills 2000-2011 (annual)	U.S. T-bonds 2000-2011 (annual)	U.S. Stocks 2000-2011 (annual)	S&P 500 1/2000-11/2011 (daily)
$T = 1$				
$y = 31$	0.02534	0.02345	0.02830	0.02871
$y = 45$	0.00349	0.00306	0.00463	0.00478
$y = 63$	0.00019	0.00016	0.00038	0.00041
$T = 2$				
$y = 31$	0.04300	0.03864	0.04898	0.04984
$y = 45$	0.00946	0.00790	0.01278	0.01323
$y = 63$	0.00115	0.00087	0.00225	0.00241
$T = 5$				
$y = 31$	0.05453	0.04723	0.06490	0.06644
$y = 45$	0.01523	0.01188	0.02216	0.02316
$y = 63$	0.00291	0.00195	0.00628	0.00679
$T = 10$				
$y = 31$	0.05595	0.04796	0.06832	0.07023
$y = 45$	0.01613	0.01230	0.02476	0.02608
$y = 63$	0.00330	0.00212	0.00793	0.00869

Table 6: Ruin probabilities for four fitted models, with investment following geometric Brownian motion (GBM) fitted to data for 2000-2011. The premium rate p is 216.7, $\lambda = 197$, and claim sizes are assumed to be standard exponentially distributed. Note that the increases in ruin probability flatten out after T = 5 years.

(2011).

Definition 1. *A function $L(x)$ is said to be* slowly varying *if*

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1, \quad \text{for all } c > 0.$$

A positive random variable S and its distribution are said to be regularly varying with (tail) index $\hat{\alpha}$ if for some $\hat{\alpha} \geq 0$, the right tail of the distribution has the representation

$$P(S > x) = L(x)x^{-\hat{\alpha}},$$

where L is a slowly varying function.

Theorem 3. *Assume that the claim size distribution function $F(x)$ is regularly varying with index α.*

(a) *In the case of the Cramér-Lundberg model the probability of ruin before time t is asymptotically given by*

$$\psi(y,t) \sim \lambda t \bar{F}(y),$$

where $\bar{F}(x) = 1 - F(x)$ is the tail distribution.

(b) *Consider a risk process of the form given in* (2)*, with investment. Let*

$$\theta = \frac{1}{2}\sigma_R^2\alpha^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(E(1+S_R)^{-\alpha} - 1\right)$$

and make the following additional assumptions:

(i) *Either $\lambda_R = 0$ or, for some $\delta > 0$, $E(1+S_R)^{-(\alpha+\delta)} < \infty$.*

(ii)

$$\theta \neq 0.$$

Then the probability of ruin before time t is asymptotically given by

$$\psi(y,t) \sim \frac{1}{\theta}\left(e^{\theta t} - 1\right)\lambda\bar{F}(y). \tag{25}$$

Proof. We first consider the case when $T = 1$. As discussed in Section 1 the inflation-adjusted risk process Y at time t is given as

$$Y_t = \bar{R}_t\left(y + \int_0^t \bar{R}_s^{-1} dP_s\right), \tag{26}$$

where $\bar{R}_t = \exp\left\{\left(\bar{r} - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}\right\}\Pi_{n=1}^{N_{R,t}}(1 + S_{R,n})$. At $t = 1$ the Lévy process P_t, defined in (1) as a jump diffusion process with negative jumps $-\sum_{i=1}^{N_t} S_i$, has Lévy measure $\nu(-\infty, -u) = \lambda\bar{F}(u)$. Consequently Theorem 4.1 in Hult and Lindskog (2011) can be applied, with P playing the role of their Lévy process Y and the process $\left\{\bar{R}_t^{-1}\right\}_{t\geq 0}$ playing the role of their cáglad strictly positive process A. With these adaptations it follows from Theorem 4.1 and Example 3.5 in Hult and Lindskog (2011) that the stated results are valid for $T = 1$.

Assume that $T \neq 1$. Since P and R are Lévy processes, changing the parameters from $p, \sigma_P, \lambda, \bar{r}, \sigma_R, \lambda_R$ to $pt, \sigma_P\sqrt{t}, \lambda t, \bar{r}t, \sigma_R\sqrt{t}$ and $\lambda_R t$ corresponds to changing the time horizon from time T to time 1, giving the stated results when $T \neq 1$. □

It is only for very large claim sizes that the Pareto distribution is intended to be a good model, as discussed in Embrechts et al. (1997). This is formalized by introducing a threshold u, which in the discussion in Chapter 6 on Danish fire claims in Embrechts et al. (1997) is put to 10. The claim sizes which are larger than u are called *exceedances*. The Pareto distribution is fitted using only this part of the data. 109 of the 2167 claims in the data set are such exceedances.

In this section we let the distribution of the claim sizes be a weighted average of a standard exponential distribution and a standard Pareto distribution. The weight assigned to the Pareto distribution corresponds to the share of claims that are exceedances, $\frac{109}{2167}$.

The standardized Pareto distribution has a density given by

$$f_2(x) = (\hat{\alpha} - 1)^{\hat{\alpha}} \hat{\alpha} (\hat{\alpha} - 1 + x)^{-(1+\hat{\alpha})}, \ x > 0, \hat{\alpha} > 1, \tag{27}$$

and tail distribution function

$$\bar{F}_2(x) = \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + x} \right)^{\hat{\alpha}}.$$

This form of Pareto distribution is called 'standardized Pareto', since the expectation value is 1 for every $\hat{\alpha}$. This distribution is regularly varying with index $\hat{\alpha}$. For $\lambda_R = 0$ asymptotic formulas for the ruin probability are given below. Asymptotically the tail of the mixed distribution we use is dominated by the Pareto part, and thus is also regularly varying.

Corollary 1. *Assume that the claim sizes follow a mix of a Pareto distribution with density, as in* (27)*, and a light-tailed distribution. Assume that a weight* $0 < w \leq 1$ *is assigned to the Pareto distribution and a weight* $1 - w$ *is assigned to the light-tailed distribution, i.e. a distribution whose moment-generating function exists in a neighborhood around zero.*

(i) *Consider the Cramér-Lundberg model, with claim sizes following a mixed distribution as described above. In this model the ruin probability is asymptotically given by*

$$\psi(y, t) \sim \lambda w t \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + y} \right)^{\hat{\alpha}}.$$

(ii) *Consider a risk process with investment of the form given in* (2)*, with claim sizes following a mixed distribution as described above, and with* $\lambda_R = 0$.

$$\hat{\theta} = \frac{1}{2}\hat{\alpha}^2 \sigma_R^2 - \hat{\alpha}\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) \neq 0.$$

Then the ruin probability is asymptotically given by

$$\psi(y, t) \approx \frac{\lambda w}{\hat{\theta}} \left(e^{\hat{\theta} t} - 1\right) \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + y} \right)^{\hat{\alpha}}. \tag{28}$$

m	128	512	2048	4196
n	105	408	1620	3237
	y=31			
Value	0.0755	0.0436	0.0376	0.0367
C.R	n.a.	2.42	2.15	2.08
	y=45			
Value	0.0298	0.0124	0.0100	0.0097
C.R	n.a.	2.98	2.32	2.17
	y=63			
Value	0.0101	0.0036	0.0031	0.0030
C.R	n.a.	4.09	2.61	2.30

Table 7: Convergence for the ruin probability in the Cramér-Lundberg model with $p = 216.7$ (corresponding to a safety loading of 0.1) and $\lambda = 197$. The claim distribution is a mixture of a standard exponential distribution and a standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$. The weight assigned to the Pareto distribution is $\frac{109}{2167}$.
.

As expected, the numerical results show that ruin probabilities based on a heavy-tailed claim size distribution model are larger than when based on a light-tailed claim size distribution. The most striking differences between the results for the heavy-tail models, given in Table 8 compared with the results for light-tailed models, given in Table 5 are found for $T = 1$ (one year). With initial capital 63 (corresponding to 32 % of the expected annual claim cost) and assuming the Cramér-Lundberg model, the ruin probability in the heavy-tail case is 17.6 times larger than in the light-tail case.

For the stock models fitted to long-term trends the ruin probabilities are about the same or even lower than for the Cramér-Lundberg models, which do not include investment risk. On the other hand, when fitted to the period 2000-2011, given in Table 9, the ruin probabilities for the stock models start to grow quickly with increasing T, in particular for $T > 2$. Especially for large initial capitals and $T \in \{5, 10\}$, the ruin probabilities for the stock models are almost twice as high as the ruin probabilities with the Cramér-Lundberg model, given in Table 8.

For the case of regularly varying claim sizes and large values of the initial capital the formulas of Theorem 3 suggest the following: If $\theta > 0$ the ruin probabilities are higher than in the Cramér-Lundberg model. If $\theta < 0$ the ruin probabilities are lower than in the Cramér-Lundberg model. More precisely, when $\theta T << 0$ the ruin probability on the time horizon T is close to $-\frac{1}{\theta}\lambda\bar{F}(y)$. When $\theta T \approx 0$ the ruin probability grows approximately linearly as a function of T for fixed (high) initial capitals. When $\theta T >> 0$ the ruin probability grows exponentially with T. Our numerical experiments support this assertion. Moreover, as can be seen from Table 8 and Table 9, the asymptotic formula for the ruin probability seems to be quite accurate, at least as long as the ruin probability from that formula is less than 0.002.

Model	Cramér-Lundberg	Geometric Brownian Motion			
Data	n.a.	U.S. T-bills	U.S. T-bonds	U.S. Stocks	S&P 500
Period	n.a.	1928-2011 (annual)	1928-2011 (annual)	1928-2011 (annual)	7/1962-12/2003 (daily)
θ	0	-0.0082	-0.03004	-0.04026	-0.09595
$T = 1$					
$y = 31$ Asym.	0.0367 (0.0095)	0.0393 (0.0095)	0.0359 (0.0094)	0.0351 (0.0093)	0.0337 (0.0091)
$y = 45$ Asym.	0.0097 (0.0046)	0.0107 (0.0046)	0.0095 (0.0045)	0.0095 (0.0045)	0.0089 (0.0044)
$y = 63$ Asym.	0.0030 (0.0024)	0.0032 (0.0023)	0.0030 (0.0023)	0.0030 (0.0023)	0.0028 (0.0022)
$y = 100$ Asym.	0.0010 (0.0009)	0.0010 (0.0009)	0.0010 (0.0009)	0.0010 (0.0009)	0.0009 (0.0009)
$T = 2$					
$y = 31$ Asym.	0.0592 (0.0190)	0.0587 (0.0188)	0.0575 (0.0184)	0.0554 (0.0183)	0.0526 (0.0173)
$y = 45$ Asym.	0.0198 (0.0092)	0.0196 (0.0091)	0.0191 (0.0089)	0.0187 (0.0088)	0.0171 (0.0083)
$y = 63$ Asym.	0.0064 (0.0047)	0.0064 (0.0047)	0.0062 (0.0046)	0.0063 (0.0045)	0.0057 (0.0043)
$y = 100$ Asym.	0.0018 (0.0019)	0.0018 (0.0019)	0.0018 (0.0018)	0.0018 (0.0018)	0.0017 (0.0017)
$T = 5$					
$y = 31$ Asym.	0.0767 (0.0047)	0.0758 (0.0465)	0.0734 (0.0441)	0.0699 (0.0430)	0.0653 (0.0377)
$y = 45$ Asym.	0.0308 (0.0229)	0.0303 (0.0224)	0.0290 (0.0213)	0.0278 (0.0207)	0.0248 (0.0182)
$y = 63$ Asym.	0.0120 (0.0118)	0.0117 (0.0115)	0.0111 (0.0109)	0.0110 (0.0107)	0.0094 (0.0094)
$y = 100$ Asym.	0.0036 (0.0047)	0.0036 (0.0046)	0.0034 (0.0044)	0.0034 (0.0043)	0.0029 (0.0037)
$T = 10$					
$y = 31$ Asym.	0.0809 (0.0950)	0.0799 (0.0912)	0.0769 (0.0821)	0.0729 (0.0782)	0.0676 (0.0611)
$y = 45$ Asym.	0.0342 (0.0458)	0.0335 (0.0440)	0.0318 (0.0395)	0.0302 (0.0377)	0.0266 (0.0294)
$y = 63$ Asym.	0.0144 (0.0236)	0.0141 (0.0226)	0.0131 (0.0204)	0.0127 (0.0194)	0.0107 (0.0152)
$y = 100$ Asym.	0.0049 (0.0094)	0.0048 (0.0090)	0.0044 (0.0081)	0.0043 (0.0077)	0.0036 (0.0061)

Table 8: Calculated and asymptotic ruin probabilities for the classical Cramér-Lundberg model, plus four models with investment following geometric Brownian motion (GBM) fitted to long-term historical trends. The premium rate is $p = 216.7$, and the expected number of claims per year is $\lambda = 187$. This corresponds to a safety load of 0.1. $y = 63$ corresponds to an initial capital equal of 32% of the expected claim cost per year. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$, as used in Embrechts et al. (1997), and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. Asymptotic ruin probabilities are given in parentheses.

Data Period	U.S. T-bills 2000-2011 (annual)	U.S. T-bonds 2000-2011 (annual)	U.S. Stocks 2000-2011 (annual)	S&P 500 1/2000-11/2011 (daily)
θ	0.00583	-0.07559	0.12539	0.14040
		$T = 1$		
$y = 31$ Asym.	0.0368 (0.0095)	0.0348 (0.0092)	0.0394 (0.0101)	0.0398 (0.0102)
$y = 45$ Asym.	0.0098 (0.0046)	0.0092 (0.0044)	0.0110 (0.0049)	0.0112 (0.0049)
$y = 63$ Asym.	0.0030 (0.0024)	0.0029 (0.0023)	0.0034 (0.0025)	0.0035 (0.0025)
$y = 100$ Asym.	0.0010 (0.0009)	0.0009 (0.0009)	0.0011 (0.0010)	0.0011 (0.0010)
		$T = 2$		
$y = 31$ Asym.	0.0595 (0.0191)	0.0549 (0.0176)	0.0649 (0.0216)	0.0658 (0.0219)
$y = 45$ Asym.	0.0199 (0.0092)	0.0179 (0.0085)	0.0234 (0.0104)	0.0239 (0.0106)
$y = 63$ Asym.	0.0065 (0.0047)	0.0058 (0.0044)	0.0080 (0.0054)	0.0083 (0.0054)
$y = 100$ Asym.	0.0019 (0.0019)	0.0017 (0.0017)	0.0022 (0.0021)	0.0022 (0.0022)
		$T = 5$		
$y = 31$ Asym.	0.0773 (0.0482)	0.0689 (0.0396)	0.0873 (0.0660)	0.0889 (0.0688)
$y = 45$ Asym.	0.0312 (0.0232)	0.0263 (0.0191)	0.0386 (0.0318)	0.0398 (0.0332)
$y = 63$ Asym.	0.0121 (0.0120)	0.0098 (0.0098)	0.0167 (0.0164)	0.0174 (0.0171)
$y = 100$ Asym.	0.0037 (0.0048)	0.0030 (0.0039)	0.0052 (0.0065)	0.0055 (0.0068)
		$T = 10$		
$y = 31$ Asym.	0.0817 (0.0978)	0.0716 (0.0667)	0.0945 (0.1895)	0.0967 (0.2075)
$y = 45$ Asym.	0.0347 (0.0471)	0.0284 (0.0321)	0.0449 (0.0913)	0.0466 (0.1000)
$y = 63$ Asym.	0.0147 (0.0243)	0.0113 (0.0165)	0.0215 (0.0470)	0.0226 (0.0515)
$y = 100$ Asym.	0.0050 (0.0097)	0.0038 (0.0066)	0.0081 (0.0188)	0.0086 (0.0206)

Table 9: Calculated and asymptotic ruin probabilities for four models, with investment following geometric Brownian motion (GBM) fitted to data from 2000-2011. The premium rate $p = 216.7$ corresponds to a safety loading of 0.1. The expected number of claims per year is $\lambda = 187$. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$ (used in Embrechts et al. (1997)) and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. Asymptotic ruin probabilities are given in parentheses.

4.5 Jumps in the investment process

Ramezani and Zeng (2007) discuss maximum likelihood estimation for the Merton model and the Kou model, as well as the geometric Brownian model (GBM). In particular they fitted these models to the S&P for daily observations for the period 1962-2003. In the previous section we used estimates from Ramezani and Zeng (2007) for the GBM model.

In both the Merton model and the Kou model the returns follow a jump-diffusion process. The Merton model was introduced in Merton (1976) and consists of letting the jumps of the returns follow a log-normal distribution. The Kou model was first discussed in Ramezani and Zeng (1998) and in Kou (2002). In this model the jumps of the log-returns follow a double exponential distribution. The Merton and Kou models are discussed in more detail in Ramezani and Zeng (2007) and in Chapter 4 in Cont and Tankov (2004). Annualized and inflation-adjusted versions of the parameter estimates for the Merton model fitted to the S&P 500 data are in Table 10, where μ and σ are the parameters of the log-normal distribution. For the Merton model the θ defined in Theorem 3 is given by

$$\theta = \frac{1}{2}\alpha^2\sigma_R^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(\exp\left(-\alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right) - 1\right).$$

Similarly, annualized and inflation-adjusted versions of the parameter estimates from Ramezani and Zeng (2007) for the Kou model are given in Table 11. In Ramezani and Zeng (2007) it is assumed that the arrival times of "good" and "bad" news (leading to positive vs. negative jumps) follow two different independent Poisson processes. These have intensities λ_u and λ_d, respectively. In Table 11 the parameter estimate for λ_R corresponds to the annualized value of the sum of their estimates for λ_u and λ_d, while the parameter q refers to the fraction $\frac{\lambda_u}{\lambda_u+\lambda_d}$. η_1 is the parameter of the exponential distribution of the positive jumps in the log-returns process. η_2 is the parameter of the exponential distribution of the size of the negative jumps in the log-returns process. If $\eta_2 > \alpha$ then θ for the Kou model is given by

$$\theta = \frac{1}{2}\alpha^2\sigma_R^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(\frac{q}{1+\frac{\alpha}{\eta_1}} + \frac{1-q}{1-\frac{\alpha}{\eta_2}} - 1\right).$$

With the parameter estimates from Ramezani and Zeng (2007) we get

$$\theta \approx -0.11.$$

Again this suggests a smaller ruin probability than with the Cramér-Lundberg model.

With these parameter estimates, values for the ruin probability in the Merton and Kou models are shown in Table 12. The table shows that, for the S&P 1962-2003 data, ruin probabilities do not differ much between models. This is not the case if we consider investment in a single stock and use parameter values from Ramezani and Zeng (2007), but this is a highly implausible strategy.

$\bar{r}$	σ_R	λ_R	μ	σ	θ
0.05674	0.127	18.5724	0.0005	0.0199	-0.06909
(0.0252)	(0.0252)	(2.6712)	(0.0009)	(0.0005)	n.a.

Table 10: Parameter estimates from Ramezani and Zeng (2007) for the Merton model fitted to daily returns of the S&P 500 1962-2003. Annualized standard errors are in parentheses. θ is defined in Theorem 3.

$\bar{r}$	σ_R	λ_R	q	η_1	η_2	θ
0.15754	0.07302	237.8376	0.4521	173.91	185.98	-0.1054
n.a.	n.a.	(41.1516)	n.a.	(0.36)	(0.38)	n.a.

Table 11: Parameter estimates from Ramezani and Zeng (2007) for the Kou model fitted to daily returns of the S&P 500 1962-2003. Annualized standard errors are in parentheses. θ is defined in Theorem 3.

Model	$T = 1$	$T = 2$	$T = 5$	$T = 10$
CL	0.0010	0.0018	0.0036	0.0049
GBM	0.0009	0.0017	0.0029	0.0036
Merton	0.0009	0.0017	0.0031	0.0036
Kou	0.0009	0.0017	0.0029	0.0039

Table 12: Ruin probabilities for the classical Cramér-Lundberg (CL) model and three investment models fitted to daily observations of S&P 500 7/1962-2003. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$ (used in Embrechts et al. (1997)) and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. The initial capital is set to $y = 100$.

5 Conclusions

The numerical experiments suggest that using precalculated Gaussian quadrature rules is an efficient way of calculating the integrals in the numerical solution of the PIDE (8). The dominant source of error seems to be the differential terms rather than the numerical integration. From a numerical point of view the main problem is the following: For the finite-difference method to be numerically stable the differential terms often have to be approximated using one-sided finite differences rather than central differences. This gives a slower convergence, in particular for moderately sized initial capitals and models where the diffusion term is small.

As for the ruin probabilities for the fitted models, they can vary substantially depending on which time period is selected for the data. The ruin probabilities are also quite sensitive to the value of the θ defined in Theorem 3, and in particular the sign of θ is critical. If $\theta > 0$ (Table 9) the effect of investments on the ruin probabilities is moderate on short time horizons, but more pronounced for longer time horizons. Our numeric results suggest that the asymptotic result in Theorem 3, based on Theorem 4.1 in Hult and Lindskog (2011), is rather accurate for short time horizons. For long time horizons, in particular for $T > 5$) the initial capital needs to very high for the formula to be a good approximation.

A possible topic for future research is to estimate inflation from the claim sizes themselves. This might require new numerical methods. It might also be interesting to approximate the small claims with a diffusion process. This would increase the ellipticity and thus allow more nodes to be approximated with central differences. Since the computation time is proportional to $(\lambda + \lambda_R)T$ the ruin probability in such a model would be easier to compute efficiently.

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6 Paper D

Existence of a classical solution of a parabolic PIDE associated with ruin probability

June 18, 2012

Abstract

In this article we will prove existence of a classical solution of the integro-differential equation for ruin probability in finite time stated in Paulsen (2008).

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1 Risk process model

In Paulsen (2008) the risk model consists of a basic risk process P_t with $P_0 = 0$, and a return on investment generating process R, with $R_0 = 0$. The risk process is defined as

$$Y_t := y + P_t + \int_0^t Y_{s-} dR_s, \tag{1.0.1}$$

with initial value $Y_0 = y$. In the above the stochastic process R_t is assumed to be a diffusion process of the form

$$R_t = rt + \sigma_R W_{R,t}, \tag{1.0.2}$$

where r and σ_R are nonnegative constants and W_R is a Brownian motion. P_t is assumed to be a jump-diffusion process of the form

$$P_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_t} S_i, \tag{1.0.3}$$

where p and σ_P are nonnegative constants and $W_{P,t}$ is a Brownian motion, N_t is a Poisson process with rate λ, and the $\{S_i\}$ are positive, independent and identically distributed random variables with distribution function F. $W_{P,t}, W_{R,t}, N_t$ and the $\{S_i\}$ are assumed to be mutually independent. The time of ruin is defined as the stopping time

$$\tau = \inf \{t : Y_t < 0\}, \tag{1.0.4}$$

with $\tau = \infty$ if Y stays nonnegative. In the case that $\sigma_P > 0$ the infinite variation of the Brownian process $W_{P,t}$ ensures that

$$\inf \{t : Y_t < 0\} = \inf \{t : Y_t \leq 0\}.$$

With τ defined as above the probability of ruin in a given finite time t is defined as

$$\psi(y, t) = P(\tau \leq t | Y_0 = y).$$

2 PIDE for the ruin probability

Let F be the distribution function of a probability measure that assigns no mass to $(-\infty, 0]$. For every $(y, t) \in (0, \infty) \times (0, T]$, let L be the parabolic differential operator

$$Lh(y,t) = \frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2)\frac{\partial^2 h(y,t)}{\partial y^2} + (p + ry)\frac{\partial h(y,t)}{\partial y},$$

and let A be the integro-differential operator

$$Ah(y,t) = Lh(y,t) + \lambda \int_0^y h(y-z,t)dF(z) - \lambda h(y,t).$$

In Paulsen (2008) it is stated that the ruin probability should be the solution of the following partial integro-differential equation (PIDE):

$$\begin{cases} \psi(y,0) = 0, & y > 0 \\ \psi(0,t) = 1, & t \in [0,T] \\ \frac{\partial \psi(y,t)}{\partial t} - A\psi(y,t) = \lambda \bar{F}(y), & (y,t) \in (0,\infty) \times (0,T]. \end{cases} \tag{2.0.5}$$

In the above $\bar{F}(y) = 1 - F(y)$ is the tail distribution function. Asymptotically a solution of equation (2.0.5) should satisfy

$$\lim_{y \to \infty} \psi(y,t) = 0, \quad t \in [0,T]. \tag{2.0.6}$$

We observe that the operator A is linear and uniformly elliptic, while the initial condition, the boundary condition, and all the coefficients are all analytic for $y > 0$. This suggests that equation (2.0.5) "should" have a smooth solution, at least if the distribution function $F(z)$ is smooth. A closer look, however, reveals a number of properties that violate the standard assumptions in the literature on PDE and PIDE problems.

- The domain is unbounded.
  Some literature, in particular on PDE's, discusses problems with unbounded domains. In general, however, these treatises require that at least the coefficients of the second space derivative be bounded. In our case the coefficient of the second space derivative is

  $$\frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2),$$

  which is obviously not bounded for $y \in (0, \infty)$, when $\sigma_R > 0$.

- Violation of compatibility condition.
  The initial condition dictates that $\lim_{y \downarrow 0} \psi(y, 0) = 0$, whereas the boundary condition dictates that $\lim_{t \downarrow 0} \psi(0, t) = 1 \neq 0$. The initial condition and the boundary condition are thus incompatible. Any solution of (2.0.5) must hence be discontinuous at the origin, which violates the requirement that a classical solution must be continuous at at the boundary.

- Asymptotic boundary condition
  In addition to the difficulties mentioned above we need to verify that, for any $t \in (0, T]$, $\lim_{y \uparrow \infty} \psi(y, t) = 0$.

The upshot of this is that standard theory does not immediately ensure existence and uniqueness of a solution of equation (2.0.5). Instead we have to rely on more indirect methods, and work mostly with an emulation of (2.0.5) on a truncated domain $(0, \kappa) \times (0, 1]$, with the more standard boundary equation $\psi(\kappa, t) = 0$ for $t \in [0, 1]$. Since there can be no *classical* solution we will in this article instead look for a solution that satisfies the requirements of a classical solution, including continuity to the boundary, except at the origin. We will call such a solution a classical solution, except at the origin. The last result in Section 3, Theorem 3.0.4 establishes the existence of such a classical solution, except at the origin, on any truncated domain.

Our objective is to establish existence on an unbounded domain, with the asymptotic boundary condition. For this we will need some estimates which we will obtain in Section 4. To derive these estimates we assume that the coefficients satisfy $\sigma_P > 0$ and either $\sigma_R = r = 0$ or $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies

$$\bar{F}(\zeta) \leq C\,(1 + \zeta)^{-\beta}\,, \quad \zeta \geq 0,$$

for some positive constants C and β.

In the last part of the article, Section 5, we will establish in Theorem 5.1.2 and Theorem 5.2.2 the existence of a classical solution on the original unbounded domain which even satisfies the asymptotic boundary condition.

3 Existence and uniqueness on a truncated domain

In this paper we will be working with the Green spaces defined in chapter VII in Garroni and Menaldi (1992). To be compatible with the definition of these spaces we will henceforth assume that $T = 1$.

In order to standardize equation (2.0.5) with $T \neq 1$ we can just substitute the parameters p, σ_P, σ_R and λ with $pT, \sigma_P\sqrt{T}, \sigma_R\sqrt{T}$ and λT. We can therefore without loss of generality assume that $T = 1$, which we will do in the rest of the paper. In order to have all the coefficients of A bounded we introduce a truncated domain $(0, \kappa)$ for y. The upper boundary condition is now in a standard form.

$$\begin{cases} \psi_\kappa(y,0) = 0, \quad y \in (0,\kappa)\,, \\ \psi_\kappa(0,t) = 1, \quad t \in [0,1]\,, \\ \psi_\kappa\,(\kappa,t) = 0, \quad t \in [0,1]\,, \\ \frac{\partial\psi_\kappa(y,t)}{\partial t} - A\psi_\kappa\,(y,t) = \lambda\bar{F}(y), \quad (y,t) \in (0,\kappa)\times(0,1]. \end{cases} \tag{3.0.7}$$

Taking a cue from Garroni and Menaldi (2002) we will look for a solution $\psi_\kappa(y,t)$ of (3.0.7) by considering the three equations

$$\begin{cases} \psi_{1,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{1,\kappa}(0,t) &= 1, \quad t \in [0,1]\,, \\ \psi_{1,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial\psi_{1,\kappa}(y,t)}{\partial t} &= \frac{1}{2}\sigma_P^2\frac{\partial^2\psi_{1,\kappa}(y,t)}{\partial y^2} + p\frac{\partial\psi_{1,\kappa}(y,t)}{\partial y}, \quad (y,t) \in (0,\kappa)\times(0,1], \end{cases} \tag{3.0.8}$$

$$\begin{cases} \psi_{2,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{2,\kappa}(0,t) &= 0, \quad t \in [0,1]\,, \\ \psi_{2,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial\psi_{2,\kappa}(y,t)}{\partial t} - L\psi_{2,\kappa} &= H_{1,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa)\times(0,1], \end{cases} \tag{3.0.9}$$

where

$$\begin{aligned} H_{1,\kappa}\,(y,t) = \frac{1}{2}\sigma_R^2 y^2\frac{\partial^2\psi_{1,\kappa}\,(y,t)}{\partial^2 y^2} &+ ry\frac{\partial\psi_{1,\kappa}\,(y,t)}{\partial y} - \lambda\psi_{1,\kappa}\,(y,t) \\ &+ \lambda\int_0^y \psi_{1,\kappa}\,(y-z,t)\,dF(z) + \lambda\bar{F}(y), \end{aligned}$$

and

$$\begin{cases} \psi_{3,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{3,\kappa}(0,t) &= 0, \quad t \in [0,1]\,, \\ \psi_{3,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial\psi_{3,\kappa}y,t)}{\partial t} - A\psi_{3,\kappa}\,(y,t) &= H_{2,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa)\times(0,1]. \end{cases} \tag{3.0.10}$$

Here

$$H_{2,\kappa}\,(y,t) = -\lambda\psi_{2,\kappa}(y,t) + \lambda\int_0^y \psi_{2,\kappa}\,(y-z,t)\,dF(z).$$

Now we focus our attention on the first of the above three equations (3.0.8). Existence and regularity of a solution to that equation can be determined from the close relation between this equation and a certain *passage time* of the Brownian motion $W_{p,t}$. Consider the following three equations.

$$\begin{cases} \psi_1^*(y,0) & = 0, \quad y > 0, \\ \psi_1^*(0,t) & = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_1^*(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_1^*(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1^*(y,t)}{\partial y^2}, \quad (y,t) \in (0,\infty) \times (0,1]. \end{cases} \tag{3.0.11}$$

$$\begin{cases} \psi_{1,\kappa}^*(y,0) & = 0, \quad y \in (0,\kappa), \\ \psi_{1,\kappa}^*(0,t) & = 1, \quad t \in [0,1], \\ \psi_{1,\kappa}^*(\kappa,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_{1,\kappa}^*(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_{1,\kappa}^*(y,t)}{\partial y^2}, \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{3.0.12}$$

and

$$\begin{cases} \psi_1(y,0) & = 0, \quad y > 0, \\ \psi_1(0,t) & = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_1(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_1(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1(y,t)}{\partial y^2} + p\frac{\partial \psi_1(y,t)}{\partial y}, \quad (y,t) \in (0,\infty) \times (0,1]. \end{cases} \tag{3.0.13}$$

Let

$$\tau_0 = \inf\{t \geq 0 : y + \sigma_P W_{P,t} < 0\},$$
$$\tilde{\tau}_0 = \inf\{t \geq 0 : y + pt + \sigma_P W_{P,t} < 0\},$$
$$\tau_\kappa = \inf\{t \geq 0 : y + \sigma_P W_{P,t} > \kappa\},$$

and let

$$\tilde{\tau}_\kappa(y) = \inf\{t \geq 0 : y + pt + \sigma_P W_{P,t} > \kappa\}.$$

Since $\psi_1^*(y,t)$ is just the probability $P(\tau_0 \leq t)$ it is well known that

$$\psi_1^*(y,t) = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sigma_P\sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{y}{\sigma_P\sqrt{2\pi}} \int_0^t s^{-\frac{3}{2}} e^{-\frac{y^2}{2\sigma_P^2 s}} ds$$

is a unique solution of equation (3.0.11). Equation (3.0.12) corresponds to the probability $P(\tau_0 \leq \min(\tau_\kappa, t))$. It is known (see exercise 2.8.11 in Karatzas and Shreve (1991)) that equation (3.0.12) has the unique solution

$$\psi_{1,\kappa}^*(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (2n\kappa + y) \int_0^t s^{-\frac{3}{2}} e^{-\frac{(2n\kappa+y)^2}{2\sigma_P^2 s}} ds.$$

Similarly, equation (3.0.13) corresponds to the probability $P(\tilde{\tau}_0 \leq t)$ and (3.0.8) corresponds to the probability $P(\tilde{\tau}_0 \leq \min(\tilde{\tau}_\kappa, t))$. Similar applications of Girsanov's theorem, as in section 3.5.C in Karatzas and Shreve (1991)), yield that

$$\psi_1(y,t) = \frac{y}{\sigma_P\sqrt{2\pi}} \int_0^t s^{-\frac{3}{2}} e^{-\frac{(y+ps)^2}{2\sigma_P^2 s}} ds \tag{3.0.14}$$

and

$$\psi_{1,\kappa}(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (2n\kappa + y) \int_0^t s^{-\frac{3}{2}} e^{-\left[\frac{(2n\kappa+y)^2}{2\sigma_P^2 s} + \hat{p}y + \frac{1}{2}\sigma_P^2\hat{p}^2 s^2\right]} ds, \tag{3.0.15}$$

where

$$\hat{p} = \frac{p}{\sigma_P^2}.$$

We will return to equation (3.0.13) and the solution (3.0.14) later in the article. Unfortunately it will turn out to be much more difficult to establish the existence of a solution of equation (3.0.9). Uniqueness, however, is relatively straightforward to establish, as outlined below.

Theorem 3.0.1. *If*

$$g_1(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$$

and

$$g_2(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$$

are two classical solutions of equation (3.0.9)*, then*

$$g_1(y,t) = g_2(y,t),$$

for every $(y,t) \in [0,\kappa] \times [0,1]$.

Proof. Since $g_1(y,t)$ and $g_1(y,t)$ are assumed to be solutions of equation (3.0.9) this follows from Theorem I.3.1 in Garroni and Menaldi (1992) by considering the differences

$$g_1(y,t) - g_2(y,t).$$

□

Before proceeding to establish existence of a solution of (3.0.9) we will first need to establish some auxiliary results and then introduce the concept of a Green function.

Proposition 3.0.1.

For every $x \in \mathbb{R}$, $t > 0$ *and for any* $\alpha, c > 0$ *and* $0 < \theta < c$

$$\sup_{t\in(0,1]} |x|^\alpha \exp\left(-c\frac{x^2}{t}\right) \le Ct^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right),$$

where

$$C = \left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right).$$

Proof. Let $(t,\theta) \in (0,1] \times (0,c)$. We observe that since

$$\left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right) t^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right) > 0,$$

there must exist some $\epsilon \in \left(0, \frac{\frac{\alpha}{2}}{\theta}\right)$ such that

$$|x|^\alpha \exp\left(-c\frac{x^2}{t}\right) < Ct^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right),$$

for every $x \in [0,\epsilon]$. Moreover, for every $x \geq \epsilon$

$$|x|^{\alpha} \exp\left(-\theta \frac{|x|^2}{t}\right) \leq \left(t^{\frac{\alpha}{2}}\right) \left(\sup_{z \in [\frac{\epsilon^2}{t},\infty)} z^{\frac{\alpha}{2}} \exp\left(-\theta z\right)\right).$$

Let $h(z) = z^{\frac{\alpha}{2}} \exp(-\theta z)$. Differentiating h we get that

$$h'(z) = z^{\frac{\alpha}{2}} \left(-\theta + \frac{\alpha}{2} z^{-1}\right) \exp\left(-\theta z\right),$$

which is positive for $z \in (0, \frac{\frac{\alpha}{2}}{\theta})$, 0 for $z = \frac{\frac{\alpha}{2}}{\theta}$ and negative for $z > \frac{\frac{\alpha}{2}}{\theta}$. Thus

$$\sup_{z \in [\frac{\epsilon^2}{t},\infty)} z^{\frac{\alpha}{2}} \exp\left(-\theta z\right) = \left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right).$$

Since t was arbitrarily chosen the result follows. □

Proposition 3.0.2. *For every $(x,t,\xi,\vartheta) \in \mathbb{R} \times (0,1] \times \mathbb{R} \times [0,t)$ and $p,q,c > 0$,*

$$\int_{\vartheta}^{t} (t-s)^{p-1} (s-\vartheta)^{q-1} \, ds = (t-\vartheta)^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

and

$$\int_{\mathbb{R}} \exp\left(-c\left[\frac{|x-z|^2}{t-s} + \frac{|z-\xi|^2}{s-\vartheta}\right]\right) dz$$
$$= \left(\frac{\pi}{c}\right)^{\frac{1}{2}} \left[\frac{(t-s)(s-\vartheta)}{t-\vartheta}\right]^{\frac{1}{2}} \exp\left(-c\frac{(x-\xi)^2}{(t-\vartheta)}\right),$$

where $\Gamma(x)$ is the Gamma function

$$\Gamma(x) := \int_0^{\infty} z^{x-1} \exp(-z) \, dz, \quad x > 0.$$

Proof. These identities are proven in section 1.1 in Garroni and Menaldi (2002). □

Proposition 3.0.3. *Let $c > 0$, $d \in \mathbb{R}$, let $-\infty < a_1 < a_2 < \infty$, $-\infty < b_1 < b_2 < \infty$ and let*

$$\mathcal{D}_{ab} := (a_1,a_2) \times (0,1] \times (b_1,b_2) \times [0,t).$$

Let $h(y,t,\xi,\vartheta)$ be a continuous function on $\mathcal{D}_{ab}$ such that $h(y,t,\xi,\vartheta)$ is differentiable with respect to t on $\mathcal{D}_{ab}$, and for some constant C

$$|h(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-d} \exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.16}$$

and

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-(d+1)} \exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$*. Then, for some constant* C

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}+(t'-\vartheta)^{-(d+\alpha)}\right]\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right) \tag{3.0.17}$$

and

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta}\right)+(t'-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right] \tag{3.0.18}$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$*, every* $t'\in(\vartheta,1]$*, and every* $\alpha\in[0,1]$*.*

Proof. Let $t_2=\max(t,t')$ and $t_1=\min(t,t')$. Assume first that

$$t_2-t_1\geq t_1-\vartheta.$$

We note that in this case

$$t_2-\vartheta\leq 2(t_2-t_1).$$

Hence, for every $\alpha\in[0,1]$

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq|h(y,t,\xi,\vartheta)|+|h(y,t',\xi,\vartheta)|$$

$$\leq 2C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+(t'-\vartheta)^{-(d+\alpha)}\exp\left(-c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right].$$

From the above it is obvious that for this case the inequality (3.0.17) also holds. Now, assume instead that

$$t_2-t_1<t_1-\vartheta.$$

We first observe that under this condition

$$t_2-\vartheta<2(t_1-\vartheta)$$

and hence we only need to prove that the inequality (3.0.17) holds. Moreover, it follows from the mean value theorem that

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|\left[(t-\vartheta)^{-(c+1)}+(t'-\vartheta)^{-(d+1)}\right]\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-d\frac{(y-\xi)^2}{t'-\vartheta}\right)\right).$$

Thus the required bound (3.0.17), and hence (3.0.18), can be obtained, since

$$|t-t'| \leq \min\left(t-\vartheta, t'-\tau\right).$$

□

Corollary 1. *Assume that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* ϑ *on* $\mathcal{D}_{ab}$, *that* (3.0.16) *holds and that*

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial\vartheta}\right| \leq C(t-\vartheta)^{-(d+1)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$. *Then, for some constant* C

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y,t,\xi,\vartheta')| \leq C|\vartheta-\vartheta'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}+(t-\vartheta')^{-(d+\alpha)}\right]\\ \times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta'}\right)\right).\end{aligned}$$

Hence

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y,t,\xi,\vartheta')| \leq C|t-t'|^{\alpha}\Bigg[(t-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta}\right)\\ +(t-\vartheta')^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta'}\right)\Bigg].\end{aligned}$$

Proposition 3.0.4. *Let* $c>0$, $d\in\mathbb{R}$, *let* $-\infty<a_1<a_2<\infty$, $-\infty<b_1<b_2<\infty$ *and let*

$$\mathcal{D}_{ab} := (a_1,a_2)\times(0,1]\times(b_1,b_2)\times[0,t).$$

Let

$$\mathcal{D}_{\bar{a}b} := [a_1,a_2]\times(0,1]\times(b_1,b_2)\times[0,t)$$

and let $h(y,t,\xi,\vartheta)$ *be a continuous function on* $\mathcal{D}_{\bar{a}b}$ *such that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* y *on* $\mathcal{D}_{ab}$. *Assume that, for some constant* C,

$$|h(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-d}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.19}$$

on $\mathcal{D}_{\bar{a}b}$ *and*

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial y}\right| \leq C(t-\vartheta)^{-\left(d+\frac{1}{2}\right)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.20}$$

on $\mathcal{D}_{ab}$. *Then, for some constant* C,

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y',t,\xi,\vartheta)| \leq& C\exp(c)|y-y'|^{\alpha}(t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)}\\ &\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y'-\xi)^2}{t-\vartheta}\right)\right)\end{aligned} \tag{3.0.21}$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{\bar{a}b}$, *every* $y'\in[a_1,a_2]$, *and for every* $\alpha\in[0,1]$.

Proof. Let $y_2 = \max(y, y')$ and $y_1 = \min(y, y')$. Assume first that

$$t - \vartheta \leq |y - y'|^2 .$$

We note that in this case

$$(t-\vartheta)^{-d} \leq (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} |y - y'|^{\alpha} ,$$

and hence in this case it follows from the bound (3.0.19) that the bound (3.0.21) holds. In the rest of the proof we will assume that

$$t - \vartheta > |y - y'|^2 .$$

Because of the continuity on $\mathcal{D}_{\bar{a}b}$ we can also assume that

$$a_1 < y_1 < y_2 < a_2.$$

We note that in this case

$$|y - y'| (t-\vartheta)^{-\left(d+\frac{1}{2}\right)} \leq |y - y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} . \tag{3.0.22}$$

Assume in addition that $\xi \notin (y_1, y_2)$. For this case it follows from the Middle Value Theorem and the bounds (3.0.20) and (3.0.22) that,

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y',t,\xi,\vartheta)| \leq & C |y-y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c \frac{(y-\xi)^2}{t-\vartheta} \right) + \exp\left(-c \frac{(y'-\xi)^2}{t-\vartheta} \right) \right). \end{aligned}$$

The last possible case is that (3.0.22) holds and that $\xi \in (y_1, y_2)$. In this case we note that

$$\min \left(\exp\left(-c \frac{(y-\xi)^2}{t-\vartheta} \right), \exp\left(-c \frac{(y'-\xi)^2}{t-\vartheta} \right) \right) \geq \exp(-c) ,$$

and hence it follows from the Middle Value Theorem and the bounds (3.0.20) and (3.0.22) that

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y',t,\xi,\vartheta)| \leq & C \exp(c) |y-y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c \frac{(y-\xi)^2}{t-\vartheta} \right) + \exp\left(-c \frac{(y'-\xi)^2}{t-\vartheta} \right) \right). \end{aligned}$$

□

Corollary 2. *Assume that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* ξ *on* $\mathcal{D}_{ab}$*, that* (3.0.19) *holds and that, for some constant* C*,*

$$\left| \frac{\partial h(y,t,\xi,\vartheta)}{\partial \xi} \right| \leq C (t-\vartheta)^{-\left(d+\frac{1}{2}\right)} \exp\left(-c \frac{(y-\xi)^2}{t-\vartheta} \right),$$

for every $(y,t,\xi,\vartheta) \in \mathcal{D}$*. Then, for some constant* C

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y,t,\xi',\vartheta)| \leq & C \exp(c) |\xi-\xi'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c \frac{(y-\xi)^2}{t-\vartheta} \right) + \exp\left(-c \frac{(y-\xi')^2}{t-\vartheta} \right) \right). \end{aligned}$$

Proposition 3.0.5. (i) *If $a, b > 0$, then*

$$\int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = \int_{\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp(-z)\, dz,$$

$$\frac{\partial}{\partial a} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = -2b^{-\frac{1}{2}} \exp\left(-\frac{a^2}{b}\right).$$

and

$$\frac{\partial}{\partial b} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = -ab^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right).$$

(ii) *If $a \in \mathbb{R}$ and $b > 0$, then for some constant C*

$$\int_0^b \left| as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) \right| ds \leq C \exp\left(-\frac{a^2}{b}\right),$$

(iii) *If $a \neq 0$ and $b > 0$, then for some constant C*

$$\left| \frac{\partial}{\partial a} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq Cb^{-\frac{1}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial^2}{\partial a^2} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq Cab^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial}{\partial b} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq Cab^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial^3}{\partial a^3} \int_0^b as^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq Cb^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{a^2}{b}\right).$$

Proof. For part (i): This can be calculated using the substitution $z = \frac{a^2}{s}$.

For part (ii): This is obvious if $a = 0$. Assume that $a \neq 0$ and that

$$|a| \geq \sqrt{b}.$$

For this case a simple calculation using the identity given in part (i) yields that the stated claim holds. In the following assume that $a \neq 0$. and that

$$|a| \leq \sqrt{b}. \tag{3.0.23}$$

Consider

$$I_1 := \int_0^{\frac{b}{2}} |a|\, s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds,$$

and

$$I_2 := \int_{\frac{b}{2}}^{b} |a|\, s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds.$$

The identity given in part (ii) yields that

$$\begin{aligned} I_1 &= \int_{2\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp(-z)\, dz \\ &\leq \exp\left(-\frac{a^2}{b}\right) \int_{2\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp\left(-\frac{1}{2}z\right) dz \\ &\leq C \exp\left(-\frac{a^2}{b}\right) \end{aligned}$$

for some constant C. Under the assumption (3.0.23) a simple calculation yields that

$$I_2 \leq \exp\left(-\frac{a^2}{b}\right).$$

The other bounds follow from Proposition 3.0.1. □

The most important concept in this article is that of a *Green function*, which we will now define, adapted to equation (3.0.9). '

Definition 3.0.1. *A function $G_{L^*,\kappa}(y,t,\xi,\vartheta)$ defined in the domain $\bar{\mathcal{D}}_\kappa$, where*

$$\begin{cases} \mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in (0,\kappa), \xi \in (0,\kappa), 0 \leq \vartheta < t \leq 1\}, \\ \partial\mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in \{0,\kappa\}, \xi \in (0,\kappa), 0 \leq \vartheta < t \leq 1\}, \\ \bar{\mathcal{D}}_\kappa = \mathcal{D}_\kappa \cup \partial\mathcal{D}_\kappa \end{cases}$$

is called a Green function on $\bar{\mathcal{D}}_\kappa$ for the differential operator L^* with Dirichlet boundary condition if it satisfies:

(i) *$G_{L^*,\kappa}(y,t,\xi,\vartheta)$ is continuous in (y,t), and locally integrable in (ξ,ϑ),*

(ii)

$$\begin{aligned} \frac{\partial G_{L^*,\kappa}(y,t,\xi,\vartheta)}{\partial t} &- L^* G_{L^*,\kappa}(y,t,\xi,\vartheta) \\ &= \delta(y-\xi)\,\delta(t-\vartheta), \quad \textit{in } \mathcal{D}_\kappa, \end{aligned}$$

(iii)

$$\lim_{t-\vartheta \downarrow 0} G_{L^*,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi), \quad \textit{in } \mathcal{D}_\kappa, \tag{3.0.24}$$

(iv)

$$G_{L^*,\kappa}(y,t,\xi,\vartheta) = 0, \quad \textit{in } \partial\mathcal{D}_\kappa,$$

In the above $\delta(y,t)$ is the Dirac measure at 0.

In order to derive existence and some regularity of a solution of equation (3.0.9) we want to use use Theorem VI.2.2 in Garroni and Menaldi (1992). This theorem, however, requires the right hand side of the equation (in our case the function $H_{1,\kappa}(y,t)$) to belong to the function space $C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$ defined below.

Definition 3.0.2. *Let $C^0([0,\kappa]\times[0,1],\mathbb{R})$ be the Banach space of bounded, real valued, continuous functions on $[0,\kappa]\times[0,1]$, with the supremum norm.*

Let $g(y,t) \in C^0([0,\kappa]\times[0,1],\mathbb{R})$. We will say that

$$g \in C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$$

or that g is Hölder continuous on $[0,\kappa]\times[0,1]$ with index α
if g has a finite value for the semi norm

$$\inf\left\{C \geq 0 : |g(y,t) - g(y',t)| \leq C|y-y'|^{\alpha}, \forall y,y' \in [0,\kappa] \ \ and \ \forall t \in [0,1]\right\}$$
$$+ \inf\left\{C \geq 0 : |g(y,t) - g(y,t')| \leq C|t-t'|^{\frac{\alpha}{2}}, \forall y \in [0,\kappa] \ \ and \ \forall t,t' \in [0,1]\right\}.$$

Alas, because of the singularity at the origin it is clear that $H_{1,\kappa}(y,t) \notin C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$ and we will have to rely on a more indirect approach. But first we need to explore a bit more the local regularity of $H_{1,\kappa}(y,t)$ on the inner domain, as we do in the next two results.

Definition 3.0.3. *Let*
$$c_0 = \frac{1}{2\sigma_P^2}.$$

Lemma 3.0.1. *There exists a constant C such that for every $(y,t) \in (0,\kappa)\times(0,1]$*

$$0 \leq \psi_{1,\kappa}(y,t) \leq C\exp\left(-c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial y}\right| \leq Ct^{-\frac{1}{2}}\exp\left(-c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial y}\right| \leq Ct^{-\frac{1}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial^2\psi_{1,\kappa}(y,t)}{\partial y^2}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial t}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

and
$$\left|\frac{\partial^3\psi_{1,\kappa}(y,t)}{\partial y^3}\right| \leq Ct^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right).$$

Proof. We first observe that in the formula (3.0.15) the singularity at the origin of $\psi_{1,\kappa}$ is taken care of by the term $n=0$, i.e. the term

$$\psi_1(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}}\int_0^t ys^{-\frac{3}{2}}e^{-c_0\frac{(y+ps)^2}{s}}ds.$$

From Leibniz' rule it follows that

$$\frac{\partial\psi_1(y,t)}{\partial t} = \frac{y}{\sigma_P\sqrt{2\pi}}t^{-\frac{3}{2}}\exp\left(-c_0\frac{(y+pt)^2}{t}\right).$$

Because of Proposition 3.0.1 we conclude that for some constant C

$$\left|\frac{\partial\psi_1(y,t)}{\partial t}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right).$$

Moreover, similar calculations as in the proof of Proposition 3.0.5 yield that for some constant C

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \le C t^{-\frac{l}{2}} \exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right)$$

for $l \in \{1,2,3\}$. Similar calculations as in the proof of Proposition 3.0.5 yield that the stated bounds hold for this term. In this calculation it is helpful to use the fact that the second derivative with respect to y can be expressed in terms of the derivative with respect to t and the first derivative with respect to y (a consequence of $\psi_1(y,t)$ being a solution of equation (3.0.13). The ratio test shows that the full series expression for $\psi_{1,\kappa}(y,t)$ given in (3.0.15) converges uniformly and thus $\psi_{1,\kappa}(y,t)$ can be differentiated term by term. For $|n| \ge 1$ we note that $(2n-\kappa y)^2 \ge \kappa^2$, so an application of Proposition 3.0.1 yields that all the other terms are smooth and sufficiently bounded for the whole series to obey the stated bounds. □

Lemma 3.0.2. *There exists a constant C such that for every $(y,t) \in (0,\kappa) \times (0,1]$, every $y_1, y_2 \in (0,\kappa)$, every $t_1, t_2 \in (0,1]$ and every $\alpha \in (0,1]$ the following bounds hold:*

$$\int_0^y \psi_{1,\kappa}(y-z,t)dF(z) + \bar{F}(y) \le C\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + \bar{F}\left(\frac{y}{2}\right), \qquad (3.0.25)$$

$$\begin{aligned}\Bigg|\int_0^{y_2} &\psi_{1,\kappa}(y_2-z,t)dF(z) + \bar{F}(y_2)\\ &- \left(\int_0^{y_1} \psi_{1,\kappa}(y_1-z,t)dF(z) + \bar{F}(y_1)\right)\Bigg|\\ \le{}& C|y_2-y_1|^{\alpha} t^{-\frac{\alpha}{2}} \qquad (3.0.26)\\ &\times\left(\exp\left(-\frac{1}{8}c_0\frac{y_1^2}{t}\right) + \exp\left(-\frac{1}{8}c_0\frac{y_2^2}{t}\right)\right.\\ &\left.+\bar{F}\left(\frac{y_1}{2}\right) + \bar{F}\left(\frac{y_2}{2}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\left|\int_0^y \psi_{1,\kappa}(y-z,t_2)dF(z) + \bar{F}(y) - \left(\int_0^y \psi_{1,\kappa}(y-z,t_1)dF(z) + \bar{F}(y)\right)\right|\\ \le C|t_2-t_1|^{\alpha}\left(t_1^{-\alpha}+t_2^{-\alpha}\right)\\ \times\left(\exp\left(-\frac{1}{8}c_0\frac{y^2}{t_1}\right) + \exp\left(-\frac{1}{8}c_0\frac{y^2}{t_2}\right) + \bar{F}\left(\frac{y}{2}\right)\right).\end{aligned} \qquad (3.0.27)$$

Proof. Let

$$\tilde{\psi}_{1,\kappa}(y,t) := \begin{cases}\psi_{1,\kappa}(y,t), & (y,t) \in [0,\kappa]\times(0,1],\\ 1, \quad , & (y,t) \in (-\infty,0)\times(0,1]\end{cases}$$

We note that, for every $t \in (0,1]$, $\psi_{1,\kappa}(0,t) = 1$, and thus $\tilde{\psi}_{1,\kappa}(y,t)$ is continuous on $(-\infty,\kappa)\times(0,1]$. Moreover, since $F(y)$ is a probability distribution it follows that, for every $(y,t) \in [0,\kappa] \times (0,1]$

$$\int_0^y \psi_{1,\kappa}(y-z,t)dF(z) + \bar{F}(y) = \int_0^\infty \tilde{\psi}_{1,\kappa}(y-z,t)dF(z).$$

Let $\tilde{y} = \min(y_2, y_1)$. From the identity above it follows that

$$\left|\int_0^{y_2} \psi_{1,\kappa}(y_2-z,t)dF(z) - \int_0^{y_1} \psi_{1,\kappa}(y_1-z,t)dF(z)\right| \\ \leq |I_1| + |I_2|$$

where

$$I_1 = \int_0^{\frac{\tilde{y}}{2}} \left(\psi_{1,\kappa}(y_2-z,t) - \psi_{1,\kappa}(y_1-z,t)\right) dF(z),$$

and

$$I_2 = \int_{\left\{z:z>\frac{\tilde{y}}{2}\right\}} \left(\psi_{1,\kappa}(y_2-z,t) - \psi_{1,\kappa}(y_1-z,t)\right) dF(z),$$

The stated bounds (3.0.26) and (3.0.27) can be obtained from considering I_1 and I_2, applying Proposition 3.0.4 and Proposition 3.0.3 and using the bounds given in Lemma 3.0.1. □

Proposition 3.0.6. *There exists a constant C such that the bounds stated below hold for every $y, y_1, y_2 \in (0,\kappa)$ and every $t, t_1, t_2 \in (0,1]$ and every $\alpha \in [0,1]$.*

$$|H_{1,\kappa}(y,t)| \leq C\left(\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + \bar{F}\left(\frac{y}{2}\right)\right),$$

$$\begin{aligned} |H_{1,\kappa}(y_2,t) - H_{1,\kappa}(y_1,t)| \leq C\,&|y_2-y_1|^\alpha t^{-\frac{\alpha}{2}} \\ &\times\left(\exp\left(-\frac{1}{8}c_0\frac{y_1^2}{t}\right) + \exp\left(-\frac{1}{8}c_0\frac{y_2^2}{t}\right)\right. \\ &\left.+\bar{F}\left(\frac{y_1}{2}\right) + \bar{F}\left(\frac{y_2}{2}\right)\right), \end{aligned} \tag{3.0.28}$$

and

$$\begin{aligned} |H_{1,\kappa}(y,t_2) - H_{1,\kappa}(y,t_1)| \leq C\,&|t_2-t_1|^\alpha \left(t_1^{-\alpha} + t_2^{-\alpha}\right) \\ &\times\left(\exp\left(-\frac{1}{8}c_0\frac{y^2}{t_1}\right) + \exp\left(-\frac{1}{8}c_0\frac{y^2}{t_2}\right) + \bar{F}\left(\frac{y}{2}\right)\right). \end{aligned} \tag{3.0.29}$$

Proof. The bounds stated above can be obtained from the bounds given in Lemma 3.0.1 and Lemma 3.0.2 and applying Proposition 3.0.4 and Proposition 3.0.3. □

Since $H_{1,\kappa}(y,t)$ is not Hölder continuous we will instead work with a sequence of Hölder continuous functions that converge to $H_{1,\kappa}(y,t)$.

Definition 3.0.4. *For every $n \in 2,3,\ldots,$ let*

$$\eta_n(t) := \begin{cases} 0, & t \in \left[0, \frac{1}{2n}\right], \\ \exp\left(\frac{1}{\frac{1}{2n}-t} + \frac{1}{\frac{1}{2n}}\right)\left(1 - \exp\left(\frac{1}{t-\frac{1}{n}}\right)\right), & t \in \left(\frac{1}{2n}, \frac{1}{n}\right), \\ 1, & t \in \left[\frac{1}{n}, 1\right], \end{cases}$$

and let

$$H_{1,\kappa,n}(y,t) := \eta_n(t) H_{1,\kappa}(y,t), \quad (0,t) \in [0,\kappa] \times [0,1] .$$

The lemma below states that, for any fixed n, the $H_{1,\kappa,n}(y,t)$ is indeed a Hölder continuous function. Because of this property we can invoke Theorem VI.2.2 in Garroni and Menaldi (1992) to establish existence of a solution of the following equation:

$$\begin{cases} \psi_{2,\kappa,n}(y,0) & = 0, \quad y \in (0,\kappa), \\ \psi_{2,\kappa,n}(0,t) & = 0, \quad t \in [0,1], \\ \psi_{2,\kappa,n}(\kappa,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial t} & -\frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \psi_{2,\kappa,n}(y,t)}{\partial y^2} - (p+ry) \frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial y} \\ & = H_{1,\kappa,n}(y,t), \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{3.0.30}$$

Moreover, Theorem VI.2.2 also gives us a representation formula for $\psi_{2,\kappa,n}(y,0)$, which we will later use to show that

$$\lim_{n\to\infty} \psi_{2,\kappa,n}(y,t)$$

is a classical solution of equation (3.0.9).

Lemma 3.0.3. *For every $n \in 2,3,\ldots,$*

(i) *$\eta_n(t)$ is differentiable on $\left(0,\frac{1}{n}\right)$, and for every $t \in [0,1]$*

$$0 \le \eta_n(t) \le 1.$$

(ii) *There exists a constant C_n, depending on n, such that, for every $\alpha \in (0,1]$, every $(y,t) \in [0,\kappa] \times [0,1]$, every $y_1, y_2 \in [0,\kappa]$ and every $t_1, t_2 \in [0,1]$*

$$|H_{1,\kappa,n}(y_1,t) - H_{1,\kappa,n}(y_2,t)| \le C_n |y_2 - y_1|^\alpha$$

and

$$|H_{1,\kappa,n}(y,t_2) - H_{1,\kappa,n}(y,t_1)| \le C_n |t_2 - t_1|^\alpha .$$

Proof. Without loss of generality we can assume that $t_2 \ge t_1$. It follows from the bounds given in Proposition 3.0.6 that there exists a constant C such that, for every $(y,t) \in [0,\kappa] \times \left[\frac{1}{2n},1\right]$, every $y_1, y_2 \in [0,\kappa]$, and every $t_1, t_2 \in \left[\frac{1}{2n},1\right]$,

$$|H_{1,\kappa}(y_2,t) - H_{1,\kappa}(y_1,t)| \le C n^{-\frac{1}{2}} |y_2 - y_1| , \tag{3.0.31}$$

and

$$|H_{1,\kappa}(y,t_2) - H_{1,\kappa}(y,t_1)| \le C n^{-1} |t_2 - t_1| . \tag{3.0.32}$$

Now, for fixed $n \in 2,3,\ldots,$ consider the function $\eta_n(t)$. An inspection yields that

$$0 < \eta_n(t) < 1$$

for every $t \in \left(\frac{1}{2n},\frac{1}{n}\right)$. Since $0 \le \eta_n(t) \le 1$ and since $\eta_n(t)$ vanishes for $t < \frac{1}{2n}$ it follows from the bound (3.0.31) that, for every $y_1, y_2 \in [0,\kappa]$,

$$|H_{1,\kappa,n}(y_2,t) - H_{1,\kappa}(y_1,t)| \le C n^{-\frac{1}{2}} |y_2 - y_1| .$$

Moreover, $H_{1,\kappa,n}(y,t)$ is a bounded function, thus, for some (other) constant C,

$$\left|H_{1,\kappa,n}(y_2,t) - H_{1,\kappa}(y_1,t)\right| \le Cn^{-\frac{1}{2}} \left|y_2 - y_1\right|^{\alpha},$$

for any $\alpha \in (0,1]$. Now, consider $\eta_n(t)$. Taking the limit we observe that

$$\lim_{t\downarrow\frac{1}{2n}} \eta_n(t) = 0,$$

while

$$\lim_{t\uparrow\frac{1}{n}} \eta_n(t) = 1,$$

thus $\eta_n(t)$ is continuous. Moreover, it can be calculated that the limit

$$\lim_{t\uparrow\frac{1}{n}} \eta_n'(t)$$

exists. Hence $\eta_n'(t)$ is bounded by some constant $\hat{C}$ on $\left(0,\frac{1}{n}\right)$. From the bound and the identities above, it follows that, for some other constant K

$$\left|\eta_n(t_2) - \eta_n(t_1)\right| \le K \left|t_2 - t_1\right|^{\alpha},$$

for any $\alpha \in (0,1]$, and thus, for some constant C_n

$$\begin{aligned} \left|H_{1,\kappa,n}(y,t_2) - H_{1,\kappa,n}(y,t_1)\right| &\le \left|H_{1,\kappa}(y,t_2)\left(\eta_n(t_2) - \eta_n(t_1)\right)\right| \\ &\quad + \left|\left(H_{1,\kappa}(y,t_2) - H_{1,\kappa,n}(y,t_1)\right)\right| \eta_n(t_1) \\ &\le C_n \left|t_2 - t_1\right|^{\alpha}. \end{aligned}$$

□

Since $H_{1,\kappa,n}(y,t)$ is Hölder continuous, we get an existence and representation result for equation (3.0.30), as stated in the theorem below.

Theorem 3.0.2. (i) *There exists a unique Green function $G_{L,\kappa}(y,t,\xi,\vartheta)$ associated with the differential operator L and Dirichlet boundary conditions on the domain $\mathcal{D}_\kappa$, i.e. satisfying the conditions in Definition 3.0.1. Furthermore, there exist positive constants C_κ and c_κ, depending on κ, such that, for $l \in \{0,1,2\}$,*

$$\left|\frac{\partial^l G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C_\kappa (t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-c_\kappa \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

and such that

$$\left|\frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| \le C_\kappa (t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_\kappa \frac{(y-\xi)^2}{(t-\vartheta)}\right).$$

(ii) *For any fixed $n \in 2,3,\ldots,$,*

$$\psi_{2,\kappa,n}(y,t) = \int_0^t \int_0^\kappa G_{L,\kappa}(y,t,\xi,\vartheta) H_{1,\kappa,n}(\xi,\vartheta) d\xi d\vartheta.$$

is a unique, bounded classical solution of equation (3.0.30).

Proof. This can be shown to follow from Theorem VI.2.1 and Theorem VI.2.2 in Garroni and Menaldi (1992). □

The next result is the first step to prove that $\psi_{2,\kappa,n}(y,t)$ converges to a solution of (3.0.9) of the form given below.

Definition 3.0.5. *Let*

$$\tilde{\psi}_{2,\kappa}(y,t) = \begin{cases} 0, & (y,t) \in (0,\kappa) \times \{0\}, \\ 0, & (y,t) \in \{0,\kappa\} \times [0,1], \\ \int_0^t \int_0^\kappa G_{L,\kappa}(y,t,\xi,\vartheta) H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta, & (y,t) \in (0,\kappa) \times (0,1]. \end{cases}$$

Lemma 3.0.4. *There exists a constant C_κ, depending on κ, such that, for any $(y_0,t_0) \in (0,\kappa) \times (0,1]$,*

(i)
$$\left|\tilde{\psi}_{2,\kappa}(y_0,t_0)\right| \le C_\kappa t_0. \tag{3.0.33}$$

Moreover, for every $(y_1,t_1) \in \{0,\kappa\} \times [0,1]$

$$\lim_{(y,t)\to(y_1,t_1)} \tilde{\psi}_{2,\kappa}(y,t) = 0.$$

(ii)
$$\tilde{\psi}_{2,\kappa}(y,t) \in C^{2,1}\left((0,\kappa) \times (0,1), \mathbb{R}\right).$$

Moreover, for $l \in \{0,1,2\}$ and $n > \frac{2}{t_0}$,

$$\left| \left.\frac{\partial^l \tilde{\psi}_{2,\kappa}(y,t_0)}{\partial y^l}\right|_{y=y_0} - \left.\frac{\partial^l \psi_{2,\kappa,n}(y,t_0)}{\partial y^l}\right|_{y=y_0} \right| \le C_\kappa \frac{t_0^{-\frac{l}{2}}}{n},$$

and

$$\left| \left.\frac{\partial \tilde{\psi}_{2,\kappa}(y_0,t)}{\partial t}\right|_{t=t_0} - \left.\frac{\partial \psi_{2,\kappa,n}(y_0,t)}{\partial t}\right|_{t=t_0} \right| \le C_\kappa \frac{t_0^{-1}}{n}.$$

Proof. For part (i): It follows from the bounds given in Theorem 3.0.2 and the boundedness of $(H_{1,\kappa}(\xi,\vartheta)$ that there exists a constant K_κ, depending on κ, such that

$$|G_{L,\kappa}(y,t,\xi,\vartheta)| \, |H_{1,\kappa}(\xi,\vartheta)| \le K_\kappa (t-\vartheta)^{-\frac{1}{2}}, \tag{3.0.34}$$

for every $(y,t,\xi,\vartheta)\,(0,\kappa) \times (0,1] \times (0,\kappa) \times [0,t)$. A calculation using the bound above yields the bound (3.0.33). Moreover, because of the bound (3.0.34), the Dominated Convergence Theorem can be invoked to yield that

$$\lim_{(y,t)\to(y_1,t_1)} \tilde{\psi}_{2,\kappa}(y,t) = 0,$$

for every $(y_1,t_1) \in \{0,\kappa\} \times [0,1]$.

For part (ii): Let $(y_0, t_0) \in (0, \kappa) \times (0, 1]$, and let

$$n \in \left\lceil \frac{2}{t_0} \right\rceil, \left\lceil \frac{2}{t_0} \right\rceil + 1, \left\lceil \frac{2}{t_0} \right\rceil + 2 \ldots, .$$

We observe that, for every $(y, t) \in (0, \kappa) \times (\frac{t_0}{2}, 1]$,

$$\tilde{\psi}_{2,\kappa}(y, t) = \psi_{2,\kappa,n}(y, t) + I_n(y, t),$$

where

$$I_n(y, t) = \int_0^{\frac{1}{n}} \int_0^{\kappa} G_{L,\kappa}(y, t, \xi, \vartheta) \left(H_{1,\kappa}(\xi, \vartheta) - H_{1,\kappa,n}(\xi, \vartheta)\right) d\xi d\vartheta.$$

It follows from Theorem 3.0.2 that $\psi_{2,\kappa,n}(y, t) \in C^{2,1}((0, \kappa) \times (0, 1), \mathbb{R})$. Furthermore, a similar calculation as in part (i) yields that

$$|I_n(y_0, t_0)| \leq C_\kappa \frac{1}{n},$$

for some constant C_κ, depending on κ.

Moreover, we note that

$$\frac{1}{n} < \frac{t_0}{2},$$

and it can be shown that the function $G_{L,\kappa}(y, t, \xi, \vartheta)$ is sufficiently regular that the partial differential operators $\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}$ and $\frac{\partial}{\partial t}$ can be taken inside the integral. Thus similar calculations as in part (i) yield that, for $l \in \{1, 2\}$,

$$\left| \frac{\partial^l I_n(y, t_0)}{\partial y^l} \right|_{y=y_0} \leq C_\kappa t_0^{-\frac{l}{2}} \frac{1}{n},$$

and

$$\left| \frac{\partial I_n(y_0, t_0)}{\partial t} \right|_{t=t_0} \leq C_\kappa t_0^{-1} \frac{1}{n},$$

for some constant C_κ, depending on κ. □

Theorem 3.0.3. *$\tilde{\psi}_{2,\kappa}(y, t)$ is a unique classical solution of equation 3.0.9. Moreover, $\tilde{\psi}_{2,\kappa}(y, t) \in C([0, \kappa] \times [0, 1], \mathbb{R})$.*

Proof. Let $(y_0, t_0) \in (0, \kappa) \times (0, 1]$, and let

$$E := \left[\frac{y_0}{2}, \frac{y_0 + \kappa}{2}\right] \times \left[\frac{3}{4} t_0, 1\right].$$

We know from Theorem 3.0.2 that, for every $n \in 2, 3, \ldots$, $\psi_{2,\kappa,n}(y, t)$ is a unique, bounded classical solution of equation (3.0.30), and, from Lemma 3.0.4, that $\psi_{2,\kappa,}(y, t) \in C^{2,1}((0, \kappa) \times (0, 1], \mathbb{R})$.

Moreover, similar bounds as those stated in Lemma 3.0.4 yield that the sequences

$$\left\{ \frac{\partial^l \psi_{2,\kappa,n}(y, t)}{\partial y^l} \right\}_{n=0}^{\infty}, \quad l \in \{0, 1, 2\}$$

converge uniformly on E to

$$\frac{\partial^l \tilde{\psi}_{2,\kappa}(y,t)}{\partial y^l}, \quad l \in \{0,1,2\},$$

and that $\frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial t}$ converges uniformly on E to $\frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial t}$. It follows from the above that, for $(y,t) \in E$

$$\begin{aligned}\frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial t} &- \left\{ \frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \tilde{\psi}_{2,\kappa}(y,t)}{\partial y^2} - (p + r y_1) \frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial y} \right\} \\ &= \lim_{n\to\infty} H_{1,\kappa,n}(y,t) \\ &= H_{1,\kappa}(y,t).\end{aligned}$$

Since (y_0,t_0) (the point used to define E) was an arbitrarily chosen point in $(0,\kappa) \times (0,1]$ it follows that

$$\begin{aligned}\frac{\partial \tilde{\psi}_{2,\kappa}(y_1,t)}{\partial t} &- \left\{ \frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \psi_{2,\kappa,n}(y,t_1)}{\partial y^2} - (p + r y) \frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial y} \right\} \\ &= H_{1,\kappa}(y,t),\end{aligned}$$

on $(y,t) \in (0,\kappa) \times (0,1]$. Lastly, we observe that by definition $\tilde{\psi}_{2,\kappa}(y,t)$ satisfies the initial condition and the boundary condition, and it follows from Lemma 3.0.4 that $\tilde{\psi}_{2,\kappa}(y,t)$ is continuous on $[0,\kappa] \times [0,1]$. □

In the following we will refer to $\tilde{\psi}_{2,\kappa}$ as $\psi_{2,\kappa}$. To obtain existence also of a solution to the last equation (3.0.10) we need $\psi_{2,\kappa}(y,t)$ to be Hölder continuous on $[0,\kappa] \times [0,1]$ with respect to both y and t, not just continuous. To obtain the Hölder continuity in t we first need the result below.

Lemma 3.0.5. *There exists a constant C_κ, depending on κ, such that, for every $t \in [0,1]$, every $y, y' \in [0,\kappa]$, $t, t' \in [0,1]$, and every $\alpha \in [0,1]$*

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y',t)| \leq C_\kappa \sqrt{t}\, |y - y'|^\alpha . \tag{3.0.35}$$

Proof. It is trivial that the bound (3.0.35) holds if $t = 0$. If $t > 0$ the bound follows from the bounds given in Theorem 3.0.2, the boundedness of $H_{1,\kappa}(y,t)$ and Proposition 3.0.4. □

Lemma 3.0.6. *There exists a constant C_κ, depending on κ, such that, for every $t_2, t_1 \in [0,1]$, every $\alpha \in [0,1]$ and every $y \in [0,\kappa]$*

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq C_\kappa\, |t_2 - t_1|^{\frac{\alpha}{2}} .$$

Proof. Let $\alpha \in [0,1]$. Without loss of generality we can assume that $t_2 > t_1$.

Assume first that

$$t_1 \leq \frac{1}{2} t_2.$$

For this case it follows from Lemma 3.0.4 and Proposition 3.0.3, that, for some constant C_κ, depending on κ,

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq C_\kappa\, (t_2 - t_1)^{\frac{\alpha}{2}} .$$

Assume instead that $t_1 > \frac{1}{2}t_2$. We then have the bound

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq |I_1| + |I_2|,$$

where

$$I_1 = \int_{t_1}^{t_2} \int_0^{\kappa} G_{L,\kappa}(y,t_2,\xi,\vartheta)\, H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta,$$

and

$$I_2 = \int_0^{t_1} \int_0^{\kappa} \left(G_{L,\kappa}(y,t_2,\xi,\vartheta) - G_{L,\kappa}(y,t_1,\xi,\vartheta)\right) H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta.$$

A similar calculation as in the proof of Lemma 3.0.4 yields that, for some constant C_κ depending on κ,

$$\begin{aligned} |I_1| &\leq C_\kappa (t_2 - t_1) \\ &\leq C_\kappa (t_2 - t_1)^{\frac{\alpha}{2}}. \end{aligned}$$

Lastly, a calculation, using the bound given in Proposition 3.0.3, yields that, for some constants $\hat{C}_\kappa$, C_κ, and c_κ depending on κ,

$$|I_2| \leq C_\kappa (t_2 - t_1)^{\frac{\alpha}{2}}.$$

□

Before proceeding with equation (3.0.10) we will need a regularity result concerning the function $H_{2,\kappa}(y,t)$, which is the right hand side of equation (3.0.10).

Lemma 3.0.7. *There exists a constant C_κ, depending on κ, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$, every $y_1, y_2 \in (0,\kappa)$ every $t_1, t_2 \in (0,1]$, and every $\alpha \in [0,]$, the following bounds hold:*

$$|H_{2,\kappa}(y_2,t) - H_{2,\kappa}(y_1,t)| \leq C_\kappa |y_2 - y_1|^{\alpha},$$

and

$$|H_{2,\kappa}(y,t_2) - H_{2,\kappa}(y,t_1)| \leq C_\kappa |t_2 - t_1|^{\frac{\alpha}{2}}.$$

Proof. Let

$$\tilde{\psi}_{2,\kappa}(y,t) := \begin{cases} \psi_{2,\kappa}(y,t), & y \in [0,\kappa], \\ 0, & y < 0. \end{cases}$$

We observe that, for every $t \in (0,1]$, $\psi_{2,\kappa}(0,t) = 0$, and that, for every $(y,t) \in (0,1]$

$$\lambda \int_0^y \psi_{2,\kappa}(y-z,t) dF(z) = \lambda \int_0^\infty \tilde{\psi}_{2,\kappa}(y-z,t) dF(z).$$

The stated bounds can be calculated using the identity above and the Hölder bounds in y and t for $\tilde{\psi}_{2,\kappa}(y-z,t)$, given in Lemma 3.0.5 and Lemma 3.0.5, respectively. □

In Garroni and Menaldi (1992) they also define Green functions for parabolic integro-differential equations. Below we have adapted definition IV.2.1 from Garroni and Menaldi (1992) to the PIDE (3.0.10). In this section we will not examine this Green function, but later, in Section (4.1.2) we will study this Green function more closely in the special case that $\sigma_R = r = 0$.

Definition 3.0.6. *A function $G_{A,\kappa}(y,t,\xi,\vartheta)$ defined in the domain $\bar{\mathcal{D}}_\kappa$, where*

$$\begin{cases} \mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in (0,\kappa), \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\ \partial\mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in \{0,\kappa\}, \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\ \bar{\mathcal{D}}_\kappa = \mathcal{D}_\kappa \cup \partial\mathcal{D}_\kappa \end{cases}$$

is called a Green function on $\bar{\mathcal{D}}_\kappa$ for the differential operator

$$\frac{\partial}{\partial t} - A,$$

with Dirichlet boundary conditions if it satisfies:

(i) $G_{A,\kappa}(y,t,\xi,\vartheta)$ *is continuous in* (y,t) *and locally integrable in* (ξ,ϑ),

(ii)

$$\frac{\partial G_{A,\kappa}(y,t,\xi,\vartheta)}{\partial t} - AG_{A,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi)\,\delta(t-\vartheta), \quad \text{in } \mathcal{D}_\kappa,$$

(iii)

$$\lim_{t-\vartheta\downarrow 0} G_{A,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi), \quad \text{in } \mathcal{D}_\kappa,$$

(iv)

$$G_{A,\kappa}(y,t,\xi,\vartheta) = 0, \quad \text{in } \partial\mathcal{D}_\kappa.$$

Theorem 3.0.4. *There exists a unique Green function $G_{A,\kappa}(y,t,\xi,\vartheta)$ associated with the integro-differential operator $\frac{\partial}{\partial t} - A$ with Dirichlet boundary conditions (i.e., satisfying the requirements of Definition 3.0.6). Let*

$$\psi_{3,\kappa}(y,t) = \begin{cases} 0 & (y,t) \in (0,\kappa) \times \{0\}, \\ 0 & (y,t) \in \{0,\kappa\} \times [0,1], \\ \int_0^t \int_0^\kappa G_{A,\kappa}(y,t,\xi,\vartheta)\, H_{2,\kappa}(\xi,\vartheta)\, d\xi d\vartheta \\ (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{3.0.36}$$

and let

$$\psi_\kappa(y,t) = \sum_{j=1}^{3} \psi_{j,\kappa}(y,t) \quad (y,t) \in [0,\kappa] \times [0,1].$$

With the definition above, for any given $\kappa > 0$ the following holds: $\psi_\kappa(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$ and $\psi_\kappa(y,t)$ is a classical solution except at the origin of the integro-differential equation (3.0.7)*, i.e.,*

$$\begin{cases} \psi_\kappa(y,0) = 0, & y \in (0,\kappa), \\ \psi_\kappa(0,t) = 1, & t \in [0,1], \\ \psi_\kappa(\kappa,t) = 0, & t \in [0,1], \\ \frac{\partial \psi_\kappa y,t)}{\partial t} - A\psi_\kappa y,t) = \lambda \bar{F}(y), & (y,t) \in (0,\kappa) \times [0,1]. \end{cases}$$

Proof. Since we have already established existence and uniqueness of equation (3.0.8) and equation (3.0.9), we only need to consider equation (3.0.10), i.e. the PIDE

$$\begin{cases} \psi_{3,\kappa}(y,0) & = 0, \quad y \in (0,\kappa)\,, \\ \psi_{3,\kappa}(0,t) & = 0, \quad t \in [0,1]\,, \\ \psi_{3,\kappa}\,(\kappa,t) & = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{3,\kappa}(y,t)}{\partial t} & -A\psi_{3,\kappa}\,(y,t) = -\lambda\psi_{2,\kappa}(y,t) + \lambda \int_0^y \psi_{2,\kappa}\,(y-z,t)\,dF(z), \\ & \quad (y,t) \in (0,\kappa) \times [0,1]\,. \end{cases} \tag{3.0.37}$$

It follows from Lemma 3.0.7) that $H_{2,\kappa}(y,t) \in C^{\frac{2}{3},\frac{1}{3}}\left([0,\kappa]\times[0,1]\right)$. Thus, existence and uniqueness will follow from Theorem VIII.2.1 in Garroni and Menaldi (1992), once we have verified that the conditions (VIII.1.2), (VIII.1.3), (VIII.1.11),(VIII.1.12, (VIII.1.14)) and (VIII.1.15) in Garroni and Menaldi (1992) all hold.

The conditions (VIII.1.2) and (VIII.1.3) concern the coefficients of differential terms of the operator A, while the conditions (VIII.1.11), (VIII.1.12, (VIII.1.14)) and (VIII.1.15)) concern the terms

$$\lambda \int_0^y \psi_{3,\kappa}(y-z,t)dF(z) - \lambda\psi_{3,\kappa}(y,t).$$

We note that none of these coefficients depend on t, and that they are all bounded and Lipschitz continuous in y on the truncated domain $[0,\kappa]\times[0,1]$. It follows that the coefficients of A are in $C^{\alpha,\frac{\alpha}{2}}\left([0,\kappa]\times[0,1]\right)$ for any $\alpha \in (0,1)$. Since we are assuming that $\sigma_P > 0$ it is obvious that the second order coefficient $\frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right)$ is bounded away from 0. From these observations it follows that the conditions (VIII.1.2) and (VIII.1.3) are satisfied.

Let $g(y,t)$ be a Borel-measurable function defined on $[0,\kappa]\times[0,1]$, and let

$$\tilde{g}(y,t) = \begin{cases} g(y,t), \quad y \in [0,\kappa]\,, \\ 0, \quad y < 0, \end{cases}$$

and let π be the finite Borel measure on $[0,\infty)$ defined by

$$\pi\left((a,b]\right) = \lambda\left(F(b) - F(a)\right), \quad b \geq 0, -\infty < a \leq b.$$

Let

$$j\,(y,t,z) = -z, \quad (y,t,z) \in [0,\kappa]\times[0,1]\times(-\infty,\infty)\,,$$

let

$$j\,(y,t,z,\theta) = \theta j\,(y,t,z)\,, \quad (y,t,z,\theta) \in [0,\kappa]\times[0,1]\times(-\infty,\infty)\times[0,1]\,,$$

and let

$$m\,(y,t,z) = 1, \quad (y,t,z) \in [0,\kappa]\times[0,1]\times[0,\infty).$$

Since F is a probability measure that assigns all its mass to $[0,\infty)$ it follows that

$$\lambda \int_0^y g(y-z,t)dF(z) - \lambda g(y,t)$$
$$= \int_0^\infty \left(g(y+j\left(y,t,z\right),t) - g(y,t)\right) m\left(y,t,z\right) d\pi\left(z\right).$$

Since both $j\left(y,t,z,\theta\right)$, and $m\left(y,t,z\right)$ are invariant of y and t it follows that conditions VIII.1.12, VIII.1.14 and condition VIII.1.15 are all satisfied. Since

$$0 \leq m\left(y,t,z\right) \leq 1,$$

and

$$\pi\left([0,\infty)\right) = \lambda,$$

it follows that the last condition, VIII.1.11, is also satisfied. Hence, it follows from Theorem VIII.2.1 in Garroni and Menaldi (1992) that $\psi_{3,\kappa}(y,t)$ as defined in (3.0.36) is a unique solution of the PIDE (3.0.10). □

4 Global estimates

So far we have shown existence and uniqueness of a classical solution except at the origin of equation (3.0.7). However, what we really want is to prove existence and uniqueness of a solution of equation (3.0.7) on the full unbounded domain, subject to an asymptotic upper boundary condition rather than a conventional Dirichlet boundary condition. Unfortunately, since so much of the conventional theory for PDE's and PIDE-s breaks down when the domain is unbounded we will not in this article be able to prove uniqueness of a solution of (3.0.7) on the full unbounded domain. The breakdown of conventional PDE-theory is also the reason we in this section will need to do extensive work with Green functions and representation formulas like the one in Definition 3.0.5. In this article we take the approach of first working with Green functions to obtain regularity bounds on the solutions of equations (3.0.8) and (3.0.9) that are independent of the upper domain boundary constant κ.

In the general case the main problem is that when the domain is not bounded, then both the first and second order coefficients go to infinity as $y \to \infty$. When $\sigma_R^2 > 0$ we deal with this problem by making the change of variable $x = \ln(1+y)$ and consider the functions

$$\hat{\psi}_{2,\kappa}(x,t) := \psi_{2,\kappa}(e^x - 1, t), \quad x \in [0, \ln(1+\kappa)] \times [0,1]$$

and

$$\hat{\psi}_{3,\kappa}(x,t) := \psi_{3,\kappa}(e^x - 1, t), \quad x \in [0, \ln(1+\kappa)] \times [0,1].$$

For now though, we will assume that $\sigma_R = r = 0$ (constant coefficients). Under this assumption regularity bounds not depending on κ can be obtained by working directly with the Green functions $G_{L^*,\kappa}\left(y,t,\xi,\vartheta\right)$ and $G_{A^*,\kappa}\left(y,t,\xi,\vartheta\right)$ and the formulas (3.0.5) and (3.0.36). The case with constant coefficients is much simpler than the other two cases, and some central ideas are considerably easier to understand in this setting. We will later see that several of these results can be recycled for the case when $\sigma_R > 0$.

To make things work on unbounded domain we will for the rest of this article make the assumption that for some $\beta > 0$ and some constant C, the tail distribution $\bar{F}$ satisfies the inequality

$$\bar{F}(\zeta) \leq C\,(1+\zeta)^{-\beta}\,. \tag{4.0.38}$$

The bounds we will obtain at the end will depend on this β. These bounds will not be sharp, but still sufficient to show that the derivatives evaluated at points bounded away from the origin are bounded, that the solution vanishes as the space variable y goes to infinity, and that the asymptotic boundary condition is thus satisfied.

4.1 Constant coefficients

4.1.1 Global estimates for a subproblem with constant coefficients

In this section we will obtain regularity estimates of the PDE (3.0.9) that are independent of the constant γ, for the special case that $\sigma_R = r = 0$. In the next section we will do the same for the PIDE (3.0.10), still assuming that $\sigma_R = r = 0$. In both cases the main tools that we want to use are representations of the solutions of the PDE (3.0.9) and the PIDE (3.0.10) in terms of Green functions. For the PDE the representation formula is given in Theorem VI.2 in Garroni and Menaldi (1992), while for the PIDE the representation formula is given in Theorem VIII.2.1. Unfortunately constructing these Green functions is quite a lot of work. In addition, since the end goal is to prove existence on an unbounded domain, we will need suitable estimates that we can later use to show that the solutions of the PDE (3.0.9) and the PIDE (3.0.10) converge in an appropriate manner, as we let the upper boundary constant γ tend to infinity.

In Garroni and Menaldi (1992) it is suggested to use *fundamental solutions*, a notion defined below, to construct Green functions for PDE problems with Dirichlet boundary conditions. We will follow this approach except that we will first focus on the construction of a Green function associated with the operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2}$. After having constructed a Green function associated with this simpler operator we will use Proposition VIII.1.2 to construct a Green function associated with the larger operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2} - p\frac{\partial}{\partial y}$. Finally, in section 4.1.2 we will use Proposition VIII.1.2 again to construct the full Green function $G_{A^*,\gamma}(y,t,\xi,\vartheta)$, still assuming that $\sigma_R = r = 0$. In section 4.2 we will show that an analogous approach, with a different variable, yields similar results when $\sigma_R > 0$ as in the case when $\sigma_R = r = 0$. The last case, when $\sigma_P = 0$ but $r > 0$, will not be treated in this article. We will use the following definition of a fundamental solution, taken from the definition in chapter IV in Garroni and Menaldi (1992).

Definition 4.1.1. *A function $\Gamma_L(y,t,\xi,\vartheta)$ defined in the domain*

$$\mathcal{D} = \{y,t,\xi,\vartheta : y,\xi \in \mathbb{R}, 0 \leq \vartheta < t \leq 1\}$$

is called a fundamental solution for the differential operator

$$\frac{\partial}{\partial t} - L$$

if it satisfies the following:

(i) $\Gamma_L(y,t,\xi,\vartheta)$ *is continuous in* (y,t) *and locally integrable in* (ξ,ϑ),

(ii)

$$\frac{\partial \Gamma_L(y,t,\xi,\vartheta)}{\partial t} - L\Gamma_L(y,t,\xi,\vartheta) = \delta(y-\xi)\,\delta(t-\vartheta)\,, \quad \text{in } \mathcal{D},$$

(iii)

$$\lim_{t-\vartheta\downarrow 0} \Gamma_L(y,t,\xi,\vartheta) = \delta(y-\xi)\,, \quad \text{in } \mathcal{D}.$$

In the above $\delta(y,t)$ *is the Dirac measure at* 0. *As discussed in section IV.1 in Garroni and Menaldi (1992) we need a further* boundedness condition, *like the one given below, to ensure uniqueness of the fundamental solution. In Garroni and Menaldi (1992) a function satisfying this condition in addition to the condition below is referred to as a* principal fundamental solution. *In this article we will, for simplicity, use this condition as part of our definition of a fundamental solution.*

(iv) *For every* $\delta > 0$, *there exists a finite positive constant* M_δ *such that*

$$|\Gamma_L(y,t,\xi,\vartheta)| \leq M_\delta, \quad \text{for } |t-\vartheta| + |y-\xi|^2 \geq \delta.$$

Condition (ii) *means that the* volume potential,

$$u(y,t) = \int_0^t \int_{-\infty}^{\infty} \Gamma_L(y,t,\xi,\vartheta)\, f(\xi,\vartheta)\, d\xi d\vartheta$$

is a classical (i.e. $C^{2,1}((-\infty,\infty)\times(0,1],\mathbb{R})$*) solution of the equation*

$$\frac{\partial u(y,t)}{\partial t} - Lu(y,t) = f(y,t)\,, \quad ,\forall y,t \in (0,1],$$

for any smooth function $f(y,t)$ *with compact support in* $\mathbb{R}\times(0,1]$. (iii) *means that for every smooth function* $\phi(y)$ *with compact support in* $\mathbb{R}$ *the* potential

$$w_\vartheta(y,t) = \int_{-\infty}^{\infty} \Gamma_L(y,t,\xi,\vartheta)\,\phi(\xi)\, d\xi$$

is a continuous and bounded function, i.e. in $C^0(\mathbb{R}\times[\vartheta,1),\mathbb{R})$, *and satisfies the limit condition*

$$\lim_{(t-\vartheta)\to 0} w_\vartheta(y,t) = \phi(y), \quad \forall y \in \mathbb{R}.$$

Now, consider the function

$$\Gamma_{\sigma_P}(y,t,\xi,\vartheta) := \frac{1}{\sqrt{2\pi(t-\vartheta)\,\sigma_P^2}} \exp\left(-\frac{(y-\xi)^2}{2\sigma_P^2(t-\vartheta)}\right), \quad (y,t,\xi,\vartheta) \in \mathcal{D}. \tag{4.1.1}$$

It is easy to verify that this function satisfies the identities and bounds stated in the results below.

Proposition 4.1.1. *For every* $(t, \vartheta) \in (0,1] \times [0,t))$ *and* $y, \xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \Gamma_{\sigma_P}(y,t,\xi,\vartheta)\, d\xi = 1.$$

Proposition 4.1.2. *For every* $(y,t,\xi,\vartheta) \in \mathcal{D}$

$$\begin{aligned}
\Gamma_{\sigma_P}(y,t,\xi,\vartheta) &= \Gamma_{\sigma_P}(y-\xi,t,0,\vartheta)\,, \\
\Gamma_{\sigma_P}(y,t,\xi,\vartheta) &= \Gamma_{\sigma_P}(y,t-\vartheta,\xi,0)\,, \\
\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi} &= \frac{y-\xi}{(t-\vartheta)\,\sigma_P^2}\Gamma_{\sigma_P}(y,t,\xi,\vartheta)\,, \\
\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y} &= -\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi}, \\
\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2} &= \frac{\Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{(t-\vartheta)\,\sigma_P^2}\left[-1+\frac{(y-\xi)^2}{(t-\vartheta)\,\sigma_P^2}\right]. \\
\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t} &= \frac{1}{2}\sigma_P^2\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2} \quad \textit{and} \\
\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \vartheta} &= -\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t}.
\end{aligned}$$

Because of Proposition 3.0.1 we also have the following bounds:

Proposition 4.1.3. *There exists a positive constant* C *such that for every*

$(y, t, \xi, \vartheta) \in \mathcal{D}$ *the following inequalities hold:*

$$\left|\Gamma_{\sigma_P}(y,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y}\right| \leq C|y-\xi|(t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi}\right| \leq C|y-\xi|(t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y \partial \xi}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial^3 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2 \partial \xi}\right| \leq C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t \partial \xi}\right| \leq C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right).$$

$$\left|\frac{\partial^3 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^3}\right| \leq C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right),$$

$$\left|\frac{\partial^4 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^3 \partial \xi}\right| \leq C(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right).$$

The most important consequence of the results above is that $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$ is a fundamental solution in the special case when $\sigma_R = p = r = 0$. Moreover, it follows from Theorem V.3.5 in Garroni and Menaldi (1992) that this fundamental solution is unique. Following the discussion in section VI.1.5 it is clear that the problem of constructing a Green function associated with the operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2$ can be reformulated as finding a solution of a PDE, as indicated in the next result.

Lemma 4.1.1. *Let $g^*_{L_0,\gamma}(y,t,\xi)$ be the unique classical solution of the equation*

$$\begin{cases} g^*_{L_0,\gamma}(y,0,\xi) & = 0, \quad y \in [0,\gamma], \\ g^*_{L_0,\gamma}(0,t,\xi) & = \Gamma_{\sigma_P}(0,t,\xi,0), \quad t \in (0,1], \\ g^*_{L_0,\gamma}(\gamma,t,\xi) & = \Gamma_{\sigma_P}(\gamma,t,\xi,0), \quad t \in (0,1], \\ \frac{\partial g^*_{L_0,\gamma}(y,t,\xi)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 g^*_{L_0,\gamma}(y,t,\xi)}{\partial y^2} \\ & \quad (y,t) \in (0,\gamma) \times (0,1], \end{cases} \tag{4.1.2}$$

and let

$$g_{L_0,\gamma}(y,t,\xi,\vartheta) := g^*_{L_0,\gamma}(y,t-\vartheta,\xi), \quad (y,t,\vartheta,\xi) \in \bar{\mathcal{D}}_\gamma.$$

Assume that for any smooth function $f(\xi,\vartheta)$ with compact support and any $(y,t) \in (0,\gamma) \times (0,1]$, and $l \in \{1,2\}$

$$\begin{aligned} \frac{\partial^l}{\partial y^l} \int_0^t \int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta) f(\xi,\vartheta) d\xi &= \int_0^t \int_0^\gamma \frac{\partial^l g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^l} f(\xi,\vartheta) d\xi, \\ \frac{\partial}{\partial t} \int_0^t \int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta) f(\xi,\vartheta) d\xi &= \int_0^t \int_0^\gamma \frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} f(\xi,\vartheta) d\xi \end{aligned} \tag{4.1.3}$$

and that for any smooth function $\phi(y)$ with compact support

$$\lim_{t-\vartheta\to 0} \int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta) \phi(\xi) d\xi = 0. \tag{4.1.4}$$

Then

$$G_{L_0,\gamma}(y,t,\xi,\vartheta) = \Gamma_{\sigma_P}(y,t,\xi,\vartheta) - g_{L_0,\gamma}(y,t,\xi,\vartheta)$$

is the Green function associated with the differential operator

$$\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2}$$

with Dirichlet boundary conditions on $(y,t) \in (0,\gamma) \times (0,T]$.

Proof. We first observe that, because of Theorem 3.0.2, existence and uniqueness of the Green function is already established.

It follows from the proof of Theorem VI.2.1 in Garroni and Menaldi (1992) that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ must satisfy the equation below, which is the same as equation VI.2.8 in Garroni and Menaldi (1992) adapted to equation (3.0.9):

$$\begin{cases} \lim_{t\downarrow\vartheta} g_{L_0,\gamma}(y,t,\xi,\vartheta) &= 0, \quad y \in (0,\gamma), \\ g_{L_0,\gamma}(0,t,\xi,\vartheta) &= \Gamma_{\sigma_P}(0,t,\xi,\vartheta), \quad t \in (\vartheta,1], \\ g_{L_0,\gamma}(\gamma,t,\xi,\vartheta) &= \Gamma_{\sigma_P}(\gamma,t,\xi,\vartheta), \quad t \in (\vartheta,1], \\ \frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} &= \frac{1}{2}\sigma_P^2 \frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^2}, \text{ for } (y,t) \in (0,\gamma) \times (\vartheta,1]. \end{cases} \tag{4.1.5}$$

Moreover, it follows from Proposition 3.0.1, Proposition 4.1.2 and Proposition 4.1.3 that for some constant C the following equality and bounds hold, for every $(y,t,\xi,\vartheta) \in \partial\mathcal{D}_\gamma$ and every $t_2, t_1 \in (\vartheta,1]$:

$$\Gamma_{\sigma_P}(y,t,\xi,\vartheta) = \Gamma_{\sigma_P}(y,t-\vartheta,\xi,0),$$

$$|\Gamma_{\sigma_P}(y,t,\xi,\vartheta)| \le C|y-\xi|^{-3}(t-\vartheta),$$

and

$$|\Gamma_{\sigma_P}(y,t_2,\xi,\vartheta) - \Gamma_{\sigma_P}(y,t_1,\xi,\vartheta)| \le C|y-\xi|^{-3}|t_2-t_1|.$$

Because of the bounds above and the smoothness of the coefficients (trivial since they are constants) of $\Gamma_{\sigma_P}(\gamma,t-\vartheta,\xi,0)$ it follows from Theorem I.2.1 in Garroni and Menaldi (1992) that, for every fixed $\xi \in (0,\xi)$, there exists a unique classical solution $g^*_{L_0,\gamma}(y,t,\xi)$ of the PDE (4.1.2). Also, this solution satisfies the boundedness condition given in part (iv)) of Definition 4.1.1 (the definition of the corresponding fundamental solution). Because of how $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ was defined it is obvious that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ also satisfies that boundedness condition.

Lastly, it follows from the symmetry property (in t and ϑ) of $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$ and the chain rule that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ is a solution of equation (4.1.5) and satisfies the other requirements in Definition 3.0.1, when $\sigma_R = r = 0$. $\square$

To solve the PDE (4.1.5) we will rely on Theorem V.5.5 in Garroni and Menaldi (1992), which in the theorem below is adapted to our situation.

Definition 4.1.2. *For*

$$g \in C([0,1],\mathbb{R})$$

let

$$P^{(1)}_{g,\gamma}(y,t) := \int_0^t \frac{1}{2}\sigma_P^2 \left.\frac{\partial \Gamma_{\sigma_P}(y,t,\eta,\vartheta)}{\partial \eta}\right|_{\eta=\gamma} g(\vartheta)\,d\vartheta, \quad y \geq 0, t \in [0,1],$$

and

$$P^{(2)}_{g}(y,t) := \int_0^t \frac{1}{2}\sigma_P^2 \left.\frac{\partial \Gamma_{\sigma_P}(y,t,\eta,\vartheta)}{\partial \eta}\right|_{\eta=0} g(\vartheta)\,d\vartheta, \quad y \geq 0, t \in [0,1].$$

For

$$\mathbf{g} = \left(g^{(1)}(t), g^{(2)}(t)\right) \in C\left([0,1],\mathbb{R}^2\right)$$

let

$$P_{\mathbf{g},\gamma}(y,t) := P^{(1)}_{g^{(1)},\gamma}(y,t) - P^{(2)}_{g^{(2)}}(y,t), \quad t \in [0,1].$$

Theorem 4.1.1. *Assume that $\sigma_R = p = r = 0$. Also assume that $\mu(t) = \left(\mu^{(1)}(t), \mu^{(2)}(t)\right) \in C\left([0,1],\mathbb{R}^2\right)$ is a solution of the integral equation*

$$\begin{cases} -\frac{1}{2}\mu^{(1)}(t) + P_{\mu,\gamma}(\gamma,t) = \Gamma_{\sigma_P}(\gamma,t,\xi,0), & t \in (0,1] \\ -\frac{1}{2}\mu^{(2)}(t) + P_{\mu,\gamma}(0,t) = \Gamma_{\sigma_P}(0,t,\xi,0), & t \in (0,1] \end{cases} \tag{4.1.6}$$

such that

$$\lim_{t \downarrow 0} \mu^{(1)}(t) = 0.$$

Then

$$P_{\mu,\gamma}(y,t)$$

is a classical solution of the PDE (4.1.5).

Proof. This follows from Theorem V.5.5 and more generally the discussion in section V.5.2 in Garroni and Menaldi (1992). $\square$

We will proceed to construct a solution of the integral equation above using the *method of successive approximations.* This entails constructing a recursively defined sequence, the sum of which is the solution of the integral equation. It will be clear from the next result that the limit of this sequence exists. It will turn out that in order to obtain regularity estimates of the entire Green function we will need regularity estimates for each "building block". We therefore include bounds for the first two derivatives with respect to t, as well as bounds for $g_{L_0,\gamma}(y,t,\xi,\vartheta)$.

Definition 4.1.3. *Let*

$$V^{(1)}_{\xi,0,\gamma}(t) := -2\Gamma_{\sigma_P}(\gamma,t,\xi,0), \quad (t,\xi)\in[0,1]\times(0,\gamma),$$

$$V^{(2)}_{\xi,0,\gamma}(t) := -2\Gamma_{\sigma_P}(0,t,\xi,0), \quad (t,\xi)\in[0,1]\times(0,\gamma) \text{ and let}$$

$$\mathbf{V}_{\xi,0,\gamma}(t,\xi) := \left(V^{(1)}_{\xi,0,\gamma}(t), V^{(2)}_{\xi,0,\gamma}(t)\right).$$

For $n \in 0,1,2,\ldots,$ *and* $(t,\xi)\in[0,1]\times(0,\gamma)$ *define*

$$\mathbf{V}_{\xi,n,\gamma}(t,\xi) = \left(V^{(1)}_{\xi,n,\gamma}(t,\xi), V^{(2)}_{\xi,n,\gamma}(t,\xi)\right)$$

recursively by

$$\begin{aligned} V^{(1)}_{\xi,n+1,\gamma}(t) &:= 2P_{\mathbf{V}_{\xi,n,\gamma},\gamma}(\gamma,t),\\ V^{(2)}_{\xi,n+1,\gamma}(t) &:= 2P_{\mathbf{V}_{\xi,n,\gamma},\gamma}(0,t)\\ \mathbf{V}_{\xi,n+1,\gamma}(t) &:= \left(V^{(1)}_{\xi,n+1,\gamma}(t), V^{(2)}_{\xi,n+1,\gamma}(t)\right). \end{aligned}$$

Let

$$U^{(1)}_{\xi,n,\gamma}(t) := \sum_{k=0}^{n} V^{(1)}_{\xi,k}(t), \quad t\in[0,1], n\in 0,1,\ldots,$$

$$U^{(2)}_{\xi,n,\gamma}(t) := \sum_{k=0}^{n} V^{(2)}_{\xi,k}, \quad n\in 0,1,\ldots,$$

let

$$\mathbf{U}_{\xi,n,\gamma}(t) := \left(U^{(1)}_{\xi,n,\gamma}(t), U^{(2)}_{\xi,n,\gamma}(t)\right), \quad n\in 0,1,\ldots,$$

let

$$U^{(1)}_{\xi,\gamma}(t) := \lim_{n\to\infty} U^{(1)}_{\xi,n,\gamma}(t), \quad t\in[0,1],$$

$$U^{(2)}_{\xi,\gamma}(t) := \lim_{n\to\infty} U^{(2)}_{\xi,n,\gamma}(t), \quad t\in[0,1],$$

and let

$$\mathbf{U}_{\xi,\gamma}(t) := \left(U^{(1)}_{\xi,\gamma}(t), U^{(2)}_{\xi,\gamma}(t)\right).$$

Lemma 4.1.2. (i) *For every* $\mathbf{g} = \left(g^{(1)}, g^{(2)}\right) \in C\left([0,1],\mathbb{R}^2\right)$

$$P_{\mathbf{g},\gamma}(\gamma,t) = -\frac{1}{2}\sigma_P^2 \int_0^t \left.\frac{\partial\Gamma_{\sigma_P}(\gamma,t,\eta,\vartheta)}{\partial\eta}\right|_{\eta=0} g^{(2)}(\vartheta)\,d\vartheta,$$

and

$$P_{\mathbf{g},\gamma}(0,t) = -\frac{1}{2}\sigma_P^2 \int_0^t \left.\frac{\partial\Gamma_{\sigma_P}(0,t,\eta,\vartheta)}{\partial\eta}\right|_{\eta=\gamma} g^{(1)}(\vartheta)\,d\vartheta.$$

(ii) $\left(P_{\mathbf{g},\gamma}(\gamma,t), P_{\mathbf{g},\gamma}(0,t)\right)$ *maps* $C\left([0,1],\mathbb{R}^2\right)$ *to* $C^2\left([0,1],\mathbb{R}^2\right)$. *Moreover, there exists a constant* C *such that for every* $t\in(0,1]$ *and* $l\in\{0,1,2\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l P_{\mathbf{g},\gamma}(\gamma,t)}{\partial t^l}\right| \le C\exp\left(-\frac{1}{2}c_0\frac{\gamma^2}{t}\right)\int_0^t \left|g^{(2)}(\vartheta)\right| d\vartheta$$

and

$$\left|\frac{\partial^l P_{\mathbf{g},\gamma}(0,t)}{\partial t^l}\right| \le C\exp\left(-\frac{1}{2}c_0\frac{\gamma^2}{t}\right)\int_0^t \left|g^{(1)}(\vartheta)\right| d\vartheta.$$

(iii) *For every* $n \in 0, \ldots,$ *and every* $t \in [0, 1]$

$$-\frac{1}{2} U^{(1)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma}(t)}(\gamma, t) = \Gamma_{\sigma_P}(\gamma, t, \xi, \vartheta) + P_{\mathbf{V}_{\xi,n,\gamma}}(\gamma, t),$$

and

$$-\frac{1}{2} U^{(2)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(0, t) = \Gamma_{\sigma_P}(0, t, \xi, \vartheta) + P_{\mathbf{V}_{\xi,n,\gamma}}(0, t).$$

(iv) *There exists a sequence* $\{k_n\}_{n=0}^{\infty}$ *of positive constants, such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that the inequalities

$$\left| V^{(1)}_{\xi,n,\gamma}(t) \right| \leq k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2} c_0 \frac{\gamma^2}{t} \right), \tag{4.1.7}$$

$$\left| V^{(1)}_{\xi,n,\gamma}{}'(t) \right| \leq k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2} c_0 \frac{\gamma^2}{t} \right),$$

$$\left| V^{(1)}_{\xi,n,\gamma}{}''(t) \right| \leq k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2} c_0 \frac{\gamma^2}{t} \right),$$

$$\left| V^{(2)}_{\xi,n,\gamma}(t) \right| \leq k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2} \frac{\gamma^2}{t} \right),$$

$$\left| V^{(2)}_{\xi,n,\gamma}{}'(t) \right| \leq k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2} \frac{\gamma^2}{t} \right),$$

and

$$\left| V^{(2)}_{\xi,n,\gamma}{}''(t) \right| \leq k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2} \frac{\gamma^2}{t} \right),$$

all hold for every $t \in [0, t]$ *and every* $n \in 1, 2 \ldots, .$

Proof. For (i): This is obvious because of Proposition 4.1.2.

For (ii): It follows from Proposition 3.0.1 and Leibniz' rule that

$$\frac{\partial P_{\mathbf{g},\gamma}(\gamma, t)}{\partial t} = \int_0^t \left. \frac{\partial^2 \Gamma_{\sigma_P}(\gamma, t, \eta, \vartheta)}{\partial t \partial \eta} \right|_{\eta=0} g^{(2)}(\vartheta)\, d\vartheta$$

and

$$\frac{\partial P_{\mathbf{g},\gamma}(0, t)}{\partial t} = \int_0^t \left. \frac{\partial^2 \Gamma_{\sigma_P}(0, t, \eta, \vartheta)}{\partial t \partial \eta} \right|_{\eta=\gamma} g^{(1)}(\vartheta)\, d\vartheta.$$

The stated bounds in part (ii) can be calculated from the identities above and the identities and bounds given in Proposition 3.0.1, Proposition 4.1.2 and Proposition 4.1.3.

For (iii): The equalities given in part (iii) obviously hold for $n = 0$. Assume that for every $k \in 0, 1, \ldots, n$

$$-\frac{1}{2}U^{(1)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(\gamma, t) = \Gamma_{\sigma_P}(\gamma, t, \xi, \vartheta) + P_{\mathbf{V}_{n,\gamma}}(\gamma, t).$$

Since by definition

$$V^{(1)}_{n+1,\gamma}(t) := 2P_{\mathbf{V}_{n,\gamma}}(\gamma, t),$$

it follows that

$$\begin{aligned}
&-\frac{1}{2}U^{(1)}_{\xi,n+1,\gamma}(t) + P_{\mathbf{U}_{\xi,n+1,\gamma},\gamma}(\gamma, t) \\
&= -\frac{1}{2}U^{(1)}_{\xi,n,\gamma} + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(\gamma, t) - \frac{1}{2}V^{(1)}_{\xi,n+1,\gamma} + P_{\mathbf{V}_{\xi,n+1,\gamma}}(\gamma, t) \\
&= \Gamma_{\sigma_P}(\gamma, t, \xi, \vartheta) + P_{\mathbf{V}_{\xi,n+1,\gamma},\gamma}(\gamma, t).
\end{aligned}$$

A similar argument yields that for every $n \in 0, 1, \ldots,$

$$-\frac{1}{2}U^{(2)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(0, t) = 1 + P_{\mathbf{V}_{\xi,n,\gamma}}(0, t).$$

For (iv): Let $m_n := \frac{1}{\Gamma\left(n+\frac{1}{2}\right)}$. We first observe that for some constant C

$$\left|V^{(1)}_{\xi,0,\gamma}(t)\right| \le Ct^{-\frac{1}{2}} \exp\left(-c_0 \frac{(\gamma - \xi)^2}{t}\right).$$

Because of the bounds given in part (ii) and the identity given in Proposition 3.0.2 it can be calculated by induction that, for some (different from above) constant C and $n \in 1, 2, \ldots,$

$$\left|V^{(2)}_{\xi,n,\gamma}(t)\right| \le C^n \frac{m_{n-1}}{m_n}.$$

Because of Proposition 3.0.2 a simple calculation yields that

$$\lim_{n\to\infty} \frac{C^{n+1} m_n}{C^n m_{n-1}} = c \lim_{n\to\infty} \frac{1}{\frac{1}{2} + n} = 0,$$

yielding the bound (4.1.7). Similar calculations also yield the other bounds given in part (iv). ☐

In Definition VII.1.1 in Garroni and Menaldi (1992) they define certain function spaces, denoted by $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$, that we we will work with in the rest of the article. Specifically we want the function $\frac{\partial P_{\mathbf{U}_{\xi,\gamma},\gamma}(y,t)}{\partial y}$ to be in the function space $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ for every $\alpha \in (0, 1)$. For that we need a few more regularity results given below.

Lemma 4.1.3. (i) *There exists a constant C such that the following inequalities are all valid for every $(y, t, \xi, \vartheta) \in \mathcal{D}_\gamma$ and every $l \in \{0, 1, 2\}$.*

$$\begin{aligned}
\frac{\partial^l g_{L_0,\gamma}(y, t, \xi, \vartheta)}{\partial y^l} \le C (t - \vartheta)^{-\frac{1+l}{2}} \Bigg\{ &\exp\left(-\frac{1}{2}c_0 \left[\frac{(y-\gamma)^2 + (\xi - \gamma)^2}{t - \vartheta}\right]\right) \\
&+ \exp\left(-\frac{1}{2}c_0\left[\frac{y^2 + \xi^2}{t}\right]\right)\Bigg\},
\end{aligned}$$

$$\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} \leq C\,(t-\vartheta)^{-\frac{3}{2}} \left\{ \exp\left(-\frac{1}{2}c_0\left[\frac{(y-\gamma)^2+(\xi-\gamma)^2}{t-\vartheta}\right]\right) + \exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

$$\frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y\partial t} \leq C\,(t-\vartheta)^{-2} \left\{ \exp\left(-\frac{1}{2}c_0\left[\frac{(y-\gamma)^2+(\xi-\gamma)^2}{t-\vartheta}\right]\right) + \exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

$$\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial \xi} \leq C\,(t-\vartheta)^{-1} \left\{ \exp\left(-\frac{1}{2}c_0\left[\frac{(y-\gamma)^2+(\xi-\gamma)^2}{t-\vartheta}\right]\right) + \exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

and

$$\frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y\partial \xi} \leq C\,(t-\vartheta)^{-\frac{3}{2}} \left\{ \exp\left(-\frac{1}{2}c_0\left[\frac{(y-\gamma)^2+(\xi-\gamma)^2}{t-\vartheta}\right]\right) + \exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

(ii) *Let*

$$G_{L_0,\gamma}\,(y,t,\xi,\vartheta) = \Gamma_{\sigma_P}\,(y,t,\xi,\vartheta) - g_{L_0,\gamma}\,(y,t,\xi,\vartheta)\,.$$

There exists a constant C such that for every $(y,t,\xi\vartheta) \in \mathcal{D}_\gamma$ the following inequalities are all valid:

$$\left|\frac{\partial^l G_{L_0,\gamma}\,(y,t,\xi,\vartheta)}{\partial y^l}\right| \leq C\,(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial G_{L_0,\gamma}\,(y,t,\xi,\vartheta)}{\partial \xi}\right| \leq C\,(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial G_{L_0,\gamma}\,(y,t,\xi,\vartheta)}{\partial t}\right| \leq C\,(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial^2 G_{L_0,\gamma}\,(y,t,\xi,\vartheta)}{\partial x\partial \xi}\right| \leq C\,(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial^2 G_{L_0,\gamma}\,(y,t,\xi,\vartheta)}{\partial x\partial t}\right| \leq C\,(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *Assume that $\sigma_R = p = r = 0$. Then for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\gamma$*

$$G_{L,\gamma}\,(y,t,\xi,\vartheta) = G_{L_0,\gamma}\,(y,t,\xi,\vartheta)\,.$$

Proof. For (i): It follows from Lemma 4.1.1, Theorem 4.1.1 and Lemma 4.1.2 that for $(y,t,\xi,\vartheta) \in \mathcal{D}_\gamma$

$$g_{L_0,\gamma}(y,t,\xi,\vartheta) = g^*_{L_0,\gamma}(y,t-\vartheta,\xi) = P_{\mathbf{U}_{\xi,\gamma}}(y,t-\vartheta)\,.$$

In the above $g^*_{L_0,\gamma}$ is defined in Lemma 4.1.1. We note that the biggest singularities of $\mathbf{U}_{\xi,\gamma}$ stem from the first term $\mathbf{V}_{\xi,0,\gamma}$. Furthermore the partial derivatives of the integral kernel $\Gamma_{\sigma_P}(y,t,\eta,\vartheta)$ are all interconnected, as indicated in Proposition 4.1.2. The stated bounds can be calculated by means of partial integration. In doing this it is helpful to consider separately the two halves of the domain of integration, corresponding to $0 < \vartheta < \frac{t}{2}$ and $\frac{t}{2} < \vartheta < t$ respectively.

For (ii): Since, for any $y, \xi \in [0,\gamma]$,

$$(y-\xi)^2 \le \min\left(y^2+\xi^2, (y-\gamma)^2 + (\xi-\gamma)^2\right), \tag{4.1.8}$$

this follows from the bounds given in part (i) and the regularity bounds of the function $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$.

For (iii): What remains for $G_{L_0,\gamma}(y,t,\xi,\vartheta)$ to be a Green function for the special case $\sigma_R = p = r$ is to show that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ satisfies the requirements (4.1.3) and (4.1.4). Because of the bounds given in part (ii) it follows that for any such smooth $f(\xi,\vartheta)$ there exists a constant C such that

$$\int_0^\gamma \left|g_{L_0,\gamma}(y,t,\xi,\vartheta) f(\xi,\vartheta)\right| d\xi \le C\Bigg(\exp\left(-\frac{1}{2}c_0\frac{(y-\gamma)^2}{t-\vartheta}\right) + \exp\left(-\frac{1}{2}c_0\frac{y^2}{t-\vartheta}\right)\Bigg),$$

and such that

$$\int_0^t\int_0^\gamma \left|\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} f(\xi,\vartheta)\right| d\xi d\vartheta \le C\sqrt{t}\left((\gamma-y)^{-1} + y^{-1}\right) \times\Bigg(\exp\left(-\frac{1}{2}c_0\frac{(y-\gamma)^2}{t-\vartheta}\right) + \exp\left(-\frac{1}{2}c_0\frac{y^2}{t-\vartheta}\right)\Bigg).$$

From these two inequalities it follows that the requirement (4.1.4) is satisfied and that

$$\frac{\partial}{\partial t}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta) f(\xi,\vartheta)\, d\xi d\vartheta = \int_0^t\int_0^\gamma \frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} f(\xi,\vartheta)\, d\xi d\vartheta.$$

Similar calculations also yield that

$$\frac{\partial^l}{\partial y^l}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta) f(\xi,\vartheta)\, d\xi d\vartheta = \int_0^t\int_0^\gamma \frac{\partial^l g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^l} f(\xi,\vartheta)\, d\xi d\vartheta,$$

for $l \in \{1,2\}$.

□

The next step is to solve another integral equation in order to construct a slightly more general Green function corresponding to $p \geq 0$. To this end we will first need to do some preparatory work that is a bit similar to what we did to solve the integral equation (4.1.6).

Definition 4.1.4. *Let*

$$Q_{\kappa,0}(y,t,\xi,\vartheta) := p\frac{\partial G_{L_0,\kappa}(y,t,\xi,\vartheta)(y,t,\xi,\vartheta)}{\partial y}, \quad (y,t,\xi,\vartheta) \in \mathcal{D}_\kappa.$$

Define the sequence of functions $\{Q_{\kappa,n}\}_{n=0}^{\infty}$ *recursively for* $n \in 1,2,\ldots,$ *and* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *by*

$$Q_{\kappa,n+1}(y,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\kappa Q_{\kappa,0}(y,t,z,s)\, Q_{\kappa,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$Q_\kappa(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{\kappa,n}(y,t,\xi,\vartheta).$$

The result below shows that the sequence defined above solves the integral equation (4.1.9). This in turn will turn out to make it possible to conclude that

$$G_{L,\kappa}(y,t,\xi,\vartheta) = G_{L_0,\kappa}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa Q_\kappa(z,s,\xi,\vartheta)\, dzds,$$

in the case that $\sigma_R = r = 0$. In addition to solving the integral equation (4.1.9). we will need some regularity results, also given below, for the limit $Q_\kappa(y,t,z,s)$. These regularity results are a part of the effort in showing that the solution $\psi_{2,\kappa}(y,t)$ has bounded first two derivatives with respect to y, and bounded derivative with respect to t.

Lemma 4.1.4. *Let* $\alpha \in (0,1)$ *and let* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $Q_\kappa \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$. *Moreover,* Q_κ *is the unique solution in* $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ *of the integral equation*

$$Q_\kappa(y,t,\xi,\vartheta) = Q_{\kappa,0}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa Q_{\kappa,0}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dzds. \tag{4.1.9}$$

(ii) *There exists a sequence* $\{k_n\}$ *of positive constants and a constant* C *such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0$$

and such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}$

$$|Q_{\kappa,n}(y,t,\xi,\vartheta)| \leq k_n (t-\vartheta)^{\frac{n-2}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)$$

and such that

$$|Q_\kappa(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *For every* $(y,t,\xi,\vartheta) \in \bar{\mathcal{D}}_\kappa$

$$Q_\kappa(y,t,\xi,\vartheta) = Q_\kappa(y,t-\vartheta,\xi,0). \tag{4.1.10}$$

(iv) *There exists a constant* C *such that, for every* $(y,t,\xi,\vartheta) \in \bar{\mathcal{D}}_\kappa$, *every* $y',\xi' \in (0,\kappa)$,*and every* $t' \in (0,t,)$ *the following inequalities are both valid:*

$$\begin{aligned} |Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y',t,\xi,\vartheta)| \le C\,|y-y'|^{\frac{1}{2}}\,(t-\vartheta)^{-\frac{5}{4}} \\ \times \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{4.1.11}$$

and

$$\begin{aligned} |Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y,t',\xi,\vartheta)| \le C\,|t-t'|^{\frac{1}{4}}\,(t'-\vartheta)^{-\frac{5}{4}} \\ \times \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{4.1.12}$$

(v)

$$\left|\frac{\partial Q_\kappa(y,t,\xi,\vartheta)}{\partial \xi}\right| \le C\,(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{8}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.1.3, that $\frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial x} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, and hence $Q_{\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $Q_{\kappa,0} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that Q_κ is the unique solution, in the function space $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, of the integral equation (4.1.9).

For (ii): It can be shown by induction, following the technique outlined in the proof of Lemma V.3.3 in Garroni and Menaldi (1992), that for some constants C and c

$$|Q_{\kappa,n}(y,t,\xi,\vartheta)| \le cC^n\frac{1}{\Gamma\left(\frac{1}{2}(n+1)\right)}(t-\vartheta)^{\frac{1}{2}n-1}\exp\left(-\frac{1}{4}\frac{(y-\xi)^2}{t-\vartheta}\right).$$

This yields the stated bounds, since

$$\lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\Gamma\left(\frac{1}{2}(n+2)\right)} = 0.$$

,

For part (iii): We first note that it is obvious that

$$Q_{\kappa,0}(y,t,\xi,\vartheta) = Q_{\kappa,0}(y,t-\vartheta,\xi,0).$$

Assume that

$$Q_{\kappa,k}(y,t,\xi,\vartheta) = Q_{\kappa,k}(y,t-\vartheta,\xi,0),$$

for $k \in 0, 1, \ldots, n$. It then follows, using the substitution $\varrho = s - \vartheta$, that

$$Q_{p,n+1}(y,t,\xi,\vartheta) = \int_{\vartheta}^{t} \int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s)\, Q_{p,n}(z,s,\xi,\vartheta)\, dz ds$$
$$= \int_{0}^{t-\vartheta} \int_{0}^{\kappa} Q_{\kappa,0}(y,t-\vartheta,z,\varrho)\, Q_{p,n}(z,\varrho,\xi,0)\, dz d\varrho,$$

and hence

$$Q_{p,n}(y,t,\xi,\vartheta) = Q_{p,n}(y,t-\vartheta,\xi,0)$$

for any $n \in 0, 1, 2, \ldots,$. Since for any $\epsilon > 0$ we can pick an N such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}$

$$\sum_{k=N}^{\infty} |Q_{p,n}(y,t,\xi,\vartheta)| < \epsilon$$

we conclude that the identity (4.1.10) holds for every $(y,t,\xi,\vartheta) \in \mathcal{D}$.

For part (iv): We observe that, if

$$t - t' \geq t' - \vartheta,$$

then the inequality (4.1.12) follows from the bounds given in part (ii). Assume instead that

$$t - t' < t' - \vartheta. \tag{4.1.13}$$

We conclude from the regularity of $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ given in Lemma 4.1.3 and the auxiliary result Proposition 4.1.3 that the inequality (4.1.12) holds for $n = 0$. Let $n \in 1, 2, \ldots,$. It is obvious that

$$Q_{\kappa,n}(y,t,\xi,\vartheta) - Q_{\kappa,n}(y,t',\xi,\vartheta) = I_{1,n} + I_{2,n},$$

where

$$I_{1,n} = \int_{\vartheta}^{t'} \int_{0}^{\kappa} \left(Q_{\kappa,0}(y,t,z,s) - Q_{\kappa,0}(y,t',z,s)\right) Q_{\kappa,n-1}(z,s,\xi,\vartheta)\, dz ds$$

and

$$I_{2,n} = \int_{t'}^{t} \int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s)\, Q_{\kappa,n-1}(z,s,\xi,\vartheta)\, dz ds.$$

Let $\{k_n\}$ be the sequence from the bound given in part (ii). It follows from the regularity of $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ and Proposition 3.0.2 that, for some constants K and C, not depending on n

$$|I_{1,n}| \leq K k_n (t-t')^{\frac{1}{4}} \int_{\vartheta}^{t'} \int_{0}^{\kappa} (t'-s)^{-\frac{5}{4}} (s-\vartheta)^{\frac{1}{2}(n-1)n-1}$$
$$\times \exp\left(-\frac{1}{4}c_0\left[\frac{(y-z)^2}{t-s} + \frac{(z-\xi)^2}{s-\vartheta}\right]\right) dz ds$$
$$\leq C k_n (t-t')^{\frac{1}{4}} (t-\vartheta)^{\frac{1}{2}n-\frac{1}{4}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Similar calculations yield that $|I_{2,n}|$ satisfies an inequality of the form given in equation (4.1.12), and that

$$Q_{\kappa}(y,t,\xi,\vartheta) - Q_{\kappa}(y',t,\xi,\vartheta)$$

satisfies an inequality of the form given in equation (4.1.11).

For (v): The real problem here is to obtain an appropriate bound for the second function in the sequence, i.e. $\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi}$, which we do below. For $n > 1$ we can obtain appropriate estimates using induction and similar calculations as in part (ii) and in the proof of Lemma V.3.1 in Garroni and Menaldi (1992). To accomplish the needed bound for $\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi}$ the most important idea is to split the domain of integration into appropriate parts. This technique is used throughout the book Garroni and Menaldi (1992) and we will tacitly (and sometimes explicitly) make use of it to obtain other bounds later on. We note that

$$\int_{\vartheta}^{t} \left| \int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds \leq \sum_{j=1}^{4} I_j,$$

where

$$I_1 = \int_{\vartheta}^{\frac{t}{2}} \left| \int_{0}^{\kappa} \left(Q_{\kappa,0}(y,t,z,s) - Q_{\kappa,0}(y,t,\xi,s)\right) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds,$$

$$I_2 = \int_{\vartheta}^{\frac{t}{2}} \left| Q_{\kappa,0}(y,t,\xi,s) - Q_{\kappa,0}\left(y,t,\xi,\frac{t}{2}\right) \right| \left| \int_{0}^{\kappa} \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds,$$

$$I_3 = \left| Q_{\kappa,0}\left(y,t,\xi,\frac{t}{2}\right) \right| \int_{\vartheta}^{\frac{t}{2}} \left| \int_{0}^{\kappa} \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds$$

and

$$I_4 = \int_{\frac{t}{2}}^{t} \left| \int_{0}^{\kappa} Q_{\kappa,0}(y,t,\xi,s) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds.$$

Because of the local Hölder-continuity of $Q_{\kappa,0}(z,s,\xi,\vartheta)$, an application of Proposition 3.0.1 and Proposition 3.0.2 yields that, for some constants C and K,

$$\begin{aligned} I_1 &\leq \int_{\vartheta}^{t} \int_{0}^{\kappa} K(t-s)^{-\frac{5}{4}} (s-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8}c_0 \left[\frac{(y-z)^2}{s-\vartheta}\right]\right) dzds \\ &\leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{8}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right). \end{aligned}$$

For I_2 and I_3 we recall that $\frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} = p \frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi \partial \xi}$ and apply Proposition 3.0.5 to obtain that these terms also obey a bound of the form stated in part (v). For I_4 there are no strong singularities and the stated bound can be obtained from a straightforward calculation. We conclude from the above that the differential operator can be taken inside the integral (the order of differentiation and integration can be interchanged) and that for some constant C

$$\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi} \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{8}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right).$$

For $n > 1$ a similar induction as in the proof of Lemma V.3.1 in Garroni and Menaldi (1992) yields that there exists a sequence of constants $\{k_n\}$ such that $\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0$ and such that

$$\left| \frac{\partial Q_{\kappa,n}(y,t,\xi,\vartheta)}{\partial \xi} \right| \leq k_n (t-\vartheta)^{\frac{n-3}{2}}.$$

Because of this property we then conclude that the sequence

$$S_n(y,t,\xi,\vartheta)=\sum_{j=0}^{n}\frac{\partial Q_{\kappa,0}(y,t,\xi,\vartheta)}{\partial\xi}$$

converges uniformly on $\mathcal{D}$, which justifies differentiating the sequence term by term. $\square$

Definition 4.1.5. *Let*

$$G_{L_1,\kappa}(y,t,\xi,\vartheta)=\int_{\vartheta}^{t}\int_{0}^{\kappa}G_{L_0,\kappa}(y,t,z,s)\,Q_\kappa(z,s,\xi,\vartheta)\,dzds.$$

Lemma 4.1.5. *There exists a constant C such that following identities and bounds are valid for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and every $l\in\{0,1,2\}$:*

(i)

$$G_{L_1,\kappa}(y,t,\xi,\vartheta)=G_{L_1,\kappa}(y,t-\vartheta,\xi,0).$$

(ii)

$$\frac{\partial^l G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}=\int_{\vartheta}^{t}\int_{0}^{\kappa}\frac{\partial^l G_{L_0,\kappa}(y,t,z,s)}{\partial x^l}Q_\kappa(z,s,\xi,\vartheta)\,dzds, \tag{4.1.14}$$

$$\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial\xi}=\int_{\vartheta}^{t}\int_{0}^{\kappa}G_{L_0,\kappa}(y,t,z,s)\,Q_\kappa(z,s,\xi,\vartheta)\,dzds,$$

$$\begin{aligned}\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial t}&=Q_\kappa(y,t,\xi,\vartheta)\\&\quad+\int_{\vartheta}^{t}\int_{0}^{\kappa}\frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial t}Q_\kappa(z,s,\xi,\vartheta)\,dzds,\end{aligned} \tag{4.1.15}$$

$$\begin{aligned}\left|\frac{\partial^l G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|&\le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),\\ \left|G_{L_1,\kappa}(y,t,\xi,\vartheta)\frac{\partial}{\partial t}\right|&\le C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),\\ \left|\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial\xi}\right|&\le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),\end{aligned} \tag{4.1.16}$$

and

$$\left|\int_{-\infty}^{\infty}G_{L_1,\kappa}(y,t,\xi,\vartheta)\,d(y-\xi)\right|\le C\sqrt{t}.$$

(iii)

$$G_{L,\kappa}(y,t,\xi,\vartheta)=G_{L_0,\kappa}(y,t,\xi,\vartheta)+G_{L_1,\kappa}(y,t,\xi,\vartheta).$$

Proof. For (i): This follows from making the substitution $\varrho=t-s$.

For (ii): As in the proof of Lemma 4.1.4 these results can be proven by splitting the domains of integration into appropriate parts. To obtain the identity (4.1.15) we will consider the functions

$$I_1(y,t,\xi,s,\vartheta) = \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\left(Q_\kappa(z,s,\xi,\vartheta) - Q_\kappa(y,s,\xi,\vartheta)\right)dz,$$

$$I_2(y,t,\xi,s,\vartheta) = \left(Q_\kappa(y,s,\xi,\vartheta) - Q_\kappa(y,t,\xi,\vartheta)\right)\int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\,dz,$$

$$I_3(y,t,\xi,s,\vartheta) = Q_\kappa(y,t,\xi,\vartheta)\int_0^\kappa \Gamma_{\sigma_P}(y,t,z,s)\,dz,$$

and

$$I_4(y,t,\xi,s,\vartheta) = -Q_\kappa(y,t,\xi,\vartheta)\int_0^\kappa g_{L_0,\kappa}(y,t,\xi,s)\,dz.$$

Because of the way $G_{L_0,\kappa}(y,t,z,s)$ was constructed it is obvious that

$$\int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\,Q_\kappa(z,s,\xi,\vartheta)\,dz = \sum_{j=1}^4 I_j(y,t,\xi,s,\vartheta).$$

Because of the local Hölder continuity of $Q_\kappa(y,t,\xi,\vartheta)$, the bounds obeyed by $g_{L_0,\kappa}(y,t,\xi,\vartheta)$ and application of Proposition 3.0.2, we see that, for some constant C

$$I_1(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-\frac{1}{2}}(t-s)^{\frac{1}{4}}(s-\vartheta)^{-\frac{3}{4}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$I_2(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-\frac{1}{2}}(t-s)^{\frac{1}{4}}(s-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\begin{aligned} I_4(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-1}(t-s)^{\frac{1}{4}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right) \\ \times\Bigg(\exp\left(-\frac{1}{4}c_0\frac{(y-\kappa)^2}{t-\vartheta}\right) \\ +\exp\left(-\frac{1}{4}c_0\frac{y^2}{t-\vartheta}\right)\Bigg). \end{aligned}$$

From these identities it is clear that, for any fixed $y,\xi\in(0,\xi)$

$$\lim_{s\to t}\left[I_1(y,t,\xi,s,\vartheta) + I_2(y,t,\xi,s,\vartheta) + I_4(y,t,\xi,s,\vartheta)\right] = 0.$$

For the last term $I_3(y,t,\xi,\vartheta)$ we get from the substitution $w = \sqrt{2c_0}\frac{y-z}{\sqrt{t-s}}$ that

$$\begin{aligned}\lim_{s\to t} I_3(y,t,\xi,s,\vartheta) &= Q_\kappa(y,t,\xi,\vartheta)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)dw \\ &= Q_\kappa(y,t,\xi,\vartheta).\end{aligned}$$

Similar calculations as above and as in the proof of Lemma 4.1.4 yield that, for some constant C, the following inequalities are all valid for $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$:

$$\begin{aligned} &\int_\vartheta^t \left| \int_0^\kappa \frac{\partial^l G_{L_0,\kappa}(y,t,z,s)}{\partial y^l} Q_\kappa(z,s,\xi,\vartheta)\,dz \right| ds \\ &\qquad \le C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right), \\ &\int_\vartheta^t \left| \int_0^\kappa \frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial \xi} Q_\kappa(z,s,\xi,\vartheta)\,dz \right| ds \\ &\qquad \le C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{4.1.17}$$

and

$$\begin{aligned} &\int_\vartheta^t \left| \int_0^\kappa \frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial t} Q_\kappa(z,s,\xi,\vartheta)\,dz \right| ds \\ &\qquad \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right). \end{aligned} \tag{4.1.18}$$

The stated identity (4.1.15) follows from the discussion above and the bound (4.1.18). The other stated bounds follow from the bound (4.1.17).

For part (iii): Since $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ is the Green function associated with the differential operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2}$ and Dirichlet boundary conditions, and Q_κ is a solution of the integral equation (4.1.9), this follows from the bounds given in part (ii). □

We are now in position to get some regularity results for the solution $\psi_{2,\kappa}(y,t)$ of the PDE (3.0.9). The representation formula given in (3.0.5) depends on the jump measure F as well as the Green function. In this article we will assume that the measure F satisfies the bound (4.0.38), and that the regularity results we get for $\psi_{2,\kappa}(y,t)$ and $\psi_{3,\kappa}(y,t)$ will depend on the values of β for which this inequality is satisfied. In this article we will not discuss what happens if we let $\beta \to \infty$.

Lemma 4.1.6. (i) *Assume that $\sigma_R = r = 0$. There exists a constant C_β, depending on the β from* (4.0.38)*, such that for $l \in \{0,1\}$*

$$\left| \frac{\partial \psi_{2,\kappa}(y,t)}{\partial y^l} \right| \le C_\beta t^{\frac{2-l}{2}} (1+y)^{-\beta}.$$

(ii) *Let $H_{1,\kappa}(y,t)$ be as in section 3.*

$$\frac{\partial^2 \psi_{2,\kappa}(y,t)}{\partial y^2} = \int_0^t \int_0^\kappa \frac{\partial^2 G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta$$

and

$$\frac{\partial \psi_{2,\kappa}(y,t)}{\partial t} = H_{1,\kappa}(y,t) + \int_0^t \int_0^\kappa \frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta.$$

(iii) *There exists a constant C_β, depending on the β such that the following inequalities are all valid:*

$$\left|\int_0^t\int_0^\kappa \frac{\partial^2\Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta\right| \le C_\beta(1+y)^{-\beta},$$
$$|H_{1,\kappa}(y,t)| + \left|\int_0^t\int_0^\kappa \frac{\partial\Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta\right| \le C_\beta(1+y)^{-\beta},$$
$$\left|\int_0^t\int_0^\kappa \frac{\partial^2 G_{1,\kappa}(y,t,\xi,\vartheta)}{\partial y^2}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta\right| \le C_\beta\sqrt{t}(1+y)^{-\beta}, \tag{4.1.19}$$

and

$$\left|\int_0^t\int_0^\kappa \frac{\partial G_{1,\kappa}(y,t,\xi,\vartheta)}{\partial t}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta\right| \le C_\beta\sqrt{t}(1+y)^{-\beta}. \tag{4.1.20}$$

Proof. Because of the representation formula given in Definition 3.0.5 and the local Hölder continuity of the function $H_{1,\kappa}(\xi,\vartheta)$, these identities and bounds follow from similar calculations as in the proof of Lemma 4.1.5. □

It is a bit more technical to obtain appropriate estimates of $\frac{\partial^2\psi^*_{2,\kappa}(y,t)}{\partial y^2}$ and $\frac{\partial\psi^*_{2,\kappa}(y,t)}{\partial t}$. In particular the proof of the next result involves a change in the order of integration.

Lemma 4.1.7. *Assume that $\sigma_R = r = 0$ and that the tail distribution of the jumps satisfies the bound (4.0.38) for some $\beta > 0$. Then there exists a constant C_β, depending on β, such that, for every $(y,t)\in(0,\kappa)\times(0,1]$, every $y'\in(y,\kappa)$, every $t'\in(0,t)$ and every $\alpha\in(0,1]$ the following inequalities are all valid:*

$$\left|\frac{\partial\psi_{2,\kappa}(y,t)}{\partial t}\right| \le C_\beta(1+y)^{-\beta}, \tag{4.1.21}$$

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y',t)| \le C_\beta(y'-y)^\alpha t^{\frac{2-\alpha}{2}}(1+y)^{-\beta},$$

and

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y,t')| \le C_\beta(t-t')^\alpha t^{1-\alpha}(1+y)^{-\beta}.$$

Proof. Thanks to Lemma 4.1.6 we only need to show that the integrals

$$\int_0^t\int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t}d\xi d\vartheta \tag{4.1.22}$$

and

$$\int_0^t\int_0^\kappa \frac{\partial^2 g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^2}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta \tag{4.1.23}$$

satisfy the stated bounds. We will do that by first showing that the order of integration can be interchanged as explained below.

Let $U^{(1)}_{\xi,\kappa}(t)$ and $U^{(2)}_{\xi,\kappa}(t)$ be the limits defined in definition 4.1.3, let $U^{(1)}_{\kappa}(\xi,t) = U^{(1)}_{\xi,\kappa}(t)$ (i.e. $U^{(1)}_{\xi,\kappa}(t)$ considered as a function of ξ as well as t) and likewise let $U^{(2)}_{\kappa}(\xi,t) = U^{(2)}_{\xi,\kappa}(t)$. Let

$$B^{(1)}_{\kappa}(s) = \int_0^{\kappa} U^{(1)}_{\kappa}(\xi,s)d\xi, \quad s \in (0,1],$$

and let

$$B^{(2)}_{\kappa}(s) = \int_0^{\kappa} U^{(2)}_{\kappa}(\xi,s)d\xi, \quad s \in (0,1].$$

We note that

$$\int_0^t \int_0^{\kappa} \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{\kappa} \int_0^{t-\vartheta} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=\kappa} U^{(1)}_{\kappa}(\xi,s) ds d\xi d\vartheta,$$

and

$$I_2 = \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{\kappa} \int_0^{t-\vartheta} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=0} U^{(2)}_{\kappa}(\xi,s) ds d\xi d\vartheta.$$

We observe that the function $\Gamma_{\sigma_P}(y,t,z,s)$ is independent of the variable ξ and that $B^{(1)}_{\kappa}(s)$ and $B^{(2)}_{\kappa}(s)$ do not depend on ϑ. Moreover, because of the bounds given in Proposition 4.1.3 and Lemma 4.1.2 we are, for fixed $(y,t) \in (0,\kappa)\times(0,1]$, free to interchange the order of integration, as in the calculation below.

$$\begin{aligned} I_1 &= \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{t-\vartheta} B^{(1)}_{\kappa}(s) \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=\kappa} ds d\vartheta \\ &= -\frac{1}{2}\sigma_P^2 \int_0^t B^{(1)}_{\kappa}(s) \int_0^{t-s} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-s,z,\vartheta)}{\partial z \partial \vartheta}\right|_{z=\kappa} d\vartheta ds \\ &= \frac{1}{2}\sigma_P^2 \int_0^t B^{(1)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\right|_{z=\kappa} ds. \end{aligned}$$

In the last step above we have also used the symmetry property between the second and fourth variables of the fundamental solution. A similar calculation yields that

$$I_2 = \frac{1}{2}\sigma_P^2 \int_0^t B^{(2)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\right|_{z=0} ds.$$

Since we also have that

$$\frac{1}{2}\sigma_P^2 \frac{\partial^2 \Gamma(y,t,\xi,\vartheta)}{\partial y^2}) = \frac{\partial \Gamma(y,t,\xi,\vartheta)}{\partial t},$$

we get the following identities:

$$\begin{aligned} &\int_0^t \int_0^{\kappa} \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \\ &= \int_0^t B^{(1)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\right|_{z=\kappa} ds \\ &\quad - \int_0^t B^{(2)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\right|_{z=0} ds, \end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \\
&= \frac{1}{2}\sigma_P^2 \int_0^t B_\kappa^{(1)}(s) \frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\bigg|_{z=\kappa} ds \\
&\quad - \frac{1}{2}\sigma_P^2 \int_0^t B_\kappa^{(2)}(s) \frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\bigg|_{z=0} ds.
\end{aligned}$$

Because of the bounds given in Proposition 4.1.3 and Lemma 4.1.2 and the inequality (4.1.8) it is straightforward to calculate that $B_\kappa^{(1)}(t)$ and $B_\kappa^{(2)}(t)$ are both bounded functions and, hence, for some constant C

$$\left| \int_0^t \int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \right| \le C,$$

and

$$\left| \int_0^t \int_0^\kappa \frac{\partial^2 g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} d\xi d\vartheta \right| \le C.$$

Because of this boundedness and the bounds and Hölder continuity of $H_{1,\kappa}(\xi,\vartheta)$, similar calculations as in the proof of Lemma 4.1.5 yield that the stated bounds are valid for the integrals (4.1.22) and (4.1.23). □

Lemma 4.1.8. *Assume that* $\sigma_R = r = 0$ *and that the tail distribution satisfies the bound* (4.0.38). *Then, for some constant* C_β, *depending on* β, *the bounds stated below all hold for every* $0 < y < y' < \kappa$, *every* $0 \le t' < t \le 1$ *and every* $\alpha \in (0,1]$*:*

$$|H_{2,\kappa}(y,t)| \le C_\beta t (1+y)^{-\beta},$$

$$|H_{2,\kappa}(y,t) - H_{2,\kappa}(y',t)|^\alpha \le C_\beta |y-y'|^\alpha t^{\frac{2-\alpha}{2}} (1+y)^{-\beta} \tag{4.1.24}$$

and

$$|H_{2,\kappa}(y,t) - H_{2,\kappa}(y,t')|^\alpha \le C_\beta (t-t')^\alpha t^{1-\alpha} (1+y)^{-\beta}. \tag{4.1.25}$$

Proof. Let

$$\tilde{\psi}_{2,\kappa}(y,t) := \begin{cases} \psi_{2,\kappa}(y,t), & (y,t) \in [0,\kappa] \times (0,1], \\ 0, & (y,t) \in (-\infty,0) \times (0,1]. \end{cases}$$

We note that for every $t \in (0,1]$

$$\lim_{y \downarrow 0} \psi_{2,\kappa}(0,t) = 0,$$

that $\tilde{\psi}_{2,\kappa}(y,t)$ is continuous on $(-\infty,\kappa) \times (0,1]$, and that

$$\int_0^y \psi_{2,\kappa}(y-z,t) dF(z) = \int_0^\infty \tilde{\psi}_{2,\kappa}(y-z,t) dF(z).$$

A similar calculation as in the proof of Lemma 3.0.2, using the identity above, the bounds given in Lemma 4.1.6 and Lemma 4.1.7, as well as the auxiliary results Proposition 3.0.4 and Proposition 3.0.3, yields that all the stated bounds hold. □

4.1.2 Global estimates for a subproblem with an integral term and constant coefficients

In the remaining part of this section we will obtain regularity estimates of the PIDE (3.0.10) that are independent of the constant κ, still assuming that $\sigma_R = r = 0$. Analogous to the previous section we will do that by working with the Green function

$$G_{A,\kappa}(y,t,\xi,\vartheta)$$

defined in Definition 3.0.6. The main idea is to construct this Green function from the Green function $G_{L,\kappa}(y,t,\xi,\vartheta)$, using the parametrix method. The first step is to construct the Green function defined below. It is known to exist and be unique because of Theorem VI.1.10 in Garroni and Menaldi (1992).

Definition 4.1.6. *Let L_λ be the differential operator*

$$L_\lambda = L - \lambda,$$

and let $G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)$ be the Green function associated with L_λ.

We will do this by first looking for a function $Q_{\lambda,\kappa}$ that solves the integral equation given in the next lemma. Also, because of the next lemma, the sequence of functions defined below is well defined.

Definition 4.1.7. *Let*

$$Q_{\lambda,\kappa,0}(y,t,\xi,\vartheta) = -\lambda G_{L,\kappa}(y,t,\xi,\vartheta),$$

and let the sequence of functions $\{Q_{\lambda,\kappa,n}\}_{n=0}^{\infty}$ be defined recursively for $n \in 1,2,\dots,$ and $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$, by

$$Q_{\lambda,\kappa,n+1}(y,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\kappa Q_{\lambda,\kappa,0}(y,t,z,s)\, Q_{\lambda,\kappa,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$Q_{\lambda,\kappa}(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{\lambda,\kappa,n}(y,t,\xi,\vartheta).$$

Lemma 4.1.9. *Assume that $\sigma_R = r = 0$ and let $\alpha \in (0,1)$ and $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *and* $Q_{\lambda,\kappa} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. *Moreover $Q_{\lambda,\kappa}$ is the unique solution in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation*

$$\begin{aligned} Q_{\lambda,\kappa}(y,t,z,\vartheta) = &-\lambda G_{L,\kappa}(y,t,\xi,\vartheta) \\ &- \lambda \int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds. \end{aligned} \tag{4.1.26}$$

(ii) $Q_{\lambda,\kappa}(y,t,\xi,\vartheta)$ *is differentiable with respect to all four variables on $\mathcal{D}_\kappa$. Furthermore, there exists a constant C such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ the following identity and inequalities are all valid for $l \in \{0,1\}$:*

$$Q_{\lambda,\kappa}(y,t,\xi,\vartheta) = Q_{\lambda,\kappa}(y,t-\vartheta,\xi,0),$$

$$\left|\frac{\partial^l Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992) and the bounds given in Proposition 4.1.3, Lemma 4.1.3 and Lemma 4.1.5, that $G_{L,\kappa}(y,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and hence $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $Q_{\lambda,\kappa}$ is the unique solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$, of the integral equation (4.1.26).

For part (ii): This can be shown using the same calculations and reasoning, based on induction and uniform convergence, as in the proof of Lemma 4.1.4 and the proof of Lemma V.3.1 in Garroni and Menaldi (1992). □

Lemma 4.1.10. *Assume that*

$$\sigma_R = r = 0.$$

(i) *For every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and* $l \in \{0,1,2\}$

$$\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= \int_0^{t-\vartheta} \int_0^\kappa G_{L,\kappa}(y,t-\vartheta,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,0)\, dzds,$$

$$\frac{\partial^l}{\partial y^l}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L,\kappa}(y,t,z,s)}{\partial y^l} Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds,$$

$$\frac{\partial}{\partial t}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= Q_{\lambda,\kappa}(y,t,\xi,\vartheta)$$
$$+ \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L,\kappa}(y,t,z,s)}{\partial t} Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds.$$

Furthermore, for some constant C

$$\left|\frac{\partial^l}{\partial y^l}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds\right|$$
$$\le C(t-\vartheta)^{\frac{1-l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L,\kappa}\left(y,t,z,s\right)Q_{\lambda,\kappa}\left(z,s,\xi,\vartheta\right)dzds\right| \\ \leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right).$$

(ii) *For every* $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$

$$G_{L_\lambda,\kappa}\left(y,t,\xi,\vartheta\right)=G_{L,\kappa}\left(y,t,\xi,\vartheta\right) \\ +\int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L,\kappa}\left(y,t,z,s\right)Q_{\lambda,\kappa}\left(z,s,\xi,\vartheta\right)dzds.$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in the proof of Lemma 4.1.5.

For (ii): Since $G_{L,\kappa}\left(y,t,\xi,\vartheta\right)$ is the Green function associated with the differential operator L and Dirichlet boundary conditions, and $Q_{\lambda,\kappa}\left(y,t,\xi,\vartheta\right)$ satisfies the integral equation (4.1.26), this can be derived from the identities and bounds given in part (i). It follows from the way the Green function was constructed that it satisfies the boundary conditions. □

After the next result we will begin the process of constructing the Green function associated with the entire operator A.

Definition 4.1.8. *Let*

$$\psi_{3,a,\kappa}(y,t):=\int_{0}^{t}\int_{0}^{\kappa} G_{L_\lambda,\kappa}\left(y,t,\xi,\vartheta\right)H_{2,\kappa}\left(\xi,\vartheta\right)d\xi d\vartheta.$$

Lemma 4.1.11. *Assume that* $\sigma_R=r=0$. *There exists a constant* C_β, *depending on* β, *such that, for every* $(y,t)\in(0,\kappa)\times(0,1]$ *and every* $l\in\{0,1,2\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l\psi_{3,a,\kappa}(y,t)}{\partial y^l}\right|\leq C_\beta t^{\frac{4-l}{2}}\left(1+y\right)^{-\beta}$$

and

$$\left|\frac{\partial\psi_{3,a,\kappa}(y,t)}{\partial t}\right|\leq C_\beta t\left(1+y\right)^{-\beta}.$$

Proof. This follows from the inequalities given in Lemma 4.1.9, Lemma 4.1.10 and Lemma 4.1.8 by making similar calculations as in the proofs of Lemma 4.1.4 and Lemma 4.1.5.

□

Definition 4.1.9. *Let*

$$Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right):=\lambda\int_{0}^{y} G_{L_\lambda,\kappa}\left(y-\zeta,t,\xi,\vartheta\right)dF\left(\zeta\right).$$

Let the sequence of functions

$$\left\{Q_{I,\kappa,n}\right\}_{n=0}^{\infty}$$

be defined inductively by

$$Q_{I,\kappa,n}(y,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\kappa} Q_{I,\kappa,0}(y,t,z,s)\, Q_{I,\kappa,n-1}(z,s,\xi,\vartheta)\, dzds,$$
$$n \in 1,2,\dots,$$

let

$$Q_{I,\kappa}(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{I,\kappa,n}(y,t,\xi,\vartheta),$$

let

$$G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta) := \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_\lambda,\kappa}(y,t,z,s)\, Q_{I,\kappa,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$G_{I_\lambda,\kappa}(y,t,\xi,\vartheta) := \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_\lambda,\kappa}(y,t,z,s)\, Q_{I,\kappa}(z,s,\xi,\vartheta)\, dzds.$$

Lemma 4.1.12. *Assume that the tail distribution of the claims satisfies the inequality* (4.0.38) *and let* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *be the Green function space defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) *For every* $\alpha \in (0,1)$

$$Q_{I,\kappa,0}(y,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}.$$

(ii) *For every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$

$$\begin{aligned} Q_{I,\kappa}(y,t,\xi,\vartheta) = {} & \lambda \int_{[0,y]} G_{L_\lambda,\kappa}(y-\zeta,t,\xi,\vartheta)\, dF(\zeta) \\ & + \lambda \int_{[0,y]} G_{I_\lambda,\kappa}(y-\zeta,t,\xi,\vartheta)\, dF(\zeta). \end{aligned} \tag{4.1.27}$$

(iii) *There exists a sequence* $\{k_n\}_{n=0}^{\infty}$ *such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\dots,$, *and every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *the following inequalities are valid:*

$$\begin{aligned} |Q_{I,\kappa,n}(y,t,\xi,\vartheta)| \le {} & k_n (t-\vartheta)^{n-\frac{1}{2}} \\ & \times \int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \\ & \times dF(\zeta_0)dF(\zeta_1),\dots,dF(\zeta_n), \end{aligned} \tag{4.1.28}$$

$$\begin{aligned}\Big| Q_{I,\kappa,n}(y,t,\xi,\vartheta) &- Q_{I,\kappa,n}(y',t,\xi,\vartheta)\Big| \\ &\le Ck_n |y-y'| (t-\vartheta)^{n-1} \\ &\quad \times \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right. \\ &\quad \left. + \exp\left(-\frac{1}{4}c_0 \frac{\left(y'-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right) \\ &\quad \times dF(\zeta_1),\dots,dF(\zeta_n), \end{aligned} \tag{4.1.29}$$

and

$$\begin{aligned} &|Q_{I,\kappa,n}(y,t,\xi,\vartheta) - Q_{I,\kappa,n}(y,t',\xi,\vartheta)| \\ &\le Ck_n |t-t'|^{\frac{1}{4}} \left(\tilde{t}-\vartheta\right)^{n-\frac{3}{4}} \\ &\quad \times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \\ &\quad \times dF(\zeta_1),\dots,dF(\zeta_n). \end{aligned} \tag{4.1.30}$$

(iv)

$$|Q_{I,\kappa}(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-\frac{1}{2}},$$

$$|Q_{I,\kappa}(y,t,\xi,\vartheta) - Q_{I,\kappa}(y',t,\xi,\vartheta)| \le C|y-y'|(t-\vartheta)^{-1},$$

and

$$|Q_{I,\kappa}(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t',\xi,\vartheta)| \le C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{3}{4}}.$$

Proof. For part (i): Let $\alpha \in (0,1)$. We first observe that it follows from Lemma VII.1.3 in Garroni and Menaldi (1992) and the bounds given in Lemma 4.1.10 that $G_{L_\lambda,\kappa} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Moreover, using similar arguments as in the proof of Theorem 3.0.4, it can be shown that all the requirements of Lemma VII.3.2 hold and hence

$$-\lambda G_{L_\lambda,\kappa}(y,t,\xi,\vartheta) + Q_{I,\kappa,0}(y,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}.$$

Since $Q_{I,\kappa,0}(y,t,\xi,\vartheta)$ is the difference between two functions that are both in the space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it is trivial to show that $Q_{I,\kappa,0}(y,t,\xi,\vartheta)$ is also in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$.

For (ii): It follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) and part (i) that, for any $\alpha \in (0,1)$, $Q_{I,\kappa}(y,t,\xi,\vartheta)$ is a solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation

$$\begin{aligned} Q_{I,\kappa}(y,t,\xi,\vartheta) &= Q_{I,\kappa,0}(y,t,\xi,\vartheta) \\ &\quad + \int_\vartheta^t \int_0^\kappa Q_{I,\kappa,0}(y,t,z,s)\, Q_{I,\kappa}(z,s,\xi,\vartheta)\, ds dz. \end{aligned}$$

Since $Q_{I,\kappa}(y,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from the Fubini Theorem that we are allowed to change the order of integration, yielding the identity (4.1.27).

For (iii): We first observe that since, for every $t > \vartheta$,

$$\lim_{y \downarrow 0} G_{L_\lambda,\kappa}(y,t,\xi,\vartheta) = 0,$$

the stated bound (4.1.28) holds for $n = 0$. A similar induction as in the proof of Lemma 4.1.4 and Lemma V.3.1 in Garroni and Menaldi (1992) yields that (4.1.28) holds for any finite n. The most important difference is that this time we need to also invoke Fubini's theorem in order to change the order of integration.

Next, we observe that, if

$$t - t' \geq t' - \vartheta,$$

then the inequality (4.1.30) follows from the bounds given in part (iii). Assume instead that

$$t - t' > t' - \vartheta.$$

Because of the regularity of $G_{L_{\kappa,\kappa}}$ and Proposition 3.0.3, it is trivial that under this assumption the inequality (4.1.30) holds for $n = 0$. Let $n \in 1, 2, \ldots,$. It is obvious that

$$Q_{I,\kappa,n}(y,t,\xi,\vartheta) - Q_{I,\kappa,n}(y,t',\xi,\vartheta) = I_{1,n} + I_{2,n},$$

where

$$I_{1,n} = \int_{\vartheta}^{t'} \int_0^{\kappa} \left(Q_{I,\kappa,0}(y,t,z,s) - Q_{I,\kappa,0}(y,t',z,s)\right) Q_{I,\kappa,n-1}(z,s,\xi,\vartheta)\, dzds$$

and

$$I_{2,n} = \int_{t'}^{t} \int_0^{\kappa} Q_{I,\kappa,0}(y,t,z,s)\, Q_{I,\kappa,n-1}(z,s,\xi,\vartheta)\, dzds.$$

Let $\{k_n\}$ be the sequence from the bound (4.1.28). It follows from the regularity of $G_{L_{\kappa,\kappa}}$, Proposition 3.0.3, the bound (4.1.28), Fubini's theorem (to allow the changing of the order of integration) and Proposition 3.0.2, that, for some constants K and C, not depending on n,

$$\begin{aligned}
|I_{1,n}| \leq K k_n (t-t')^{\frac{1}{4}} & \int_0^{\infty} \ldots, \int_0^{\infty} \int_{\vartheta}^{t'} (t'-s)^{-\frac{3}{4}} (s-\vartheta)^{n-\frac{1}{2}} \\
& \times \int_0^{\kappa} \int_0^{\infty} \exp\left(-\frac{1}{4} c_0 \frac{(y-z-\zeta)^2}{t-s}\right) \\
& \times \exp\left(-\frac{1}{4} c_0 \frac{\left(z-\xi-\sum_{j=0}^{n-1} \zeta_j\right)^2}{s-\vartheta}\right) dzds \\
& \times dF(\zeta)\, dF(\zeta_0) \ldots, dF(\zeta_{n-1}) \\
\leq C 2^n k_n |t-t'|^{\frac{1}{4}} (t'-\vartheta)^{n-\frac{3}{4}} & \\
& \times \exp\left(-\frac{1}{4} c_0 \frac{\left(y-\xi-\sum_{j=0}^{n} \zeta_j\right)^2}{t-\vartheta}\right) \\
& \times dF(\zeta_0), \ldots, dF(\zeta_n).
\end{aligned}$$

Similar calculations yield that

$$|I_{2,n}| \leq C2^n k_n |t-t'|^{\frac{1}{4}} (t'-\vartheta)^{n-\frac{3}{4}} \times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \times dF(\zeta_0),\ldots,dF(\zeta_n)$$

and that

$$Q_{I,\kappa,n}(y,t,\xi,\vartheta) - Q_{I,\kappa,n}(y',t,\xi,\vartheta)$$

satisfies an inequality of the form given in part (4.1.29).

For part (iv): Since F is a probability distribution this follows from the bounds given in part (iii). □

Lemma 4.1.13. *There exists a sequence* $\{k_n\}_{n=0}^{\infty}$ *such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\ldots,$ *every* $l \in \{0,1,2\}$ *and every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *the following identities and inequalities are valid:*

(i)

$$\frac{\partial^l G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial y^l} = \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} Q_{I,\kappa,n}(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial t} = Q_{I,\kappa,n}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L_\lambda,\kappa}(y,t,z,s)}{\partial t} Q_{I,\kappa,n}(z,s,\xi,\vartheta)\,dzds.$$

(ii)

$$\left|\frac{\partial^l G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial y^l}\right| \leq k_n (t-\vartheta)^{n+\frac{1-l}{2}} \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \times dF(\zeta_1),\ldots,dF(\zeta_n),$$

$$\left|\frac{\partial G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial t}\right| \leq k_n (t-\vartheta)^{n-\frac{1}{2}} \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \times dF(\zeta_1),\ldots,dF(\zeta_n),$$

and

$$
\begin{aligned}
\left|G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)\right| \le k_n\left(\min(y,\kappa-y)\right)^{\frac{1}{2}}(t-\vartheta)^{n+\frac{1}{4}} \\
\times \int_0^\infty \cdots \int_0^\infty \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right. \\
+\exp\left(-\frac{1}{4}c_0 \frac{\left(\kappa-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\
\left.+\exp\left(-\frac{1}{4}c_0 \frac{\left(\xi+\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right) \\
\times dF(\zeta_1),\dots,dF(\zeta_n).
\end{aligned}
$$

(iii)

$$
\frac{\partial^l G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l} = \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} Q_{I,\kappa}(z,s,\xi,\vartheta)\,dzds,
$$

and

$$
\begin{aligned}
\frac{\partial G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t} &= Q_{I,\kappa}(y,t,\xi,\vartheta) \\
&+ \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L_\lambda,\kappa}(y,t,z,s)}{\partial t} Q_{I,\kappa}(z,s,\xi,\vartheta)\,dzds,
\end{aligned}
$$

Furthermore, there exists a constant C such that for every (y,t,ξ,ϑ), and every $l \in \{0,1,2\}$

$$
\left|\frac{\partial^l G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C(t-\vartheta)^{\frac{1-l}{2}},
$$

and

$$
\left|\frac{\partial G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{1}{2}}.
$$

(iv) *For every $\xi \in (0,\kappa)$ and $0 \le \vartheta < t \le 1$*

$$
G_{A,\kappa}(y,t,\xi,\vartheta) = G_{L_\lambda,\kappa}(y,t,\xi,\vartheta) + G_{I_\lambda,\kappa}(y,t,\xi,\vartheta).
$$

Proof. For part (i): These identities follow from similar calculations as in the proof of Lemma 4.1.5, using the bounds given in Lemma 4.1.12.

For part (ii): It follows from the identities in part (i) that, for every finite n

$$
\frac{\partial^2 G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} = \sum_{j=1}^3 I_{j,n},
$$

where

$$I_{1,n}=\int_{\vartheta}^{t}\int_{0}^{\kappa}\frac{\partial^{2}G_{L_{\lambda},\kappa}\left(y,t,z,s\right)}{\partial y^{2}}\left(Q_{I,\kappa,n}\left(z,s,\xi,\vartheta\right)-Q_{I,\kappa,n}\left(y,s,\xi,\vartheta\right)\right)dzds,$$

$$I_{2,n}=\int_{\vartheta}^{t}\left(Q_{I,\kappa,n}\left(y,s,\xi,\vartheta\right)-Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right)\right)\int_{0}^{\kappa}\frac{\partial^{l}G_{L_{\lambda},\kappa}\left(y,t,z,s\right)}{\partial y^{l}}dzds,$$

and

$$I_{3,n}=Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right)\int_{\vartheta}^{t}\int_{0}^{\kappa}\frac{\partial^{l}G_{L_{\lambda},\kappa}\left(y,t,z,s\right)}{\partial y^{l}}dzds.$$

A calculation using the bounds given in Lemma 4.1.10, Lemma 4.1.12 and Proposition 3.0.2, and invoking the Fubini's theorem to change the order of integration, yields that, for some constant C, not depending on n

$$\begin{aligned}\left|I_{1,n}\right| &\leq Ck_{n}\left(t-\vartheta\right)^{-\frac{1}{2}}\int_{0}^{\infty}\dots\int_{0}^{\infty}\exp\left(-\frac{1}{4}c_{0}\frac{\left(y-\sum_{j=0}^{n}\zeta_{j}\right)^{2}}{t-\vartheta}\right)\\ &\quad\times\int_{\vartheta}^{t}\left(t-s\right)^{-\frac{1}{2}}\left(s-\vartheta\right)^{n-\frac{1}{2}}dsdF\left(\zeta_{1}\right)\dots dF\left(\zeta_{n}\right),\\ &\leq C\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+1\right)}k_{n}\left(t-\vartheta\right)^{n-\frac{1}{2}}\\ &\quad\times\int_{\vartheta}^{t}\left(t-s\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_{0}\frac{\left(y-\sum_{j=0}^{n}\zeta_{j}\right)}{t-\vartheta}\right)\\ &\quad\times dF\left(\zeta_{1}\right)\dots dF\left(\zeta_{n}\right),\end{aligned}$$

where $\{k_{n}\}_{n=0}^{\infty}$ is a sequence of positive constants, such that

$$\lim_{n\to\infty}\frac{k_{n+1}}{k_{n}}=0.$$

It follows from the above that the stated bound for

$$\left|\frac{\partial^{2}G_{I_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)}{\partial y^{2}}\right|$$

is valid for $|I_{1,n}|$. Similar calculations yield that bounds of the form given in the claim are also valid for $|I_{2,n}|$ and $|I_{3,n}|$, and thus the stated bound for

$$\left|\frac{\partial^{2}G_{I_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)}{\partial y^{2}}\right|$$

is valid. Other calculations along these lines also yield that the stated bounds for

$$\left|\frac{\partial^{l}G_{I_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)}{\partial y^{l}}\right|,\quad l\in\{0,1\},$$

and

$$\left|\frac{\partial G_{I_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)}{\partial t}\right|,$$

are also valid.

For part (iii): This follows from uniform convergence and similar considerations as in the proof of Lemma 4.1.5.

For part iv: Since $G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)$ is the Green function associated with the differential operator $G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)$, with Dirichlet boundary conditions, and because of the properties given in part (iii), the only property that remains to be shown is that, for every $\xi \in (0,\kappa)$ and $0 \leq \vartheta < t \leq 1$,

$$\lim_{y\to 0} G_{I_\lambda,\kappa}(y,t,\xi,\vartheta) = \lim_{y\to\kappa} G_{I_\lambda,\kappa}(y,t,\xi,\vartheta) = 0.$$

Since $G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)$ is continuous and vanishes at $y=0$ and $y=\kappa$, a similar calculation as in the proof Proposition 3.0.4 yields that, for some constant C, the following bound is valid for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$:

$$|G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)| \leq C \min\left(\sqrt{y}, \sqrt{\kappa - y}\right)(t-\vartheta)^{-\frac{3}{4}}.$$

Because of this inequality and the bound on $Q_{I,\kappa}(y,t,\xi,\vartheta)$, the Dominated Convergence Theorem can be applied to yield the inequality stated in part (iv). □

Theorem 4.1.2. *Assume that* $\sigma_R = r = 0$ *and that the bound* (4.0.38) *on the tail distribution function* $\bar{F}$ *holds. Then there exist constants* C *and* C_β*, where* C_β *depends on* β*, such that for every* $n \in 0,1,\ldots,$ *and every* $l \in \{0,1,2\}$ *the following inequalities are all valid:*

$$\begin{aligned}\left|\frac{\partial^l \psi_{3,\gamma}(y,t)}{\partial y^l}\right| &\leq C_\beta t^{\frac{4-l}{2}}(1+y)^{-\beta},\\ \left|\frac{\partial \psi_{3,\gamma}(y,t)}{\partial t}\right| &\leq C_\beta t(1+y)^{-\beta},\end{aligned} \tag{4.1.31}$$

$$|\psi_{3,\gamma}(y,t)| \leq C t^{\frac{7}{4}} \min(y, \gamma - y)^{\frac{1}{2}}, \tag{4.1.32}$$

$$\begin{aligned}\left|\frac{\partial^l \psi_{3,\gamma}(y,t)}{\partial y^l}\right| &\leq C_\beta t^{\frac{4-l}{2}}(1+y)^{-\beta},\\ \left|\frac{\partial \psi_{3,\gamma}(y,t)}{\partial t}\right| &\leq C_\beta t(1+y)^{-\beta},\end{aligned} \tag{4.1.33}$$

$$\begin{aligned}\left|\frac{\partial^l \psi_{\gamma}(y,t)}{\partial y^l}\right| &\leq C\left(t^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + t^{\frac{2-l}{2}} C_\beta (1+y)^{-\beta}\right) \quad \textit{and}\\ \left|\frac{\partial \psi_{\gamma}(y,t)}{\partial t}\right| &\leq C\left(t^{-1}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta(1+y)^{-\beta}\right).\end{aligned} \tag{4.1.34}$$

Proof. For every $n \in 0,1,\ldots,$ and $(y,t) \in [0,\gamma]\times[0,1]$ let

$$\psi_{3,b,\gamma,n}(y,t) = \int_0^t \int_0^\gamma G_{I_\lambda,\gamma,n}(y,t,\xi,\vartheta)\, H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta$$

and let

$$\psi_{3,b,\gamma}(y,t) = \int_0^t \int_0^\gamma G_{I_\lambda,\gamma}(y,t,\xi,\vartheta)\, H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta.$$

It follows from Lemma 4.1.11 that the inequalities given in equation (4.1.31) are valid for $\psi_{3,a,\gamma}(y,t)$ (defined in Definition 4.1.8). Thus, to establish these bounds, what remains is to show that they also hold for $\psi_{3,b,\gamma}(y,t)$. Once this is done, it will follow from the already established regularity properties of $\psi_{1,\gamma}(y,t)$ and $\psi_{2,\gamma}(y,t)$ that the inequalities given in equation (4.1.34) are all valid. This can be done in 3 steps.

The first step is to use the bounds given in Lemma 4.1.13 to show that, for every $(y,t) \in (0,\gamma) \times (0,1]$, every finite $n \in 0,1,\ldots,$ and every $l \in \{0,1,2\}$

$$\frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} = \int_0^t \int_0^\gamma \frac{\partial^l G_{I_\lambda,\gamma,n}}{\partial y^l} H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta \tag{4.1.35}$$

and

$$\frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t} \int_0^t \int_0^\gamma \frac{\partial G_{I_\lambda,\gamma,n}}{\partial t} H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta. \tag{4.1.36}$$

The second step is to establish that there exists a sequence $\{k_n\}$ of positive numbers and a constant C_β, depending on β, such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every $(y,t) \in (0,\gamma) \times (0,1]$, every $n \in 0,1,\ldots,$ and every $l \in \{0,1,2\}$

$$\left| \frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} \right| \leq C_\beta t^{n+\frac{6-l}{2}} (1+y)^{-\beta}, \tag{4.1.37}$$

and

$$\left| \frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t} \right| \leq C_\beta t^{n+2} (1+y)^{-\beta}. \tag{4.1.38}$$

The last step is to establish that

$$\frac{\partial^l \psi_{3,b,\gamma}(y,t)}{\partial y^l} = \sum_{n=0}^{\infty} \frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} \tag{4.1.39}$$

and that

$$\frac{\partial \psi_{3,b,\gamma}(y,t)}{\partial t} = \sum_{n=0}^{\infty} \frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t}. \tag{4.1.40}$$

It follows from the identity (4.1.36), the regularity bounds obeyed by $G_{I_\lambda,\gamma,n}$ stated in Lemma 4.1.13, and Fubini's theorem, that there exists a sequence $\{k_n\}_{n=0}^\infty$ and a constant C_β, depending on β, such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0.$$

Also

$$\left|\frac{\partial\psi_{3,\gamma,b,n}(y,t)}{\partial t}\right| \leq k_n \int_0^\infty \cdots \int_0^\infty \int_0^t (t-\vartheta)^{n-\frac{1}{2}}$$
$$\times \int_0^\gamma \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) H_{2,\gamma}(\xi,\vartheta)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\ldots dF(\zeta_n)$$
$$\leq I_{1,n}+I_{2,n}+I_{3,n},$$

where

$$I_{1,n} = C_\beta k_n (1+y)^{-\beta} \int_0^\infty \cdots \int_0^\infty \int_0^t (t-\vartheta)^{n-\frac{1}{2}}\, \vartheta$$
$$\times \int_{\frac{y}{2}}^\gamma \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\ldots dF(\zeta_n),$$

$$I_{2,n} = k_n \exp\left(-\frac{1}{128}c_0\frac{y^2}{t}\right) \int_{\left\{\zeta_0,\ldots,\zeta_n\geq 0:\sum_{j=0}^n \zeta_j \leq \frac{y}{4}\right\}}$$
$$\times \int_0^t (t-\vartheta)^{n-\frac{1}{2}}\, \vartheta$$
$$\times \int_0^{\frac{y}{2}} \exp\left(-\frac{1}{8}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\ldots dF(\zeta_n),$$

and

$$I_{3,n} = k_n t^{n+2} \int_{\left\{\zeta_0,\ldots,\zeta_n\geq 0:\sum_{j=0}^n \zeta_j > \frac{y}{4}\right\}} dF(\zeta_0)\ldots dF(\zeta_n).$$

From the above it is clear that the stated bounds holds for $I_{1,n}$ and $I_{2,n}$. Moreover, we observe that it follows from the assumption (4.0.38) on the tail distribution function $\bar{F}$ that, for every $\zeta_0,\zeta_1,\ldots,\zeta_n \geq 0$,

$$\Pi_{j=0}^\infty \bar{F}(\zeta_0)\,\bar{F}(\zeta_1)\ldots,\bar{F}(\zeta_n) \leq C^n \left[(1+\zeta_0)(1+\zeta_1)\cdots\times(1+\zeta_n)\right]^{-\beta}$$
$$\leq C^n \left(1+\sum_{j=0}^n \zeta_j\right)^{-\beta}.$$

Since $\lim_{n\to\infty}\frac{k_{n+1}}{k_n}=0$, it follows from this inequality that the stated bound (4.1.38) also holds for $I_{3,n}$. Similar calculations also yield that the bounds given in equation (4.1.37) and (4.1.32) also hold. Similar reasoning, also based on uniform convergence, as in the proof of Lemma 4.1.5, yields that the differentiation can be done term by term, as indicated in the identities (4.1.39) and (4.1.40). □

4.2 Unbounded coefficients

4.2.1 Global estimates for a subproblem with unbounded coefficients

In this section we will study the equation (3.0.9)) when $\sigma_R > 0$. We will assume that σ_R is positive, and not look into the case $\sigma_R = 0, r > 0$. In much the same way as we did in Section 4.1.1 and Section 4.1.2, we will do this by obtaining bounds for some Green functions, denoted $\hat{G}_{\hat{L},\kappa}$ and $\hat{G}_{\hat{A},\kappa}$, with the above assumption. Because the coefficients of L are not bounded on $(0, \infty)$ it is very hard to prove directly the existence of the fundamental solution associated with L. This is only one of the number of problems that arise when σ_R is positive. Instead of working with the original Green function we will work with something we call an *auxiliary* Green function.

The basic idea is to consider the function

$$\hat{\psi}_{2,\hat{\kappa}}(x,t) := \psi_{2,\kappa}(e^x - 1, t), \quad (x,t) \in [0, \ln(\kappa+1)].$$

From the definition above it is obvious that

$$\psi_{2,\kappa}(y,t) = \hat{\psi}_{2,\hat{\kappa}}(x,t) \quad (y,t) \in [0,\kappa].$$

and the chain rule yields the result below.

Definition 4.2.1. *Let*

$$\hat{a}_{1,1}(x) := \frac{1}{2}\left(\sigma_p^2 e^{-2x} + \sigma_R^2 \left(1 - e^{-x}\right)^2\right), \quad x \geq 0,$$

let

$$\hat{a}_1(x) := \left(pe^{-x} + r\left(1 - e^{-x}\right)\right) - \hat{a}_{1,1}(x), \quad x \geq 0,$$

and let

$$\hat{L} := \left(\hat{a}_{1,1}(x)\frac{\partial^2}{\partial x^2} + \hat{a}_1(x)\frac{\partial}{\partial x}\right), \quad x \geq 0.$$

Lemma 4.2.1. *Let the function $H_{1,\kappa}$ be as in Section 3. $\hat{\psi}_{2,\hat{\kappa}}(x,t)$ is the unique solution of the PDE*

$$\begin{cases} \hat{\psi}_{2,\hat{\kappa}}(x,0) &= 0, \quad x \in (0, \ln(1+\kappa)), \\ \hat{\psi}_{2,\hat{\kappa}}(0,t) &= 0, \quad t \in [0,1], \\ \hat{\psi}_{2,\hat{\kappa}}(ln(1+\kappa), t) &= 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t} - \hat{L}\hat{\psi}_{2,\hat{\kappa}}(x,t) &= H_{1,\kappa}(e^x - 1, t), \quad (x,t) \in (0, \ln(1+\kappa)) \times (0,1]. \end{cases} \tag{4.2.1}$$

Proof. Let

$$x = \ln(1+y).$$

From the definition and the chain rule it follows that

$$\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x} = e^x \frac{\partial \psi_{2,\kappa}(y,t)}{\partial y}\bigg|_{y=e^x-1},$$

and that

$$\frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2} = e^{2x}\left(\left.\frac{\partial \psi_{2,\kappa}(y,t)}{\partial y}\right|_{y=e^x-1} + \left.\frac{\partial^2 \psi_{2,\kappa}(y,t)}{\partial y^2}\right|_{y=e^x-1}\right).$$

The claim follows from the identities above, the maximum theorem (similar to the uniqueness of the PDE (3.0.9)) and $\psi_{2,\kappa}(y,t)$ being a solution of the PDE (3.0.9). □

Crucially the coefficients of the differential operator $\hat{L}$ are bounded on $(0,\infty)$. As we shall see this property enables us to obtain regularity estimates for $\hat{\psi}_{2,\hat{\kappa}}(x,t)$ similar to those we obtained for the PDE (3.0.9), where we assumed constant coefficients.

Starting with the representation formula below, much of what will follow will resemble the discussion in sections 4.1.1.

Definition 4.2.2. *Let*

$$\hat{\kappa} := \ln(1+\kappa),$$

and let

$$\begin{cases} \mathcal{D}_{\hat{\kappa}} = \{x,t,\xi,\vartheta : x,\xi \in (0,\hat{\kappa}), 0 \le \vartheta < t \le\}, \\ \partial\mathcal{D}_{\hat{\kappa}} = \{x,t,\xi,\vartheta : x,\xi \in \{0,\hat{\kappa}\}, 0 \le \vartheta < t \le 1\}, \\ \bar{\mathcal{D}}_{t,\hat{\kappa}} = \mathcal{D}_{\hat{\kappa}} \cup \partial\mathcal{D}_{\hat{\kappa}}. \end{cases}$$

Theorem 4.2.1. *There exists a unique Green function $\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)$ associated with the differential operator $\hat{L}$ and Dirichlet boundary conditions on the domain $\mathcal{D}_{\hat{\kappa}}$, i.e. satisfying the conditions in Definition 3.0.1 with L replaced by $\hat{L}$ and κ replaced by $\hat{\kappa}$. Furthermore, for every $(y,t) \in (0,\hat{\kappa}) \times (0,1]$*

$$\hat{\psi}_{2,\hat{\kappa}}(x,t) = \int_0^t \int_0^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)\, H_{1,\kappa}(e^{\xi}-1,\vartheta) d\xi d\vartheta.$$

Proof. This can be shown using arguments similar to those that lead to the result in Theorem 3.0.3. □

In the next section we will discuss the regularity of a function we will refer to as the *auxiliary* fundamental solution. Similar to what we did in Section 4.1.1 and Section 4.1.2, we will use this function to construct the Green function $\hat{G}_{\hat{L},\hat{\kappa}}$. After that a similar calculation as in Section 4.1.2 will yield estimates of the derivatives of $\hat{\psi}_{2,\hat{\kappa}}(x,t)$, which in turn can be used to obtain estimates of the derivatives of $\psi_{2,\kappa}(x,t)$. We will construct the Green function $\hat{G}_{\hat{L},\hat{\kappa}}$ by first constructing a fundamental solution and a Green function associated with, not the differential operator $\hat{L}$, but a "smaller" equation, that only includes the second order term. We then use the Green function associated with the second order term as building material for $\hat{G}_{\hat{L},\hat{\kappa}}$, similar to our construction of the Green function for the whole operator A from a simpler equation (assuming constant coefficients) in section 4.1.2. For technical reasons (we want to invoke Theorem V.1.3.5 and Theorem V.5.5) we will define a fundamental solution associated with an extension of the second order term $\hat{a}_{1,1}(x)\frac{\partial^2}{\partial x^2}$ to the whole line, that preserves the differentiability, uniform ellipticity and boundedness of the coefficients.

Definition 4.2.3. *Let*

$$\hat{a}_{1,1}^{*}(x) = \begin{cases} \hat{a}_{1,1}(x) & x \geq 0, \\ \frac{1}{2}\sigma_p^2 + \left[(1-e^x)\sigma_P^2 + \frac{1}{2}\sigma_P^2 x^2 e^x\right] + \frac{1}{2}\left(\sigma_R^2 + 2\sigma_P^2\right) x^2 e^x, & x < 0, \end{cases},$$

let

$$\hat{a}_1^{*}(x) = \begin{cases} \hat{a}_1(x) & x \geq 0, \\ \hat{a}_1(0) & x < 0, \end{cases},$$

let

$$\hat{L}_0 := \hat{a}_{1,1}^{*}(x)\frac{\partial^2}{\partial x}$$

and let

$$\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)$$

be the fundamental solution associated with the differential operator $\hat{L}_0$.

It can be calculated that the extended second order coefficient $\hat{a}_{1,1}^{*}$ and the first order coefficient $\hat{a}_1$ (restricted to $x > 0$) are smooth, uniformly elliptic and bounded. We state these properties in the next result without giving a proof.

Proposition 4.2.1. *The extended second order coefficient $\hat{a}_{1,1}^{*}$ and the restricted $\hat{a}_1(x)$ are bounded and two times continuously differentiable, on the real line and for positive x, respectively. Furthermore, for every $x \geq 0$*

$$\frac{1}{2}\left[\frac{\sigma_p^2\sigma_R^2}{\sigma_p^2+\sigma_R^2}\right] \leq \hat{a}_{1,1}(x) \leq \frac{1}{2}\max\left(\sigma_P^2,\sigma_R^2\right)$$

and, for some constant C, the following inequalities are valid for $x > 0$:

$$\begin{aligned} |\hat{a}_1(x)| &\leq C \\ |\hat{a}_{1,1}{}'(x)| &\leq C \\ |\hat{a}_1{}'(x)| &\leq C \\ |\hat{a}_{1,1}{}''(x)| &\leq C \\ |\hat{a}_1{}''(x)| &\leq C. \end{aligned}$$

Definition 4.2.4. *For $(x,t,\xi,\vartheta) \in \mathcal{D}$ let*

$$\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) := \frac{1}{\sqrt{2\pi(t-\vartheta)\hat{a}_{1,1}^{*}(\xi)}}\exp\left(-\frac{(x-\xi)^2}{4\hat{a}_{1,1}^{*}(\xi)(t-\vartheta)}\right), \tag{4.2.2}$$

and let

$$\hat{c}_0 := \frac{1}{2}\max\left(\sigma_P^2,\sigma_R^2\right).$$

Basic calculations yield that the function defined above has certain properties that we state in the next two results without giving a proof. Because of these basic properties it follows that $\hat{\Gamma}_{\hat{L}_0}$ is a fundamental solution associated with the extended second order term, as stated in Lemma 4.2.2 below. The main idea that be inferred from these results is that the principal term can be split into two terms, where the first term behaves very much like the principal term in the case of constant coefficients, while the second term has a weaker singularity than the first. We will use these properties primarily when, as part of the effort to construct the auxiliary Green function, we want do integration by parts similar to what we relied on in the proof of Lemma 4.1.3.

Proposition 4.2.2. *For every* $(x,t,\xi,\vartheta)\in\mathcal{D}$

$$
\begin{aligned}
\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) &= \hat{\Gamma}_{\hat{L}_0}(x,t-\vartheta,\xi,0),\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x} &= -\frac{x-\xi}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta),\\
\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2} &= \frac{\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\left[-1+\frac{(x-\xi)^2}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\right],\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} &= \hat{a}^*_{1,1}(\xi)\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2},\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial \xi} &= -\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\\
&\qquad+\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\frac{\hat{a}^{*\,\prime}_{1,1}(\xi)}{\hat{a}^*_{1,1}(\xi)}\left[-\frac{1}{2}+\frac{1}{4\hat{a}^*_{1,1}(\xi)}\frac{(x-\xi)^2}{t-\vartheta}\right],\\
\frac{\partial^3\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2\partial\xi} &= \frac{1}{\hat{a}^*_{1,1}(\xi)}\left[\frac{\hat{a}^{*\,\prime}_{1,1}(\xi)}{\hat{a}^*_{1,1}(\xi)}\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\vartheta}-\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial\vartheta}\right].
\end{aligned}
$$

Proposition 4.2.3. *There exists a positive constant* C *such that, for every* $(x,t,\xi,\vartheta)\in\mathcal{D}\times\mathbb{R}$*, the following inequalities hold:*

$$
\begin{aligned}
\left|\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\right| &\le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\right| &\le C|x-\xi|(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2}\right| &\le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t^2}\right| &\le C(t-\vartheta)^{-\frac{5}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),
\end{aligned}
$$

$$
\left|\int_{-\infty}^{\infty}\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\,d(x-\xi)-1\right|\le C\sqrt{t-\vartheta},
$$

$$
\begin{aligned}
\left|\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi}+\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\right| &\le C(t-\vartheta)^{-\frac{1}{2}}\\
&\qquad\times\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x\partial\xi}+\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2}\right| &\le C|x-\xi|(t-\vartheta)^{-\frac{3}{2}}\\
&\qquad\times\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),
\end{aligned}
$$

and

$$\left|\frac{\partial^3\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial x^2} - \frac{1}{\hat{a}^*_{1,1}(x)}\left\{\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial\vartheta} + \frac{\hat{a}^*_1(x)}{\hat{a}^*_{1,1}(x)}\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\vartheta}\right\}\right|$$
$$\leq C(t-\vartheta)^{-\frac{1}{2}}\left(1+\frac{|x-\xi|}{t-\vartheta}\right)\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right).$$

Lemma 4.2.2. *$\hat{\Gamma}_{\hat{L}_0}$ is the unique (principal) Fundamental solution associated with the equation $\hat{L}_0$.*

Proof. This can be calculated using the identities and inequalities given in Proposition 4.2.2 and Proposition 4.2.3. □

Analogous to the construction of the Green function $G_{L,\hat{\kappa}}$ in Section 4.1.1 (where we assumed constant coefficients), we can construct the Green function associated with just the second order term by solving the PDE given in the next result below.

Lemma 4.2.3. *Let $\hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t,\xi)$ be the unique classical solution of the equation*

$$\begin{cases} \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(0,0,\xi) & = 0, \quad x\in[0,\hat{\kappa}], \\ \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(0,t,\xi) & = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0), \quad t\in(0,1], \\ \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(\kappa,t,\xi,0) & = \hat{\Gamma}_{\hat{L}_0}(\kappa,t,\xi,0), \quad t\in(0,1], \\ \frac{\partial\hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t,\xi)}{\partial t} & = \hat{a}_{1,1}(x)\frac{\partial^2\hat{g}^*_{\hat{L}_0,\hat{\kappa}}}{\partial x^2}(x,t,\xi), \quad (x,t)\in(0,\hat{\kappa})\times(0,1]. \end{cases} \tag{4.2.3}$$

Let

$$\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta) := \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t-\vartheta,\xi), \quad (x,t,\vartheta,\xi)\in\bar{\mathcal{D}}_{\hat{L}_0,\hat{\kappa}}.$$

Assume in addition that for any smooth function $f(\xi,\vartheta)$ with compact support, any $(x,t)\in(0,\kappa)\times(0,1]$, and $l\in\{1,2\}$

$$\begin{aligned} \frac{\partial^l}{\partial x^l}\int_0^t\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)f(\xi,\vartheta)\,d\xi &= \int_0^t\int_0^{\hat{\kappa}}\frac{\partial^l\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}f(\xi,\vartheta)\,d\xi, \\ \frac{\partial}{\partial t}\int_0^t\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)f(\xi,\vartheta)\,d\xi &= \int_0^t\int_0^{\hat{\kappa}}\frac{\partial\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t}f(\xi,\vartheta)\,d\xi, \end{aligned} \tag{4.2.4}$$

and that for any smooth function $\phi(y)$ with compact support

$$\lim_{t-\vartheta\to 0}\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)\,\phi(\xi)d\xi = 0. \tag{4.2.5}$$

Then

$$\hat{G}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta) = \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)$$

is the unique Green function associated with the differential operator $\hat{L}_0$ and Dirichlet boundary conditions.

Proof. Because of the symmetry property between the variables t and ϑ this follows from reasoning similar to the proof of Lemma 4.1.1. □

It follows from the lemma following after the definitions below that the sequences and series in the definitions below are actually well defined.

Definition 4.2.5. *For*

$$g \in C\left([0,1],\mathbb{R}\right)$$

let

$$\hat{P}^{(1)}_{g,\hat{\gamma}}\left(x,t\right) := \int_0^t \hat{a}_{1,1}\left(\hat{\gamma}\right) \frac{\partial \hat{\Gamma}_{\hat{L}_0}\left(x,t,\eta,\vartheta\right)}{\partial \eta}\bigg|_{\eta=\hat{\gamma}} g\left(\vartheta\right) d\vartheta, \quad y \geq 0, t \in [0,1]$$

and

$$\hat{P}^{(2)}_{g}\left(x,t\right) := \int_0^t \hat{a}_{1,1}\left(0\right) \frac{\partial \hat{\Gamma}_{\hat{L}_0}\left(x,t,\eta,\vartheta\right)}{\partial \eta}\bigg|_{\eta=0} g\left(\vartheta\right) d\vartheta, \quad y \geq 0, t \in [0,1],$$

and for

$$\mathbf{g} = \left(g^{(1)}\left(t\right), g^{(2)}\left(t\right)\right) \in C\left([0,1],\mathbb{R}^2\right)$$

let

$$\hat{P}_{\mathbf{g},\hat{\gamma}}\left(x,t\right) := \hat{P}^{(1)}_{g^{(1)},\gamma}\left(x,t\right) - \hat{P}^{(2)}_{g^{(2)}}\left(x,t\right), \quad t \in [0,1].$$

Definition 4.2.6. *Let*

$$\hat{V}^{(1)}_{\xi,0,\gamma}\left(t\right) := -2\hat{\Gamma}_{\hat{L}_0}\left(\hat{\gamma},t,\xi,0\right), \quad (t,\xi) \in [0,1] \times (0,\hat{\gamma}),$$

$$\hat{V}^{(2)}_{\xi,0,\hat{\gamma}}\left(t\right) := -2\hat{\Gamma}_{\hat{L}_0}\left(0,t,\xi,0\right), \quad (t,\xi) \in [0,1] \times (0,\hat{\gamma}), \ \textit{and}$$

$$\hat{\mathbf{V}}_{\xi,0,\hat{\gamma}}\left(t,\xi\right) := \left(\hat{V}^{(1)}_{\xi,0,\gamma}\left(t\right), \hat{V}^{(2)}_{\xi,0,\hat{\gamma}}\left(t\right)\right).$$

For $n \in 0,1,2,\ldots,$ *define*

$$\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}} = \left(\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}\left(t,\xi\right), \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}\left(t,\xi\right)\right)$$

recursively by

$$\begin{aligned}
&\hat{V}^{(1)}_{\xi,n+1,\hat{\gamma}}\left(t\right) := 2\hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}},\hat{\gamma}}\left(\hat{\gamma},t\right), \quad t \in [0,1], \\
&\hat{V}^{(2)}_{\xi,n+1,\hat{\gamma}}\left(t\right) := 2\hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}},\hat{\gamma}}\left(0,t\right). \quad t \in [0,1], \quad n \in 0,1,\ldots, \quad t \in [0,1], \\
&\hat{\mathbf{V}}_{\xi,n+1,\hat{\gamma}}\left(t\right) := \left(\hat{V}^{(1)}_{\xi,n+1,\hat{\gamma}}\left(t\right), \hat{V}^{(2)}_{\xi,n+1,\hat{\gamma}}\left(t\right)\right), \quad n \in 0,1,\ldots, \quad t \in [0,1].
\end{aligned}$$

Let

$$\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}\left(t\right) := \sum_{k=0}^{n} \hat{V}^{(1)}_{\xi,k}\left(t\right), \quad t \in [0,T], n \in 0,1,\ldots,$$

$$\hat{U}^{(2)}_{\xi,n,\hat{\gamma}}\left(t\right) := \sum_{k=0}^{n} \hat{V}^{(2)}_{\xi,k}, \quad n \in 0,1,\ldots,$$

let

$$\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}\left(t\right) := \left(\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}\left(t\right), \hat{U}^{(2)}_{\xi,n,\hat{\gamma}}\left(t\right)\right), \quad n \in 0,1,\ldots,$$

let

$$\hat{U}^{(1)}_{\xi,\hat{\gamma}}\left(t\right) := \lim_{n\to\infty} \hat{U}^{(1)}_{\xi,n,\hat{\gamma}}(t), \quad t \in [0,1],$$

$$\hat{U}^{(2)}_{\xi,\hat{\gamma}}(t) := \lim_{n\to\infty} \hat{U}^{(2)}_{\xi,n,\hat{\gamma}}(t), \quad t \in [0,1],$$

and let

$$\hat{\mathbf{U}}_{\xi,\hat{\gamma}}(t) := \left(\hat{U}^{(1)}_{\xi,\hat{\gamma}}(t), \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t)\right).$$

Lemma 4.2.4. (i) *For every $n \in 0, 1, \ldots$, $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t)$ and $\hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t)$ are continuous on $[0,1]$ and differentiable on $(0,1]$, and the same holds for $\hat{U}^{(1)}_{\xi,\hat{\gamma}}$ and $\hat{U}^{(2)}_{\xi,\hat{\gamma}}$. Furthermore, there exists a sequence of positive constants $\{k_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k_n < \infty$, and a constant C, such that for every $t \in (0,1]$ and $l \in \{0,1,2\}$ the identities and inequalities stated below are all valid:*

$$-\frac{1}{2}\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}}(\hat{\gamma},t) = \hat{\Gamma}_{\hat{L}_0}(\hat{\gamma},t,\xi,0) + \hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}}}(\hat{\gamma},t),$$

$$-\frac{1}{2}\hat{U}^{(2)}_{\xi,n,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}}(0,t) = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0) + \hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}}}(0,t),$$

$$\left|\frac{\partial^l \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{\frac{n-1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{\frac{n-1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le C t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le C t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right).$$

(ii) *For every fixed $\xi \in (0,\hat{\gamma})$, $\hat{\mathbf{U}}_{\xi,\hat{\gamma}}(t)$ is a solution of the integral equation*

$$\begin{cases} -\frac{1}{2}\hat{\mathbf{U}}^{(1)}_{\xi,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(\hat{\gamma},t) = \hat{\Gamma}_{\hat{L}_*}(\hat{\gamma},t,\xi,0), & t \in (0,1], \\ -\frac{1}{2}\hat{\mathbf{U}}^{(2)}_{\xi,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(0,t) = \hat{\Gamma}_{\hat{L}_*}(0,t,\xi,0), & t \in (0,1]. \end{cases} \tag{4.2.6}$$

(iii) *There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$*

$$\left|\hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(x,t)\right| \le C(t-\vartheta)^{-\frac{1}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0 \frac{(x-\hat{\gamma})^2 + \frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left. + \exp\left(-\frac{1}{2}\hat{c}_0 \frac{x^2 + \frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

(iv) *$P_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}}}(x,t)$ is the classical solution of the PDE (4.2.3).*

Proof. Because of the symmetry property between the variables t and ϑ and the bounds given in Proposition 4.2.2 and Proposition 4.2.3, the lemma follows from Theorem V.5.5 in Garroni and Menaldi (1992) and similar calculations as in the proof of Lemma 4.1.2. □

Definition 4.2.7. *For every $n \in 0, 1, \ldots,$ and $(\xi, t) \in (0, \gamma) \times (0, 1]$ let*

$$\hat{V}^{(1)}_{\hat{\gamma}}(\xi, t) := \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t),$$

let

$$\hat{V}^{(2)}_{\hat{\gamma}}(\xi, t) := \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t),$$

let

$$\hat{U}^{(1)}_{\hat{\gamma}}(\xi, t) := \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t),$$

and let

$$\hat{U}^{(2)}_{\hat{\gamma}}(\xi, t) := \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t).$$

Proposition 4.2.4. (i) *For every $n \in 0, 1, 2, \ldots,$ $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)$ and $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)$ are differentiable with respect to ξ on $(\xi, t) \in (0, \hat{\gamma}) \times (0, 1]$. Furthermore, there exists a constant C, and a sequence of positive constants $\{k_n\}_{n=0}^{\infty}$ such that,*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every $(\xi, t) \in (0, \gamma) \times (0, 1]$, and $l \in \{0, 1\}$ the following inequalities are all valid:

$$\left|\frac{\partial \hat{V}^{(1)}_{\xi,0,\gamma}(\xi, t)}{\partial \xi}\right| \le Ct^{-\frac{1}{2}}\left(1 + \frac{|\hat{\gamma} - \xi|}{t}\right)\exp\left(-\hat{c}_0\frac{(\hat{\gamma} - \xi)^2}{t}\right),$$

$$\left|\frac{\partial \hat{V}^{(2)}_{\xi,0,\hat{\gamma}}(\xi, t)}{\partial \xi}\right| \le Ct^{-\frac{1}{2}}\left(1 + \frac{\xi}{t}\right)\exp\left(-\hat{c}_0\frac{\xi^2}{t}\right),$$

$$\left|\frac{\partial^{1+l} \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)}{\partial \xi \partial t^l}\right| \le k_n t^{\frac{n}{2}-(1+l)}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(\hat{\gamma} - \xi)^2}{t}\right)$$

and

$$\left|\frac{\partial^{1+l} \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(\xi, t)}{\partial \xi \partial t^l}\right| \le k_n t^{\frac{n}{2}-(1+l)}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\xi^2}{t}\right).$$

(ii) *$\hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)$ and $\hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)$ are differentiable with respect to ξ on $(\xi, t) \in (0, \gamma) \times (0, T)$. Moreover, there exists a constant C such that, for every $(\xi, t) \in (0, \gamma) \times (0, 1]$,*

$$\left|\frac{\partial \hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)}{\partial \xi}\right| \le Ct^{-\frac{1}{2}}\left(1 + \frac{|\hat{\gamma} - \xi|}{t}\right)\exp\left(-\frac{1}{4}\hat{c}_0\frac{(\hat{\gamma} - \xi)^2}{t}\right),$$

$$\left|\frac{\partial \hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)}{\partial \xi}\right| \le Ct^{-\frac{1}{2}}\left(1 + \frac{\xi}{t}\right)\exp\left(-\frac{1}{4}\hat{c}_0\frac{\xi^2}{t}\right),$$

$$\left|\frac{\partial^2 \hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)}{\partial \xi \partial t}\right| \le Ct^{-2}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(\hat{\gamma} - \xi)^2}{t}\right)$$

and

$$\left|\frac{\partial^2 \hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)}{\partial \xi \partial t}\right| \le Ct^{-2}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\xi^2}{t}\right).$$

Proof. For part (i): It follows from Proposition 4.2.2 that a bound of this form holds for $n = 0$. The claim can be established by exploiting the symmetry property between t and ϑ, and doing a similar induction as in Lemma 4.1.2. The main problem is the singularity at $t = \vartheta$, but this is only a problem for the first few terms in the sequence.

For part (ii): This can be established from part (i) and the uniform convergence of the derivatives of $\hat{U}_{\hat{\gamma}}^{(n)}(\xi, t)$ as $n \to \infty$. □

Lemma 4.2.5. *There exists a constant C such that, for every $(x, t, \vartheta, \xi) \in \bar{\mathcal{D}}_{\hat{\gamma}}$, and every $l \in \{0, 1, 2\}$ the following inequalities are all valid:*

(i)

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-1}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x \partial t}\right| \le C(t-\vartheta)^{-2}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi \partial x}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

(ii) *For every $(x, t, \xi, \vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$*

$$\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)$$

is the Green function associated with the differential operator $\hat{L}_0$ and Dirichlet boundary conditions.

(iii)

$$\left|\frac{\partial^l \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial y^l}\right| \leq C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi}\right| \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial^2 \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x\partial \xi}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial^2 \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x\partial t}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

Proof. For (i): It follows from the inequalities in Proposition 4.2.3 that the derivatives of $\hat{\Gamma}_{\hat{L}_{0,*}}(x,t,\xi,\vartheta)$ can be written as the sum of terms which behave like the fundamental solution with constant coefficients discussed in Section 4.1.1. Thus we can calculate bounds for the derivatives using integration by parts as in the proof of Lemma 4.1.3. Some extra terms have a weaker singularity. A calculation along these lines yields the stated inequalities.

For part (ii): Because of Lemma 4.2.2 this follows from similar calculations as in the proof of part (iii) of Lemma 4.1.3.

For part (iii): Since, for any $x,\xi \in [0,\hat{\gamma}]$,

$$(x-\xi)^2 \leq \min\left(x^2+\xi^2, (x-\hat{\gamma})^2+(\xi-\hat{\gamma})^2\right),$$

this follows from the bounds given in part (i) and the regularity bounds of the function $\hat{\Gamma}_{\hat{L}_0}$. □

Proposition 4.2.5. *There exists a constant C such that, for every $(x,t) \in (0,\hat{\gamma}) \times (0,1]$:*

$$\left|\int_0^t\int_0^{\hat{\gamma}} \frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x^2} d\xi d\vartheta\right| \leq C\Bigg(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2}{t}\right) + \exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2}{t}\right)\Bigg)$$

and

$$\left|\int_0^t\int_0^{\hat{\gamma}} \frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta\right| \leq C\Bigg(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2}{t}\right) + \exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2}{t}\right)\Bigg).$$

Proof. Let

$$\hat{B}_{\hat{\gamma}}^{(1)}(s) = \int_0^{\hat{\gamma}} U_{\hat{\gamma}}^{(1)}(\xi, s) d\xi, \quad s \in (0,1],$$

and let

$$\hat{B}_{\hat{\gamma}}^{(2)}(s) = \int_0^{\hat{\gamma}} U_{\hat{\gamma}}^{(2)}(\xi, s) d\xi, \quad s \in (0,1].$$

A similar calculation as in the proof of Lemma 4.1.7 yields that

$$\int_0^t \int_0^{\hat{\gamma}} \frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta = I_1 - I_2,$$

where

$$I_1 = \int_0^t B^{(1)}(s) \int_0^{t-\vartheta} \hat{a}_{1,1}(\hat{\gamma}) \left. \frac{\partial^2 \hat{\Gamma}_{\hat{L}_0}(x, t-\vartheta, \eta, s)}{\partial t \partial \eta} \right|_{\eta=\hat{\gamma}} d\vartheta ds,$$

and

$$I_2 = \int_0^t B^{(2)}(s) \int_0^{t-\vartheta} \hat{a}_{1,1}(0) \left. \frac{\partial^2 \hat{\Gamma}_{\hat{L}_0}(x, t-\vartheta, \eta, s)}{\partial t \partial \eta} \right|_{\eta=0} d\vartheta ds.$$

Calculating the integrals above, using the bounds given in Proposition 4.2.3, yields that the stated inequalities are valid. □

Analogous to what we did in Section 4.1.1, we will construct the Green function $\hat{G}_{\hat{L},\hat{\gamma}}$, associated with the entire differential operator $\hat{L}$ and Dirichlet boundary condition, by solving an integral equation.

Definition 4.2.8. *Let*

$$\hat{Q}_{\hat{\gamma},0}(x,t,\xi,\vartheta) := \hat{a}_1^*(x) \frac{\partial \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x}, \quad (x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}.$$

Let the sequence of functions $\left\{\hat{Q}_{\hat{\gamma},n}\right\}_{n=0}^{\infty}$ *be defined recursively for* $n \in 1, 2, \ldots,$ *and* $(x,t,\xi,\vartheta) \in \mathcal{D}_\gamma$ *by*

$$\hat{Q}_{\hat{\gamma},n+1}(x,t,\xi,\vartheta) = \int_\vartheta^t \int_0^{\hat{\gamma}} \hat{Q}_{\hat{\gamma},0}(x,t,z,s) \hat{Q}_{\hat{\gamma},n}(z,s,\xi,\vartheta) dz ds,$$

and let

$$\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{\hat{\gamma},n}(x,t,\xi,\vartheta).$$

Lemma 4.2.6. *Assume that* $\sigma_R > 0$. *Let* $\alpha \in (0,1)$ *and let* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $\hat{Q}_{\hat{\gamma}} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$. *Moreover,* $\hat{Q}_{\hat{\gamma}}$ *is the unique solution in* $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ *of the integral equation*

$$\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{Q}_{\hat{\gamma},0}(x,t,\xi,\vartheta) + \int_\vartheta^t \int_0^{\hat{\gamma}} \hat{Q}_{\hat{\gamma},0}(x,t,z,s) \hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta) dz ds. \tag{4.2.7}$$

(ii) *There exists a constant C such that, for every $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$, every $x', \in (0,\hat{\gamma})$, and every $t' \in (0,t,)$ the following identities and inequalities are all valid:*

$$\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{Q}_{\hat{\gamma}}(x,t-\vartheta,\xi,0),$$

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\gamma}}(x',t,\xi,\vartheta)\right| \leq C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\gamma}}(x,t',\xi,\vartheta)\right| \leq C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii)

$$\left|\frac{\partial\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial\xi}\right| \leq C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.2.5, that $\frac{\partial\hat{G}_{L,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, and hence $\hat{Q}_{\hat{\gamma},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $\hat{Q}_{\hat{\gamma},0} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $\hat{Q}_{\hat{\gamma}}$ is the unique solution in the function space $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ of the integral equation (4.2.7).

For part (ii): It follows from similar calculations as in the proofs of Lemma 4.1.12, that these regularity bounds hold for the function $\sum_{j=0}^{n}\hat{Q}_{\hat{\gamma},j}(x,t,\xi,\vartheta)$, for any n. Furthermore, it can be shown that these sums converge uniformly.

For part (iii): We first observe that this bound holds for $\frac{\partial\hat{Q}_{\hat{\gamma},0}(x,t,\xi,\vartheta)}{\partial\xi}$. A similar calculation as in the proof of Lemma 4.1.4 part (v) yields that, for some constant C

$$\left|\frac{\partial\hat{Q}_{\hat{\gamma},1}(x,t,\xi,\vartheta)}{\partial\xi}\right| \leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

For $n \in 2,3,\ldots$ it can be shown by induction that the functions $\frac{\partial\hat{Q}_{\hat{\gamma},n}(x,t,\xi,\vartheta)}{\partial\xi}$ are less singular, and that the sum $\sum_{j=0}^{n}\frac{\partial\hat{Q}_{\hat{\gamma},n}(x,t,\xi,\vartheta)}{\partial\xi}$ converges uniformly on $\mathcal{D}_{\hat{\gamma}}$, thus allowing the sum to be differentiated term by term. □

Lemma 4.2.7. *There exists a constant C such that following identities and bounds are valid for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\gamma}}$ and every $l \in \{0,1,2\}$:*

(i)

$$\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds = \int_{0}^{t-\vartheta}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t-\vartheta,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,0)\,dzds,$$

$$\frac{\partial^l}{\partial x^l}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds = \int_{\vartheta}^{t}\int_{0}^{\gamma}\frac{\partial^l\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)}{\partial x^l}\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds,$$

$$\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds = \hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) + \int_{\vartheta}^{t}\int_{0}^{\gamma}\frac{\partial\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)}{\partial t}\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds,$$

$$\left|\frac{\partial^l}{\partial x^l}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds\right| \le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds\right| \le C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial}{\partial \xi}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds\right| \le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\int_{-\infty}^{\infty}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzdsd(x-\xi)\right| \le C\sqrt{t}.$$

(ii)

$$\hat{G}_{\hat{L},\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta) + \int_{\vartheta}^{t}\int_{0}^{\hat{\gamma}}\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,z,s)\,\hat{Q}_{\hat{\gamma}}(z,s,\xi,\vartheta)\,dzds.$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in Lemma 4.1.5.

For part (ii): Since $\hat{G}_{\hat{L}_0,\hat{\gamma}}$ is the Green function associated with the differential operator $\hat{L}_0$ and Dirichlet boundary conditions, and $\hat{Q}_{\hat{\gamma}}$ is a solution of the integral equation (4.2.7), this follows from the bounds given in part (i). □

After the next result we will finally be ready to obtain regularity bounds on $\psi_2(y,t)$ (using the original variable y).

Proposition 4.2.6. *Assume that $\sigma_R > 0$, and that the tail distribution $\bar{F}$ satisfies the bound* (4.0.38)*. Let the function $H_{1,\kappa}$ be as in section 3. Then there exists a constant C_β, depending on β, such that, for every $x' > x > 0$ and every $1 \geq t > t' > 0$, the following inequalities are valid:*

(i)
$$\left|H_{1,\kappa}\left(e^x - 1, t\right)\right| \leq C_\beta e^{-\beta x},$$
and for every $\alpha \in (0,1]$
$$\left|H_{1,\kappa}\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^{x'} - 1, t'\right)\right| \leq C_\beta \left(t - t'\right)^\alpha \left(t'\right)^{-\alpha} e^{-\beta x}.$$

(ii) *For every $\alpha \in (0, \min(1,\beta))$*
$$\left|H_{1,\kappa}\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^x - 1, t\right)\right| \leq C_\beta \left|x - x'\right|^\alpha t^{-\frac{\alpha}{2}} \exp\left(-\left(\beta - \alpha\right)\right).$$

Proof. For part (i): These inequalities follow trivially from the bounds given in Proposition 3.0.6.

For part (ii): Assume first that
$$x' - x \geq \frac{1}{2}.$$
For this case the stated inequality is trivially true because of the bound on the function $H_{1,\kappa}$ itself given in Proposition 3.0.6. Assume instead that
$$x' - x < \frac{1}{2}.$$
We observe that in this case
$$e^{x'} - e^x \leq \frac{4}{3} e^x \left(x' - x\right).$$
Because of this bound and the bounds in Proposition 3.0.6 it can be calculated that the stated bound holds even for this case. ☐

Lemma 4.2.8. *There exists a constant C_β, depending on β, such that, for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ and every $y \in (0,\kappa)$ the following inequalities are valid:*

(i)
$$\int_0^t \left|\int_0^{\hat{\kappa}} \frac{\partial^2 \hat{\Gamma}_{\hat{L}_{*,0}}\left(x,t,\xi,\vartheta\right)}{\partial x^2} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi\right| d\vartheta \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$
$$\int_0^t \left|\int_0^{\hat{\kappa}} \frac{\partial \hat{\Gamma}_{\hat{L}_{*,0}}\left(x,t,\xi,\vartheta\right)}{\partial t} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi\right| d\vartheta \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$
$$\left|\int_0^t \int_0^{\kappa} \frac{\partial^2 \hat{g}_{\hat{L}_{*,0},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^2} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta\right| \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\int_0^t\int_0^\kappa \frac{\partial \hat{g}_{\hat{L}_*,0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta\right| \le C_\beta \exp\left(-\frac{1}{2}\beta x\right).$$

(ii) *For every $l \in \{1,2\}$, the following identities are all valid:*

$$\frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l} = \int_0^t\int_0^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta,$$

$$\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t} = H_{1,\kappa}\left(e^x - 1, t\right) + \int_0^t\int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta.$$

(iii) *For every $l \in \{0,1\}$, every $(x',t') \in (x,\kappa)\times(0,t)$ and every $\alpha \in (0,\frac{1}{2}]$ the following bounds are all valid:*

$$\left|\frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{2-l}{2}} \exp(-\beta x),$$

$$\left|\frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta t^{\frac{2-l}{2}} \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x',t)\right| \le C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp(-\beta x),$$

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x,t')\right| \le C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp(-\beta x),$$

and

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t)\right| \le C_\beta \left(\min\left(x, \hat{\kappa}-x\right)\right)^{\frac{1}{2}} t^{\frac{3}{4}}.$$

(iv) *There exists a constant C_β such that for every $(y,t) \in (0,\kappa)\times(0,T]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \psi_{2,\kappa}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}} (1+y)^{-(l+\beta)},$$

$$\left|\frac{\partial^2 \psi_{2,\kappa}(y,t)}{\partial y^2}\right| \le C_\beta (1+y)^{-\frac{1}{2}\beta},$$

$$\left|\frac{\partial \psi_{2,\kappa}(y,t)}{\partial t}\right| \le C_\beta (1+y)^{-\frac{1}{2}\beta}.$$

Proof. For parts (i)-(ii): We first observe that for any $a,b>0$ there exists a constant C depending on a and b such that for any $x,\xi>0$ and $0\le\vartheta<t\le 1$

$$\exp\left(-a\frac{(x-\xi)^2}{t-\vartheta}\right)\exp(-b\xi) \le C\exp\left(-\frac{1}{2}a\frac{(x-\xi)^2}{t-\vartheta}\right)\exp(-bx). \qquad (4.2.8)$$

The identities and bounds given in part (i) and part (ii) follow from the bound above and similar calculations as in the proof of Lemma 4.1.6 and Lemma 4.1.5.

For part (iii): This follows from the bound (4.2.8), the bounds given in part (i), and the identities and bounds given in Proposition 4.2.2, Proposition 4.2.3, Lemma 4.2.5, Lemma 4.2.7 and Proposition 4.2.5.

For part (iv): Since

$$\psi_{2,\kappa}(y,t) = \hat{\psi}_{2,\hat{\kappa}}(\ln(1+y),t),$$

this follows from the bounds given in part (ii) and the chain rule. □

Lemma 4.2.9. *Assume that $\sigma_R > 0$ and that the tail distribution satisfies the bound* (4.0.38)*. Let the function $H_{2,\kappa}$ be as in section 3. Then, for some constant C_β, depending on β, the bounds stated below all hold for every $0 < x < x' < \hat{\kappa}$, every $0 \le t' < t \le 1$ and every $\alpha \in (0, \frac{1}{2}\min(\beta,1)]$:*

$$|H_{2,\kappa}(e^x - 1, t)| \le C_\beta t \exp(-\beta x),$$

$$|H_{2,\kappa}(e^x - 1, t) - H_{2,\kappa}(e^x - 1, t')| \le C_\beta (t-t')^\alpha t^{1-\alpha} \exp(-\beta x),$$

and

$$\left|H_{2,\kappa}(e^x - 1, t) - H_{2,\kappa}\left(e^{x'} - 1, t\right)\right| \le C_\beta (x'-x)^\alpha t^{\frac{2-\alpha}{2}} \exp(-(\beta-\alpha)x).$$

Proof. It follows from similar calculations as in Proposition 4.2.6 that the stated inequalities are valid for $\psi_{2,\kappa}(y,t)$. Similar calculations as in Lemma 4.1.8 yield that the bound inequalities are valid for $H_{2,\kappa}(e^x - 1, t)$. □

4.2.2 Regularity estimates for for a subproblem with an integral term and unbounded coefficients

In this section we will consider the function

$$\hat{\psi}_{3,\hat{\kappa}}(x,t) := \psi_{3,\kappa}(e^x - 1, t), \quad (x,t) \in [0, \ln(\kappa+1)], \quad (x,t) \in [0,\hat{\kappa}] \times [0,1],$$

where as before $\hat{\kappa} = \ln(1+\kappa)$. Since $\psi_{3,\kappa}(e^x - 1, t)$ is a classical solution of the PIDE 3.0.10 it follows from the chain rule that the function $\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a solution of a different PIDE defined in the result below.

Definition 4.2.9. *Let the operator $\hat{A}$ be defined for any function $g(x,t) \in C^{2,1}((0,\hat{\kappa}) \times (0,1])$ as*

$$\hat{A}g(x,t) = \hat{L}g(x,t) - \lambda g(x,t) + \lambda \int_0^{e^x-1} g(\ln(e^x - \zeta), t)\, dF(\zeta).$$

Lemma 4.2.10. *Let the function $H_{2,\kappa}$ be as in section 3. $\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a classical solution of the PIDE*

$$\begin{cases} \hat{\psi}_{3,\hat{\kappa}}(x,0) & = 0, \quad x \in (0,\hat{\kappa}), \\ \hat{\psi}_{3,\hat{\kappa}}(0,t) & = 0, \quad t \in [0,1], \\ \hat{\psi}_{3,\hat{\kappa}}(\hat{\kappa},t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t} & -\hat{A}\hat{\psi}_{3,\hat{\kappa}}(x,t) \\ & = H_{2,\kappa}(e^x - 1, t), \quad (x,t) \in (0, \ln(1+\kappa)) \times (0,1]. \end{cases} \tag{4.2.9}$$

Proof. This is similar to Lemma 4.2.1. □

Theorem 4.2.2. *$\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a unique classical solution of the PIDE (4.2.9)). Furthermore, there exists a unique Green function $\hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)$ associated with the differential operator $\hat{L}$ and Dirichlet boundary conditions on the domain $\mathcal{D}_{\hat{\kappa}}$, i.e. satisfying the conditions in Definition 3.0.1 with L replaced by $\hat{L}$ and κ replaced by $\hat{\kappa}$. Furthermore, for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$*

$$\hat{\psi}_{3,\hat{\kappa}}(x,t) = \int_0^t \int_0^{\hat{\kappa}} \hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta) \, H_{2,\kappa}(e^{\xi} - 1, \vartheta) d\xi d\vartheta.$$

Proof. This can be shown using similar arguments as those that lead to the result in Theorem 3.0.3. The most important difference is that in this case we define the function $j(x,t,\zeta)$ as

$$j(x,t,\zeta) = \begin{cases} -x + \ln(e^x - \zeta), & (x,t,\zeta) \in [0,\hat{\kappa}] \times [0,1] \times [0, e^x - 1] \\ -x + e^x, & (x,t,\zeta) \in [0,\hat{\kappa}] \times [0,1] \times (e^x - 1, \infty). \end{cases}$$

It can be shown that $j(x,t,\zeta)$ is continuously differentiable with respect to x on $[0,\hat{\kappa}]$, and that, for $x \in [0,\hat{\kappa}]$,

$$0 \leq \frac{\partial j(x,t,\zeta)}{\partial x} < \infty,$$

thus satisfying the requirement (VIII.1.23) in Garroni and Menaldi (1992). □

Analogous to what we did in Section 4.1.2 we will construct the Green function $\hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)$ in two steps. The first step is to use the Green function and Proposition VIII.1.2 to construct a Green function $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ associated with the differential operator

$$\hat{L} - \lambda,$$

and the second step is to do the same once again to construct the full Green function from $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$, as was the case in Section 4.1.2.

Definition 4.2.10. *Let*

$$\hat{Q}_{\lambda,\hat{\kappa},0} = -\lambda \hat{G}_{\hat{L},\hat{\kappa}},$$

and let the sequence of function $\left\{\hat{Q}_{\lambda,\hat{\kappa},n}\right\}_{n=0}^{\infty}$ be defined inductively for $n \in 1,2,\ldots,$ and $(x,t,\xi,\vartheta) \in \mathcal{D}_\kappa$, by

$$\hat{Q}_{\lambda,\hat{\kappa},n}(x,t,\xi,\vartheta) = \int_\vartheta^t \int_0^{\hat{\kappa}} \hat{Q}_{\lambda,\hat{\kappa},0}(x,t,z,s) \, \hat{Q}_{\lambda,\hat{\kappa},n}(z,s,\xi,\vartheta) \, dzds,$$

and let

$$\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{\lambda,\hat{\kappa},n}(x,t,\xi,\vartheta).$$

Lemma 4.2.11. *Assume that $\sigma_R > 0$. Let $\alpha \in (0,1)$ and let $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *and* $\hat{Q}_{\lambda,\hat{\kappa}} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$*. Moreover* $\hat{Q}_{\lambda,\hat{\kappa}}$ *is the unique solution in* $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *of the integral equation*

$$\begin{aligned} \hat{Q}_{\lambda,\hat{\kappa}}(x,t,z,\vartheta) = & -\lambda \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta) \\ & -\lambda \int_{\vartheta}^{t} \int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\, \hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds. \end{aligned} \tag{4.2.10}$$

(ii) $\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta)$ *is differentiable with respect to all four variables. Furthermore, there exists a constant* C*, such that, for every* $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\kappa}}$*, the following identities and inequalities are all valid:*

$$\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta) = \hat{Q}_{\lambda,\hat{\kappa}}(x,t-\vartheta,\xi,0),$$

$$\left| \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial x}(x,t,\xi,\vartheta) \right| \leq C (t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta} \right),$$

$$\left| \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial \xi}(x,t,\xi,\vartheta) \right| \leq C (t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta} \right),$$

and

$$\left| \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial t}(x,t,\xi,\vartheta) \right| \leq C (t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta} \right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.2.5 and Lemma 4.2.7, that $\hat{G}_{L,\hat{\kappa}}(x,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and hence $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $\hat{Q}_{\lambda,\hat{\kappa}}$ is the unique solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation (4.2.10).

For part (ii): This can be shown using the same calculations and reasoning as in the proof of Lemma 4.1.4, based on induction, the symmetry property between the t and ϑ variable, and uniform convergence. □

Lemma 4.2.12. *Assume that* $\sigma_R > 0$.

(i) *For every* $(x,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and* $l \in \{0,1,2\}$

$$\begin{aligned} & \int_{\vartheta}^{t} \int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\, \hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds \\ & \quad = \int_{0}^{t-\vartheta} \int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t-\vartheta,z,s)\, \hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,0)\, dzds, \end{aligned}$$

$$\begin{aligned} & \frac{\partial^l}{\partial x^l} \int_{\vartheta}^{t} \int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\, \hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds \\ & \quad = \int_{\vartheta}^{t} \int_{0}^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)}{\partial x^l} \hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds, \end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\\
&\qquad=\hat{Q}_{\lambda,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)\\
&\qquad\quad+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\frac{\partial\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)}{\partial t}\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds.
\end{aligned}
$$

Furthermore, for some constant C

$$
\begin{aligned}
&\left|\frac{\partial^{l}}{\partial x^{l}}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right|\\
&\qquad\leq C\left(t-\vartheta\right)^{\frac{1-l}{2}}\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right|\\
&\qquad\leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right).
\end{aligned}
$$

(ii) *For some constant* C

$$
\begin{aligned}
\left|\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right| &\leq C\left(\min\left(x,\hat{\kappa}-x\right)\right)^{\frac{1}{2}}\\
&\quad\times\left(t-\vartheta\right)^{\frac{1}{4}}\\
&\quad\times\left(\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right)\right.\\
&\quad+\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\xi^{2}}{t-\vartheta}\right)\\
&\quad\left.+\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(\hat{\kappa}-\xi\right)^{2}}{t-\vartheta}\right)\right).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)&=\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)\\
&\quad+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds.
\end{aligned}
$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in Lemma 4.1.4 and Lemma 4.1.5.

For part (ii): This can be calculated from the bounds given in Lemma 4.2.7.

For part (iii): Since $\hat{G}_{\hat{L},\hat{\kappa}}$ is the Green function associated with the differential operator $\hat{L}$ it can be derived from the bounds given in part (i) and (ii) that

$$\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds$$

is the unique (principal) Green function associated with the differential operator

$$\hat{L}-\lambda$$

and Dirichlet boundary conditions. □

Definition 4.2.11. *For $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$ let*

$$\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)=\lambda\int_{0}^{e^{x}-1}\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}\left(\ln\left(e^{x}-\zeta\right),t,\xi,\vartheta\right)dF(\zeta),$$

and let the sequence of functions

$$\left\{\hat{Q}_{I,\hat{\kappa},n}\right\}_{n=0}^{\infty}$$

be defined inductively by

$$\hat{Q}_{I,\hat{\kappa},n}(x,t,\xi,\vartheta)=\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{Q}_{I,\hat{\kappa},0}(x,t,z,s)\,\hat{Q}_{I,\hat{\kappa},n-1}(z,s,\xi,\vartheta)\,dzds,$$
$$n\in 1,2,\ldots.$$

Let

$$\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)=\sum_{n=0}^{\infty}\hat{Q}_{I,\hat{\kappa},n}(x,t,\xi,\vartheta)$$

and

$$\hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)=\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{I,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds. \tag{4.2.11}$$

Proposition 4.2.7. *Assume that $\sigma_R>0$ and that the tail distribution $\bar{F}$ satisfies the inequality (4.0.38). Then there exists a constant C_β, depending on β, such that the following inequalities are valid for every $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$, every $(x',t')\in(x,\hat{\kappa})\times[0,t)$ and every $\alpha\in\left(0,\min\left(\frac{1}{2},\frac{\beta}{2}\right)\right)$:*

$$\left|\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)\right|\leq C_\beta(t-\vartheta)^{-\frac{1}{2}}\times\left(\exp\left(-\left(\frac{1}{32}c_0(x-\xi)^2+2\beta|x-\xi|\right)\right)+\exp(-\beta x)\right). \tag{4.2.12}$$

$$\begin{aligned}&\left|\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)-\hat{Q}_{I,\hat{\kappa},0}(x,t',\xi,\vartheta)\right|\\&\quad\leq C_\beta(t-t')^{\frac{1}{4}}(t'-\vartheta)^{-\frac{3}{4}}\\&\qquad\times\left(\exp\left(-\left(\frac{1}{32}c_0(x-\xi)^2+2\beta|x-\xi|\right)\right)\right.\\&\qquad\left.+\exp(-\beta x)\right).\end{aligned} \tag{4.2.13}$$

Also

$$\begin{aligned}&\left|\hat{Q}_{I,\hat{\kappa},0}\left(x,t,\xi,\vartheta\right)-\hat{Q}_{I,\hat{\kappa},0}\left(x',t,\xi,\vartheta\right)\right|\\&\quad\le C_{\beta}\left|x-x'\right|^{\alpha}\left(t-\vartheta\right)^{-\frac{1+\alpha}{2}}\left(\exp\left(-\left(\frac{1}{32}c_0\left(x-\xi\right)^2+2\beta\left|x-\xi\right|\right)\right)\right.\\&\qquad+\exp\left(-\left(\frac{1}{32}c_0\left(x'-\xi\right)^2+2\beta\left|x'-\xi\right|\right)\right)\\&\qquad\left.+\exp\left(-\left(\beta-\alpha\right)x\right)\right).\end{aligned} \tag{4.2.14}$$

Proof. It is obvious that the stated bounds hold if

$$|x-\xi|\le 1.$$

Furthermore, it follows from the bounds given in Lemma 4.2.5, Lemma 4.2.7 and Lemma 4.2.12 that there exists a constant C such that, for every $\zeta\in[0,e^x-1]$ and $(y,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$,

$$\left|\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right)\right|\le C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\left(\ln\left(e^x-\zeta\right)-\xi\right)^2\right). \tag{4.2.15}$$

Thus a simple calculation yields that, if

$$x\le\xi-1 \tag{4.2.16}$$

then for some (other) constants K and C and a constant C_{β}, depending on β,

$$\begin{aligned}\left|\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right)\right|&\le K\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(\ln\left(e^x-\zeta\right)-\xi\right)^2}{t-\vartheta}\right)\\&\le C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\left(x-\xi\right)^2\right).\end{aligned}$$

Assume that

$$x>\xi+1. \tag{4.2.17}$$

Another simple calculation yields that, if (4.2.17) holds and

$$\zeta\le e^x-e^{\frac{1}{2}(x+\xi)},$$

then

$$\left(\ln\left(e^x-\zeta\right)-\xi\right)^2\ge\frac{1}{4}\left(x-\xi\right)^2,$$

while for any ζ such that

$$\zeta>e^x-e^{\frac{1}{2}(x+\xi)},$$

it follows from the assumed inequalities (4.2.17) and (4.0.38) that

$$\bar{F}\left(\zeta\right)\le C_{\beta}e^{-\beta x}.$$

From the inequalities above it is clear that the inequality (4.2.12) holds. The inequalities (4.2.13) and (4.2.14) follow from similar calculations as above and as in the proof of Proposition4.2.6. □

Lemma 4.2.13. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ obeys the bound* (4.0.38).

(i) *Let $\alpha \in (0,1)$ and let $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992). Then $\hat{Q}_{I,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and $\hat{Q}_{I,\hat{\kappa}} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Moreover $\hat{Q}_{I,\hat{\kappa}}$ is the unique solution in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation*

$$\begin{aligned}\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta) =& \lambda \int_0^{e^x-1} \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right) dF(\zeta) \\ &+ \lambda \int_0^{e^x-1} \hat{G}_{I,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right) dF(\zeta).\end{aligned} \tag{4.2.18}$$

(ii) *There exists a constant C_β, depending on β such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$, and every $(x',t) \in (x,\hat{\kappa}) \times (\vartheta,t)$*

$$\left|\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \exp\left(-\beta\left|x-\xi\right|\right), \tag{4.2.19}$$

$$\left|\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \times \Big(\exp\left(-2\beta\left|x-\xi\right|\right) + \exp\left(-\beta x\right)\Big), \tag{4.2.20}$$

$$\begin{aligned}\left|\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta) - \hat{Q}_{I,\hat{\kappa}}(x,t',\xi,\vartheta)\right| \le& C_\beta (t-t')^{\frac{1}{4}} (t'-\vartheta)^{-\frac{3}{4}} \\ &\times \Big(\exp\left(-2\beta\left|x-\xi\right|\right) \\ &+ \exp\left(-\beta x\right)\Big),\end{aligned} \tag{4.2.21}$$

and

$$\begin{aligned}\left|\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta) - \hat{Q}_{I,\hat{\kappa}}(x',t,\xi,\vartheta)\right| \le& C_\beta \left|x-x'\right|^\alpha (t-\vartheta)^{-\frac{1+\alpha}{2}} \\ &\times \Big(\exp\left(-2\beta\left|x-\xi\right|\right) \\ &+ \exp\left(-2\beta\left|x'-\xi\right|\right) \\ &+ \exp\left(-\left(\beta-\alpha\right)x\right)\Big).\end{aligned} \tag{4.2.22}$$

(iii) *For every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1,2\}$ the following identities are all valid:*

$$\frac{\partial^l \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l} = \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}(x,t,z,\vartheta)}{\partial x^l} \hat{Q}_{I,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds,$$

and

$$\begin{aligned}\frac{\partial \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} =& \hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta) \\ &+ \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}(x,t,z,\vartheta)}{\partial t} \hat{Q}_{I,\hat{\kappa}}(z,s,\xi,\vartheta)\, dzds.\end{aligned}$$

(iv) *There exists a constant C_β depending on β such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C_\beta (t-\vartheta)^{\frac{1-l}{2}} \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big),$$

$$\left|\frac{\partial^2 \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^2}\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right),$$

$$\left|\frac{\partial \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t}\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right)$$

and

$$\left|\hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)\right| \le C_\beta \min(x,\hat{\kappa}-x)^{\frac{1}{4}} (t-\vartheta)^{\frac{1}{4}} \left(\exp\left(-\frac{1}{2}\beta|x-\xi|\right) + \exp\left(-\frac{1}{2}\beta\xi\right) + \exp\left(-\frac{1}{2}\beta(\hat{\kappa}-\xi)\right)\right).$$

(v)

$$\hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta) = \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}(x,t,\xi,\vartheta) + \hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta).$$

Proof. For part (i): Because of the bounds obeyed by $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ this follows from a similar argument as in the proof of Lemma 4.1.12.

For part (ii): Because of the bound (4.2.8) and the bounds given in Proposition 4.2.7, this follows from similar calculations, based on induction and uniform convergence, as in the proof of Lemma 4.1.12. □

For part (iii) *and part* (iv)*: This follows from similar calculations as in the proofs of Lemma 4.1.5 and Lemma 4.2.12, and using the bounds given in part* (ii) *and the bound* (4.2.8)*.*

For part (v)*: Since $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ is the Green function associated with the differential operator*

$$\hat{L} - \lambda,$$

this follows from the bounds and identities given in part (i)-(iv)*.*

Theorem 4.2.3. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies the bound* (4.0.38)*.*

(i) *For every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ and every $l \in \{1,2\}$*

$$\frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l} = \int_0^t \int_0^\infty \frac{\partial^l \hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l} H_{2,\hat{\kappa}}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta$$

and

$$\frac{\partial \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t} = H_{2,\hat{\kappa}}\left(e^{x}-1,\vartheta\right) + \int_0^t \int_0^\infty \frac{\partial \hat{G}_{\hat{A},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial t} H_{2,\hat{\kappa}}\left(e^{\xi}-1s,\vartheta\right) d\xi d\vartheta.$$

(ii) *There exists a constant* C_β*, depending on* β*, such that, for every* $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ *and every* $l \in \{0,1\}$ *the following inequalities are all valid*

$$\left|\frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l}\right| \leq C_\beta t^{\frac{4-l}{2}} \exp\left(-\beta x\right),$$

$$\left|\frac{\partial^2 \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^2}\right| \leq C_\beta t \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t}\right| \leq C_\beta t \exp\left(-\frac{1}{2}\beta x\right)$$

and

$$\left|\hat{\psi}_{3,\hat{\kappa}}(x,t)\right| \leq C_\beta t \min\left(x,\hat{\kappa}-x\right).$$

(iii) *There exists a constant* C_β*, depending on* β*, such that, for every* $(y,t) \in (0,\kappa) \times (0,1]$ *and every* $l \in \{0,1\}$

$$\left|\frac{\partial^l \psi_{3,\kappa}\left(y,t\right)}{\partial y^l}\right| \leq C_\beta t^{\frac{4-l}{2}} \left(1+y\right)^{-(\beta+l)}$$

$$\left|\frac{\partial^2 \psi_{3,\kappa}\left(y,t\right)}{\partial y^2}\right| \leq C_\beta t \left(1+y\right)^{-\left(\frac{1}{2}\beta+2\right)}$$

and

$$\left|\frac{\partial \psi_{3,\kappa}\left(y,t\right)}{\partial t}\right| \leq C_\beta t \left(1+y\right)^{-\frac{1}{2}\beta}$$

(iv) *There exists a constant* C *and a constant* C_β*, depending on* β*, such that, for every* $(y,t,) \in (0,\kappa) \times (0,1]$ *and* $l \in \{0,1\}$*,*

$$\left|\psi_{3,\kappa}\left(x,t\right)\right| \leq C_\beta t \min\left(y,\kappa-y\right),$$

$$\left|\frac{\partial^l \psi_{\kappa}\left(y,t\right)}{\partial y^l}\right| \leq C t^{-\frac{l}{2}} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta t^{\frac{2-l}{2}} \left(1+y\right)^{-(\beta+l)},$$

$$\left|\frac{\partial^2 \psi_{\kappa}\left(y,t\right)}{\partial y^2}\right| \leq C t^{-1} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta \left(1+y\right)^{-\left(\frac{1}{2}\beta+2\right)},$$

and

$$\left|\frac{\partial \psi_{\kappa}\left(y,t\right)}{\partial t}\right| \leq C t^{-1} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta \left(1+y\right)^{-\frac{1}{2}\beta}.$$

Proof. For part (i)-(ii): This can be calculated from the representation formula given in Theorem 4.2.2 the bounds on $H_{2,\kappa}$ given in Lemma 4.1.8 and the bounds on the Green functions given in Lemma 4.2.12 and Lemma 4.2.13.

For part-(iii)-(iv): These bounds follows from the bounds given in part (ii), the bounds already obtained for for $\psi_{1,\kappa}$ and $\psi_{2,\kappa}$, the Middle value theorem and the chain rule. □

5 Existence on an unbounded domain

In this section we will finally prove the existence of a classical solution, except at the origin, of the equation

$$\begin{cases} \psi(y,0) = 0, \quad y > 0, \\ \psi(0,t) = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi(y,t) = 0, \quad t \in [0,1], \\ \frac{\partial \psi(y,t)}{\partial t} - A\psi(y,t) = \lambda \bar{F}(y), \quad (y,t) \in (0,\infty)\times(0,1]. \end{cases} \tag{5.0.23}$$

Analogous to what we did in Section 3 we will look for a solution $\psi(y,t)$ of (5.0.23) by considering the three equations

$$\begin{cases} \psi_1(y,0) & = 0, \quad y > 0, \\ \psi_1(0,t) & = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_1(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_1(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1(y,t)}{\partial y^2} + p\frac{\partial \psi_1(y,t)}{\partial y}, \quad (y,t) \in (0,\infty)\times(0,1], \end{cases} \tag{5.0.24}$$

$$\begin{cases} \psi_2(y,0) & = 0, \quad y > 0, \\ \psi_2(0,t) & = 0, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_2(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_2(y,t)}{\partial t} - L\psi_2 & = H_1(y,t), \quad (y,t) \in (0,\infty)\times(0,1], \end{cases} \tag{5.0.25}$$

where

$$\begin{aligned} H_1(y,t) = \frac{1}{2}\sigma_R^2 y^2 \frac{\partial^2 \psi_1(y,t)}{\partial^2 y^2} + ry\frac{\partial \psi_1(y,t)}{\partial y} - \lambda\psi_1(y,t) \\ + \lambda \int_0^y \psi_1(y-z,t)\, dF(z) + \lambda \bar{F}(y), \end{aligned}$$

and

$$\begin{cases} \psi_3(y,0) & = 0, \quad y > 0, \\ \psi_3(0,t) & = 0, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_3(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_3 y,t)}{\partial t} - A\psi_3(y,t) & = H_2(y,t), \quad (y,t) \in (0,\infty)\times(0,1], \end{cases} \tag{5.0.26}$$

where

$$H_2(y,t) = -\lambda\psi_2(y,t) + \lambda \int_0^y \psi_2(y-z,t)\, dF(z).$$

As discussed in section (3) we already have a solution for the first equation, given as

$$\psi_1(y,t) = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sigma_P \sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{y}{\sigma_P \sqrt{2\pi}} \int_0^t s^{-\frac{3}{2}} e^{-\frac{(y+\hat{p})^2}{2\sigma_P^2 s}} ds. \tag{5.0.27}$$

Since we also have the representation formula (3.0.15) we immediately get the regularity result given below.

Lemma 5.0.14. (i) *There exists a constant C such that for every $(y,t) \in (0,\infty)\times(0,1]$, every $l \in \{0,1,2,3\}$ and every $m \in \{0,1,2\}$, the following identity and inequalities are all valid:*

$$\frac{\partial \psi_1(y,t)}{\partial t} = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1(y,t)}{\partial y^2} + p\frac{\partial \psi_1(y,t)}{\partial y},$$

$$0 < \psi_1(y,t) < 1,$$

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \leq Ct^{-\frac{l}{2}} \exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

$$\left|\frac{\partial \psi_1(y,t)}{\partial t}\right| \leq Ct^{-1} \exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

$$\left|y^m\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \leq Ct^{-\frac{l-m}{2}} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right),$$

$$\left|\frac{\partial^{1+m} y^m \psi_1(y,t)}{\partial t \partial y^m}\right| \leq Ct^{-1} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right).$$

(ii) *There exists a constant C such that for every $(y,t) \in (0,\kappa)\times(0,1]$ and every $l \in \{0,1,2,3\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l} - \frac{\partial^l \psi_{1,\kappa}(y,t)}{\partial y^l}\right| \leq C \exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right),$$

$$\left|\frac{\partial \psi_1(y,t)}{\partial t} - \frac{\partial^l \psi_{1,\kappa}(y,t)}{\partial t}\right| \leq C \exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right).$$

Proof. For (i): This follows from similar calculations as described in Lemma 3.0.1.

For (ii): For every $(y,t) \in (0,\kappa)\times(0,1]$ the symmetry properties of the function $\Gamma_{\sigma_P,p}$ yield that

$$\begin{aligned}\psi_{1,\kappa}(y,t) - \psi_1(y,t) = \sigma_P^2 \Bigg\{ &\int_0^t \left.\frac{\partial \Gamma_{\sigma_P,p}(y-\xi,s,0,0)}{\partial \xi}\right|_{\xi=\kappa} U_\kappa^{(1)}(t-s)\, d\vartheta \\ &- \int_0^t \left.\frac{\partial \Gamma_{\sigma_P,p}(y-\xi,s,0,0)}{\partial \xi}\right|_{\xi=0} \\ &\times \Big(U_\kappa^{(2)}(t-s) - U(t-s)\Big)\, ds \Bigg\},\end{aligned} \tag{5.0.28}$$

from which the stated bounds can be calculated using integration by parts. □

In a way that is analogous to the discussion in section 4 we will need regularity results for the functions $H_1(y,t)$ and $H_2(y,t)$. Because of the result above we immediately get the regularity result below, which is very similar to Proposition 3.0.6.

Lemma 5.0.15. *Assume that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38)*.*

(i) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\infty)$, every $y' > 0$, every $t' \in (0,t)$ and every $\alpha \in (0,1]$ the following inequalities are all valid:*

$$\begin{aligned} |H_1(y,t)| &\leq C_\beta \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y',t)| &\leq C_\beta \left|y-y'\right|^\alpha t^{-\frac{\alpha}{2}} \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y,t')| &\leq C_\beta \left(t-t'\right)^\alpha t'^{-\alpha} \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y,t')| &\leq C_\beta \left(t-t'\right)^\alpha t'^{-\alpha} \left(1+y\right)^{-\beta}, \end{aligned}$$

and, for every $\alpha \in \left(0, \min\left(1, \frac{\beta}{2}\right)\right)$, and $x, x' > 0$

$$\left| H_1\left(e^x - 1, t\right) - H_1\left(e^{x'} - 1, t\right) \right| \leq C_\beta \left|x - x'\right|^\alpha t^{-\frac{\alpha}{2}} \exp\left(-\frac{\beta x}{2}\right).$$

(ii) *There exists a constant C, such that, for every $(y,t) \in (0,\kappa)$, every $y' \in (0,\kappa)$, every $t' \in (t,1)$ and every $\alpha \in (0,1]$*

$$\begin{aligned} |(H_1(y,t) - H_{1,\kappa}(y,t))| &\leq C \exp\left(-\frac{1}{4} c_0 \frac{\kappa^2}{t}\right), \\ |(H_1(y,t) - H_{1,\kappa}(y,t)) - (H_1(y',t) - H_{1,\kappa}(y',t))| &\leq C \left|y - y'\right|^\alpha \\ &\quad \times \exp\left(-\frac{1}{8} c_0 \frac{\kappa^2}{t}\right), \\ |(H_1(y,t) - H_{1,\kappa}(y,t)) - (H_1(y,t') - H_{1,\kappa}(y,t'))| &\leq C \left(t - t'\right)^\alpha \\ &\quad \times \exp\left(-\frac{1}{8} c_0 \frac{\kappa^2}{t}\right). \end{aligned}$$

and, for every $x, x' \in (0, \ln(1+\kappa))$,

$$\begin{aligned} &\left| \left(H_1\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^x - 1, t\right)\right) - \left(H_1\left(e^{x'} - 1, t\right) - H_{1,\kappa}\left(e^{x'} - 1, t\right)\right) \right| \\ &\qquad \leq C \left|x - x'\right|^\alpha \exp\left(-\frac{1}{16} c_0 \frac{\kappa^2}{t}\right). \end{aligned}$$

Proof. For (i): This follows from the bounds given in 5.0.14 and similar calculations as in Lemma 3.0.2 and Proposition 4.2.6.

For (ii): This follows from the bounds given in Lemma 5.0.14 and similar calculations as in Lemma 4.1.8 and Proposition 4.2.6. □

5.1 Constant coefficients

In this section we will again assume that $\sigma_R = r = 0$. The main idea is to show that, for any sequence $\{\kappa_n\}_{n=0}^\infty$ such that

$$\lim_{n\to\infty} \kappa_n = \infty,$$

the sequences of functions $\{\psi_{2,\kappa_n}\}_{n=0}^{\infty}$ and $\{\psi_{3,\kappa_n}\}_{n=0}^{\infty}$ and their derivatives converge uniformly to solutions ψ_2 and ψ_3 and their derivatives of equations (5.0.25) and (5.0.26), respectively.

Definition 5.1.1. *For $\xi > 0$ and $t \in (0,1]$ let*

$$V_{\xi,0}(t) := -2\Gamma_{\sigma_P}(0,t,\xi,0),$$

and for $n \in \{0,1,2\ldots,\}$, let

$$V_{\xi,n+1}(t) := -2P^{(2)}_{V_{\xi,n}}(0,t).$$

Let

$$U_\xi(t) := \sum_{n=0}^{\infty} V_{\xi,n}(t).$$

Lemma 5.1.1. *Assume that $\sigma_R = r = 0$.*

(i) *U_ξ is differentiable. Furthermore, there exists a constant C such that for every $t \in (0,1]$, every $\xi > 0$ the following identity and inequalities are all valid:*

$$-\frac{1}{2}U_\xi(t) - P^{(2)}_{U_{\xi,n}} = \Gamma_{\sigma_P}(0,t,\xi,0),$$

$$|U_\xi(t)| \le Ct^{-\frac{1}{2}}\exp\left(-\frac{1}{2}c_0\frac{\xi^2}{t}\right)$$

and

$$\left|U'_\xi(t)\right| \le Ct^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{\xi^2}{t}\right).$$

(ii) *There exists a constant C such that for every $t \in (0,1]$, every $\xi \in (0,\kappa)$*

$$\left|U_\xi(t) - U^{(2)}_{\xi,\kappa}(t)\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right),$$

and

$$\left|U'_\xi(t) - U^{(2)\prime}_{\xi,\kappa}(t)\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right).$$

(iii) *For every fixed $\xi > 0$*

$$g^*_{L_0}(y,t) := -P^{(2)}_{U_\xi}(y,t)$$

is a classical solution of the PDE

$$\begin{cases} g^*_{L_0}(y,0,\xi) & = 0, \quad y > 0, \\ g^*_{L_0}(0,t,\xi) & = \Gamma_{\sigma_P}(0,t,\xi,0), \quad t \in (0,1], \\ \lim_{y\to\infty} g^*_{L_0}(y,t,\xi) & = 0, \\ \frac{\partial g^*_{L_0}(y,t,\xi)}{\partial t} & = Lg^*_{L_0}(y,t,\xi), \quad (y,t) \in (0,\infty)\times(0,1], \end{cases}$$

Proof. For parts (i) and (ii): We observe that

$$V_{\xi,0}(t) = V^{(2)}_{\xi,0,\kappa}(t).$$

The stated identity and inequalities follow from similar calculations, based on induction and uniform convergence, as in Lemma 4.1.2.

For part (iii): Let $\{\kappa_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \kappa_n = \infty,$$

and consider the sequence of functions

$$\left\{g^*_{L_0,\kappa_n}(y,t,\xi)\right\}_{n=0}^{\infty}.$$

It follows from the bounds given in Lemma 4.1.3 that there exists a constant C such that, for any n such that $\kappa_n > \xi + 1$

$$\left|g^*_{L_0,\kappa_n}(y,t,\xi)\right| \le C\left(1+\xi^{-2}\right)\exp\left(-\frac{1}{8}c_0\frac{\xi^2}{t}\right).$$

Thus, $g^*_{L_0,\kappa_n}$ satisfies the initial condition. Also, for any $t_0 \in [0,1]$

$$\begin{aligned}\lim_{(y,t)\to(0,t_0)} g^*_{L_0}(y,t,\xi) &= \lim_{n\to\infty}\lim_{(y,t)\to(0,t_0)} g^*_{L_0,\kappa_n}(y,t,\xi)\\ &= \begin{cases}\Gamma_{\sigma_P}(0,t_0,\xi,0), & t_0>0\\ 0, & t_0=0.\end{cases}\end{aligned}$$

The uniqueness follows from Theorem I.3.1 in Garroni and Menaldi (1992) and similar arguments as in the proof of Theorem 3.0.1. □

Definition 5.1.2. *For every $y,\xi>0$ and $0\le\vartheta<t\le 1$ let*

$$g_{L_0}(y,t,\xi,\vartheta) = -P^{(2)}_{U_\xi}(y,t-\vartheta).$$

Lemma 5.1.2. *Assume that $\sigma_R = r = 0$.*

(i) *There exists a constant C such that for every $y,\xi>0$ and $0\le\vartheta<t\le 1$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)$$

and

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2+\frac{1}{2}\xi^2}{t-\vartheta}\right).$$

(ii) *There exists a constant C such that for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l} + \frac{\partial^l}{\partial y^l}P^{(2)}_{U^{(2)}_{\xi,\kappa}}(y,t-\vartheta)\right| \le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}c_0\frac{\kappa^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t} + \frac{\partial}{\partial t}P^{(2)}_{U^{(2)}_{\xi,\kappa}}(y,t-\vartheta)\right| \le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}c_0\frac{\kappa^2+\xi^2}{t-\vartheta}\right),$$

$$\begin{aligned}&\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l} - \frac{\partial^l g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|\\ &\le Ct^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right),\end{aligned}$$

and

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t} - \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right|$$
$$\leq Ct^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(\kappa-y)^2 + \frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ the following inequalities are all valid:*

$$\int_0^t \int_\kappa^\infty \left|\frac{\partial^2 g_{L_0}(y,t,\xi,\vartheta)}{\partial y^2}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}c_0 \frac{\kappa^2}{t}\right),$$

and

$$\int_0^t \int_\kappa^\infty \left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}c_0 \frac{\kappa^2}{t}\right).$$

(iv) *There exists a constant C such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$*

$$\left|\int_0^t \int_0^\kappa \left(\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l} - \frac{\partial^l g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right) d\xi d\vartheta\right|$$
$$\leq Ct^{\frac{2-l}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(\kappa-y)^2}{t}\right),$$

and such that

$$\left|\int_0^t \int_0^\kappa \left(\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t} - \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right) d\xi d\vartheta\right|$$
$$\leq C \exp\left(-\frac{1}{2}c_0 \frac{(\kappa-y)^2}{t}\right).$$

Proof. This follows from similar calculations as in the proof of Lemma 4.1.3 and also the bounds given in Lemma 4.1.3. □

Definition 5.1.3. *For every $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ let*

$$G_{L_0}(y,t,\xi,\vartheta) = \Gamma_{\sigma_P}(y,t,\xi,\vartheta) - g_{L_0}(y,t,\xi,\vartheta),$$

let

$$Q_0(y,t,\xi,\vartheta) := p\frac{\partial G_{L_0}(y,t,\xi,\vartheta)}{\partial x},$$

let the sequence of function $\{Q_n\}_{n=0}^\infty$ be defined inductively for $n \in 1,2,\ldots,$ by

$$Q_n(y,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\infty Q_0(y,t,z,s)\, Q_{n-1}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$Q(y,t,\xi,\vartheta) = \sum_{n=0}^\infty Q_n(y,t,\xi,\vartheta).$$

Lemma 5.1.3. *Assume that $\sigma_R > 0$ and let $\alpha \in (0,1)$.*

(i) *Q solves the integral equation*

$$Q(y,t,\xi,\vartheta) = Q_0(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty Q_0(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds. \tag{5.1.1}$$

(ii) *There exists a constant C, such that, for every $y, y', \xi > 0$ and every $0 \le \vartheta < t' < t \le 1$ the following identities and inequalities are all valid:*

$$Q(y,t,\xi,\vartheta) = Q(y,t-\vartheta,\xi,0),$$

$$|Q(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$|Q(y,t,\xi,\vartheta) - Q(y',t,\xi,\vartheta)| \le C|y-y'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$|Q(y,t,\xi,\vartheta) - Q(y,t',\xi,\vartheta)| \le C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

(iii) *There exists a constant C, such that, for every $(y,t,\xi,\vartheta) \in \bar{\mathcal{D}}_\kappa$, every $y', \in (0,\kappa)$,and every $t' \in (0,t,)$ the following identities and inequalities are all valid:*

$$|Q(y,t,\xi,\vartheta) - Q_\kappa(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-1} \times \exp\left(-\frac{1}{4}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

$$|Q(y,t,\xi,\vartheta) - Q(y',t,\xi,\vartheta) - (Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y',t,\xi,\vartheta))| \le C|y-y'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

and

$$|Q(y,t,\xi,\vartheta) - Q(y,t',\xi,\vartheta) - (Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y,t',\xi,\vartheta))| \le C|t-t'|^{\frac{1}{4}}(t-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

Proof. This follows from similar calculations and reasoning as in the proofs of Lemma 4.1.4. □

Proposition 5.1.1. (i) *For every* $y, \xi > 0$ *and* $0 \le \vartheta < t \le 1$ *and* $l \in \{0,1,2\}$

$$\frac{\partial^l}{\partial y^l} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds$$
$$= \int_\vartheta^t \int_0^\infty \frac{\partial^l G_{L_0}(y,t,z,s)}{\partial y^l} Q(z,s,\xi,\vartheta)\, dzds,$$

and

$$\frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds$$
$$= Q(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial G_{L_0}(y,t,z,s)}{\partial t} Q(z,s,\xi,\vartheta)\, dzds.$$

(ii) *There exists a constant* C *such that for every* $y, \xi > 0$ *and every* $0 \le \vartheta < t \le 1$

$$\left| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \right|$$
$$\le C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \right|$$
$$\le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

(iii) *There exists a constant* C *such that for every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and every* $l \in \{0,1,2\}$

$$\left| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \left(\int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \right.\right.$$
$$\left.\left. - \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dz \right) \right|$$
$$\le C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{8} c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \left(\int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dz \right.\right.$$
$$\left.\left. - \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dz \right) ds \right|$$
$$\le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{8} c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

Proof. Because of the regularity bounds obeyed by Q, given in Lemma 5.1.3, this follows from similar calculations as in the proof of Lemma 4.1.5. □

Definition 5.1.4. *Let*

$$G_L(y,t,\xi,\vartheta) := G_{L_0}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds,$$

and let

$$\psi_2(y,t) := \int_0^t \int_0^\infty G_L(y,t,\xi,\vartheta)\,H_{1,\kappa}(y,\vartheta)\,d\xi d\vartheta.$$

Theorem 5.1.1. *Assume that $\sigma_R = r = 0$ and that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38)*.*

(i) *For every $y > 0$, every $t \in (0,1]$ and every $l \in \{1,2\}$*

$$\frac{\partial^l \psi_2(y,t)}{\partial y^l} = \int_0^t \int_0^\infty \frac{\partial^l G_L(y,t,\xi,\vartheta)}{\partial y^l} H_1(\xi)\,d\xi d\vartheta$$

and

$$\frac{\partial \psi_2(y,t)}{\partial t} = H_1(y,t) + \int_0^t \int_0^\infty \frac{\partial G_L(y,t,\xi,\vartheta)}{\partial t} H_1(\xi,\vartheta)\,d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $y > 0$ and $t \in (0,1]$ and every $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}}(1+y)^{-\beta},$$

$$\left|\frac{\partial \psi_2(y,t)}{\partial t}\right| \le C_\beta (1+y)^{-\beta}$$

and, for every $y' > y$, $t' \in (0,t)$ and $\alpha \in (0,1]$

$$|\psi_2(y,t) - \psi_2(y',t)| \le C_\beta (y'-y)^\alpha t^{\frac{2-\alpha}{2}}(1+y)^{-\beta},$$

and

$$|\psi_2(y,t) - \psi_2(y,t')| \le C_\beta (t-t')^\alpha t^{1-\alpha}(1+y)^{-\beta}.$$

(iii) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$ and every $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l} - \frac{\partial^l \psi_{2,\kappa}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}}(1+\kappa)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{(\kappa-y)^2}{t}\right),$$

$$\left|\frac{\partial \psi_2(y,t)}{\partial t}\right| \le C_\beta (1+\kappa)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{(\kappa-y)^2}{t}\right).$$

Also, for every $y' \in (y,\kappa)$, $t' \in (0,t)$ and $\alpha \in (0,1]$

$$\begin{aligned}&\left|\left(\psi_2\left(y,t\right)-\psi_{2,\kappa}\left(y,t\right)\right)-\left(\psi_2\left(y',t\right)-\psi_{2,\kappa}\left(y',t\right)\right)\right|\\&\quad\le C_\beta\left(y'-y\right)^\alpha t^{\frac{2-\alpha}{2}}\left(1+\kappa\right)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{\left(\kappa-y'\right)^2}{t}\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\left(\psi_2\left(y,t\right)-\psi_{2,\kappa}\left(y,t\right)\right)-\left(\psi_2\left(y,t'\right)-\psi_{2,\kappa}\left(y,t'\right)\right)\right|\\&\quad\le C_\beta\left(t-t'\right)^\alpha t^{1-\alpha}\left(1+\kappa\right)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{\left(\kappa-y\right)^2}{t}\right).\end{aligned}$$

(iv) *$\psi_2\left(y,t\right)$ is the unique classical solution of the PDE* (5.0.25)*.*

Proof. For (i) and (ii): These follow from the regularity bounds obeyed by $H_1(y,t)$ and $H_1(y,t)-H_{1,\kappa}(y,t)$, given in Lemma 5.0.15 and similar calculations as in the proof of Lemma 4.1.5.

For (iii): This can be calculated from the bounds given in Lemma 5.1.2, that are obeyed by $H_1(y,t)-H_{1,\kappa}(y,t)$, and examining the three cases

$$\xi \le \frac{1}{2}\kappa,$$

$$\frac{1}{2}\kappa < \xi \le \kappa,$$

and

$$\xi > \kappa.$$

For (iv): It follows from Lemma 5.1.1 and part (i) that $\psi_2(y,t)$ satisfies the equation (5.0.25) on the inner domain. It follows from part (ii) that $\psi_2(y,t)$ satisfies the asymptotic boundary condition. Similar reasoning as in the proof of Lemma 5.1.1 yields that, for every $t_0 \in (0,1]$,

$$\lim_{(y,t)\to(0,t_0)}\psi_2(y,t)=0.$$

□

Definition 5.1.5. *For $y>0$ and $t\in(0,1]$ let*

$$H_2(y,t) := -\lambda\psi_2\left(y,t\right)+\lambda\int_0^y\psi_2\left(y-z,t\right)dF(z).$$

Proposition 5.1.2. *Assume that $\sigma_R = r = 0$ and that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38)*.*

(i) *Then there exists a constant C_β, depending on β, such that, for every $y>0$, $t,\alpha\in(0,1]$ and $y'>0$*

$$\begin{aligned}\left|H_2(y,t)\right| &\le C_\beta\left(1+y\right)^{-\beta},\\ \left|H_2(y,t)-H_2(y',t)\right| &\le C_\beta\left(y'-y\right)^\alpha t^{\frac{2-\alpha}{2}}\left(1+y\right)^{-\beta},\end{aligned}$$

and such that

$$|H_2(y,t) - H_2(y,t')| \leq C_\beta \left(t-t'\right)^\alpha {t'}^{1-\alpha} \left(1+y\right)^{-\beta}.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$ every $(y',t') \in (y,\kappa) \times (0,t)$ and $\alpha \in (0,1]$*

$$|H_2(y,t) - H_{2,\kappa}(y,t)| \leq C_\beta t \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa - y\right)^2}{t}\right),$$

$$\begin{aligned}&|(H_2(y,t) - H_{2,\kappa}(y,t)) - (H_2(y',t) - H_{2,\kappa}(y',t))| \\ &\quad \leq C_\beta \left(y'-y\right)^\alpha t^{\frac{2-\alpha}{2}} \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa - y'\right)^2}{t}\right),\end{aligned}$$

and

$$\begin{aligned}&|(H_2(y,t) - H_{2,\kappa}(y,t)) - (H_2(y,t') - H_{2,\kappa}(y,t'))| \\ &\quad \leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa - y\right)^2}{t}\right).\end{aligned}$$

Proof. This follows from the bounds given in Theorem 5.1.1 and similar calculations as in Lemma 4.1.8. □

We will now proceed to show existence of the equation (5.0.25) analogous to the results in Section 4.1.2.

Definition 5.1.6. *Let*

$$Q_{\lambda,0}\left(y,t,\xi,\vartheta\right) = -\lambda G_L\left(y,t,\xi,\vartheta\right),$$

and let the sequence of functions $\{Q_{\lambda,n}\}_{n=0}^\infty$ be defined inductively for $n \in 1,2,\ldots,$ and $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ by

$$Q_{\lambda,n}\left(y,t,\xi,\vartheta\right) = \int_\vartheta^t \int_0^\infty Q_{\lambda,0}\left(y,t,z,s\right) Q_{\lambda,n-1}\left(z,s,\xi,\vartheta\right) dzds,$$

and let

$$Q_\lambda\left(y,t,\xi,\vartheta\right) = \sum_{n=0}^\infty Q_{\lambda,n}\left(y,t,\xi,\vartheta\right).$$

Lemma 5.1.4. (i) *Q_λ is a solution of the integral equation*

$$\begin{aligned}Q_\lambda\left(y,t,z,\vartheta\right) &= -\lambda G_L\left(y,t,\xi,\vartheta\right) \\ &\quad - \lambda \int_\vartheta^t \int_0^\infty G_L\left(y,t,z,s\right) Q_\lambda\left(z,s,\xi,\vartheta\right) dzds.\end{aligned} \tag{5.1.2}$$

(ii) *$Q_\lambda\left(y,t,\xi,\vartheta\right)$ is differentiable with respect to y,t and ϑ on $(0,\infty) \times (0,1] \times (0,\infty) \times [0,t)$. Furthermore, there exists a constant C such that for every*

$y,\xi>0$ and $0\le\vartheta<t\le 1$ the following identity and inequalities are all valid for $l\in\{0,1\}$

$$Q_\lambda(y,t,\xi,\vartheta)=Q_\lambda(y,t-\vartheta,\xi,0),$$

$$\left|\frac{\partial^l Q_\lambda(y,t,\xi,\vartheta)}{\partial y^l}\right|\le C(t-\vartheta)^{-\frac{1+l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial Q_\lambda(y,t,\xi,\vartheta)}{\partial t}\right|\le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and every $l\in\{0,1\}$ the following inequalities are all valid:*

$$\begin{aligned}\left|\frac{\partial^l Q_\lambda(y,t,\xi,\vartheta)}{\partial y^l}-\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|&\le C(t-\vartheta)^{-\frac{1+l}{2}}\\&\times\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2}{t-\vartheta}\right)\\&\times\exp\left(-\frac{1}{8}c_0\frac{(\kappa-\xi)^2}{t-\vartheta}\right),\end{aligned}\tag{5.1.3}$$

and

$$\begin{aligned}\left|\frac{\partial Q_\lambda(y,t,\xi,\vartheta)}{\partial t}-\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right|&\le C(t-\vartheta)^{-\frac{3}{2}}\\&\times\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2}{t-\vartheta}\right)\\&\times\exp\left(-\frac{1}{8}c_0\frac{(\kappa-\xi)^2}{t-\vartheta}\right).\end{aligned}\tag{5.1.4}$$

Proof. For part (i): Similar calculations as in Lemma 4.1.4, based on induction and uniform convergence yield that the inequalities given in part (i) are all valid. In particular, it can be shown that there exists a sequence $\{k_n\}_{n=0}^{\infty}$ of positive constants such that $\lim_{n\to\infty}\frac{k_{n+1}}{k_n}=0$ and such that

$$|Q_{\lambda,n}(y,t,\xi,\vartheta)|\le k_n(t-\vartheta)^{\frac{2n-1}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Also by induction it can be shown that, for every $n\in 1,2,\dots,$

$$\begin{aligned}\sum_{j=0}^{n}Q_{\lambda,j}(y,t,z,\vartheta)=&-\lambda G_L(y,t,\xi,\vartheta)\\&-\lambda\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\sum_{j=0}^{n}Q_{\lambda,j}(z,s,\xi,\vartheta)\,dzds\\&+\lambda\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_{\lambda,n}(z,s,\xi,\vartheta)\,dzds.\end{aligned}$$

Because of the bounds obeyed by $Q_{\lambda,n}(z,s,\xi,\vartheta)$ it follows that

$$\sum_{j=0}^{n} Q_{\lambda,j}(y,t,\xi,\vartheta)$$

converges uniformly to a solution of the integral equation (5.1.2).

For part (ii): It follows from the regularity bounds obeyed by the Green function $G_L(y,t,\xi,\vartheta)$ derived in section 4.1.1 that the stated bounds hold for $Q_{\lambda,0}(y,t,\xi,\vartheta)$ and $Q_{\lambda,0}(y,t,\xi,\vartheta) - Q_{\lambda,\kappa,0}(y,t,\xi,\vartheta)$. Similar calculations, based on induction and uniform convergence as in the proof of Lemma 4.1.4 yields that these bounds also hold for the limits $Q_\lambda(y,t,\xi,\vartheta)$ and $Q_\lambda(y,t,\xi,\vartheta) - Q_{\lambda,\kappa}(y,t,\xi,\vartheta)$.

For part (iii): We first note that it follows from the bounds given in that, for some constant C, $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1\}$

$$\left| \frac{\partial^l Q_{\lambda,0}(y,t,\xi,\vartheta)}{\partial y^l} - \frac{\partial^l Q_{\lambda,0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l} \right| \leq C(t-\vartheta)^{-\frac{1+l}{2}} \times \exp\left(-\frac{1}{2}c_0 \frac{(\kappa-y)^2 + \frac{1}{2}(\kappa-\xi)^2}{t-\vartheta} \right).$$

Furthermore, an application of Proposition 3.0.2 yields that, for some constant C

$$\int_\kappa^\infty \exp\left(-\frac{1}{4}c_0 \left(\frac{(y-z)^2}{t-s} + \frac{(z-\xi)^2}{s-\vartheta} \right) \right) dz \leq C \exp\left(-\frac{1}{8}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta} \right) (t-\vartheta)^{-\frac{1}{2}} (t-s)^{\frac{1}{2}} (s-\vartheta)^{\frac{1}{2}}.$$

Similar calculations as in Lemma 4.1.4, based on induction, uniform convergence the symmetry property between there t and ϑ and the bound above, yield that the stated bounds (5.1.3) and (5.1.4) all hold. □

Definition 5.1.7. *For $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ let*

$$G_{L_\lambda}(y,t,\xi,\vartheta) := G_L(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dzds.$$

Lemma 5.1.5. *Assume that $\sigma_R = r = 0$.*

(i) *For every $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ and $l \in \{1,2\}$*

$$\frac{\partial^l}{\partial y^l} \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dzds = \int_\vartheta^t \int_0^\infty \frac{\partial^l G_L(y,t,z,s)}{\partial y^l} Q_\lambda(z,s,\xi,\vartheta)\, dzds,$$

and

$$\frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dzds = Q_\lambda(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial G_L(y,t,z,s)}{\partial t} Q_\lambda(z,s,\xi,\vartheta)\, dzds.$$

(ii) *There exists a constant C such that, for every $y, \xi > 0$ and $0 \le \vartheta < t \le 1$ the following inequalities are all valid:*

$$\left|\frac{\partial^l}{\partial y^l}\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dzds\right| \le C(t-\vartheta)^{\frac{1-l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dzds\right| \le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that, for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l}{\partial y^l}\int_\vartheta^t\left(\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dz - \int_0^\infty G_{L,\kappa}(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dz\right)ds\right| \le C(t-\vartheta)^{\frac{1-l}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\left(\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dz - \int_0^\infty G_{L,\kappa}(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dz\right)ds\right| \le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right).$$

Proof. This follows from the bounds given in Lemma 5.1.4 and Lemma 5.1.2 and similar calculations as in the proof of Lemma 4.1.5. □

Definition 5.1.8. *Let*

$$Q_{I,0}(y,t,\xi,\vartheta) := \lambda\int_0^y G_{L_\lambda}(y-\zeta,t,\xi,\vartheta)\,dF(\zeta).$$

Let the sequence of functions

$$\{Q_{I,n}(y,t,\xi,\vartheta)\}_{n=0}^\infty$$

be defined inductively by

$$Q_{I,n}(y,t,\xi,\vartheta) = \int_\vartheta^t\int_0^\infty Q_{I,0}(y,t,z,s)\,Q_{I,n-1}(z,s,\xi,\vartheta)\,dzds,\quad n \in 1,2,\ldots.$$

Let

$$Q_I(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{I,n}(y,t,\xi,\vartheta).$$

Let

$$G_{I_\lambda,n}(y,t,\xi,\vartheta) := \int_\vartheta^t \int_0^\infty G_{L_\lambda}(y,t,z,s)\, Q_{I,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$G_{I_\lambda}(y,t,\xi,\vartheta) := \int_\vartheta^t \int_0^\infty G_{L_\lambda}(y,t,z,s)\, Q_I(z,s,\xi,\vartheta)\, dzds.$$

Lemma 5.1.6. *Assume that* $\sigma_R = r = 0$ *and that the inequality* (4.0.38) *holds.*

(i) *There exists a sequence* $\{k_n\}_{n=0}^{\infty}$ *such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\ldots,$ *every* $y,\xi > 0$ *and* $0 \le \vartheta < t \le 1$,

$$\begin{aligned} |Q_{I,n}(y,t,\xi,\vartheta)| &\le k_n (t-\vartheta)^{n-\frac{1}{2}} \\ &\quad \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\ &\quad \times dF(\zeta_0)dF(\zeta_1),\ldots,dF(\zeta_n), \end{aligned} \tag{5.1.5}$$

$$\begin{aligned} &\Big| Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,n}(y',t,\xi,\vartheta) \Big| \\ &\quad \le C k_n |y-y'| (t-\vartheta)^{n-1} \\ &\quad\quad \times \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \right. \\ &\quad\quad \left. + \exp\left(-\frac{1}{4}c_0 \frac{\left(y'-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right) \\ &\quad\quad \times dF(\zeta_1),\ldots,dF(\zeta_n). \end{aligned} \tag{5.1.6}$$

Also

$$\begin{aligned} |Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,n}(y,t',\xi,\vartheta)| &\le C k_n |t-t'|^{\frac{1}{4}} \left(\tilde{t}-\vartheta\right)^{n-\frac{3}{4}} \\ &\quad \times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\ &\quad \times dF(\zeta_1),\ldots,dF(\zeta_n), \end{aligned} \tag{5.1.7}$$

(ii)

$$|Q_I(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-\frac{1}{2}},$$

$$|Q_I(y,t,\xi,\vartheta) - Q_I(y',t,\xi,\vartheta)| \le C|y-y'|(t-\vartheta)^{-1},$$

and

$$|Q_I(y,t,\xi,\vartheta) - Q_I(y,t',\xi,\vartheta)| \le C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{3}{4}}.$$

(iii) *For every* (y,t,ξ,ϑ)

$$\begin{aligned} Q_I(y,t,\xi,\vartheta) = \lambda \int_0^y G_{L_\lambda}(y-\zeta,t,\xi,\vartheta)\,dF(\zeta) \\ + \lambda \int_0^y G_{I_\lambda}(y-\zeta,t,\xi,\vartheta)\,dF(\zeta). \end{aligned} \tag{5.1.8}$$

Proof. For part (i)-(ii): This follows from similar calculations as in Lemma 4.1.12.

For part (iii): This follows from similar calculations as in the proof of Lemma 5.1.4. □

Lemma 5.1.7. *Assume that $\sigma_R = r = 0$. There exists a constant C such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and every $(y',t') \in (y,\kappa)\times(0,t)$ the following inequalities are all valid:*

$$\begin{aligned} |Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-\frac{1}{2}} \\ \times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2+(\kappa-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{5.1.9}$$

$$\begin{aligned} &|(Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)) - (Q_I(y',t,\xi,\vartheta) - Q_{I,\kappa}(y',t,\xi,\vartheta))| \\ &\quad \le C(y'-y)^{\frac{1}{2}}(t-\vartheta)^{-\frac{3}{4}}\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y')^2+(\kappa-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{5.1.10}$$

and

$$\begin{aligned} &|(Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)) - (Q_I(y,t',\xi,\vartheta) - Q_{I,\kappa}(y,t',\xi,\vartheta))| \\ &\quad \le C(t-t')^{\frac{1}{4}}t'^{-\frac{3}{4}}\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2+(\kappa-\xi)^2}{t-\vartheta}\right). \end{aligned} \tag{5.1.11}$$

Proof. For $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $n \in 0,1,\ldots,$ let

$$\Delta Q_{I,n}(y,t,\xi,\vartheta) := Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta).$$

and let

$$\Delta Q_I(y,t,\xi,\vartheta) := Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta).$$

Because of the bounds given in Lemma 5.1.2 similar calculations as in Proposition 3.0.3 and Proposition 3.0.4 yield that

$$\Delta Q_{I,0}(y,t,\xi,\vartheta),$$

$$\Delta Q_{I,0}\left(y,t,\xi,\vartheta\right)-\Delta Q_{I,0}\left(y',t,\xi,\vartheta\right)$$

and

$$\Delta Q_{I,0}\left(y,t,\xi,\vartheta\right)-\Delta Q_{I,0}\left(y,t',\xi,\vartheta\right)$$

obey bounds of the stated form (5.1.10). Similar calculations, based on induction and uniform convergence, as in the proofs of Lemma4.1.4 and Lemma (5.1.4) part (iii), yield that the stated regularity bounds for $\Delta Q_I\left(y,t,\xi,\vartheta\right)$ all hold. □

Lemma 5.1.8. *Assume that $\sigma_R = r = 0$. There exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty}\frac{k_{n+1}}{k_n}=0,$$

and such that, for every finite $n \in 0,1,\ldots,$, every $l \in \{0,1,2\}$, $y,\xi > 0$ and $0 \le \vartheta < t \le 1$ the following identities and inequalities are valid:

(i)

$$\frac{\partial^l G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)}{\partial y^l}=\int_\vartheta^t\int_0^\infty\frac{\partial^l G_{L_\lambda}\left(y,t,z,s\right)}{\partial y^l}Q_{I,n}\left(z,s,\xi,\vartheta\right)dzds,$$

and

$$\begin{aligned}\frac{\partial G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)}{\partial t}&=Q_{I,n}\left(y,t,\xi,\vartheta\right)\\&\quad+\int_\vartheta^t\int_0^\infty\frac{\partial G_{L_\lambda}\left(y,t,z,s\right)}{\partial t}Q_{I,n}\left(z,s,\xi,\vartheta\right)dzds.\end{aligned}$$

(ii)

$$\begin{aligned}\left|\frac{\partial^l G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)}{\partial y^l}\right|&\le k_n\left(t-\vartheta\right)^{n+\frac{1-l}{2}}\\&\quad\times\int_0^\infty\cdots\int_0^\infty\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi-\sum_{j=0}^n\zeta_j\right)^2}{t-\vartheta}\right)\\&\quad\times dF(\zeta_1),\ldots,dF(\zeta_n),\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)}{\partial t}\right|&\le k_n\left(t-\vartheta\right)^{n-\frac{1}{2}}\\&\quad\times\int_0^\infty\cdots\int_0^\infty\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi-\sum_{j=0}^n\zeta_j\right)^2}{t-\vartheta}\right)\\&\quad\times dF(\zeta_1),\ldots,dF(\zeta_n)\end{aligned}$$

and

$$\left|G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)\right| \leq k_n y^{\frac{1}{2}}\left(t-\vartheta\right)^{n+\frac{1}{4}}$$
$$\times \int_0^\infty \cdots \int_0^\infty \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) + \exp\left(-\frac{1}{4}c_0 \frac{\left(\xi+\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right)$$
$$\times\, dF(\zeta_1),\ldots,dF(\zeta_n).$$

(iii)
$$\frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l} = \int_\vartheta^t \int_0^\infty \frac{\partial^l G_{L_\lambda}\left(y,t,z,s\right)}{\partial y^l} Q_{I,\kappa}\left(z,s,\xi,\vartheta\right) dzds$$

and

$$\frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t} = Q_{I,\kappa}\left(y,t,\xi,\vartheta\right) + \int_\vartheta^t \int_0^\infty \frac{\partial G_{L_\lambda}\left(y,t,z,s\right)}{\partial t} Q_{I,\kappa}\left(z,s,\xi,\vartheta\right) dzds.$$

Furthermore, there exists a constant C such that for every (y,t,ξ,ϑ), and every $l \in \{0,1,2\}$

$$\left|\frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l}\right| \leq C\left(t-\vartheta\right)^{\frac{1-l}{2}},$$

and

$$\left|\frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t}\right| \leq C\left(t-\vartheta\right)^{-\frac{1}{2}}.$$

(iv) *For every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$*

$$\left|\frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l} - \frac{\partial^l G_{I_{\lambda,\kappa}}\left(y,t,\xi,\vartheta\right)}{\partial y^l}\right| \leq C\left(t-\vartheta\right)^{\frac{1-l}{2}} \exp\left(-\frac{1}{8}c_0 \frac{\left(\kappa-y\right)^2+\left(\kappa-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t} - \frac{\partial G_{I_{\lambda,\kappa}}\left(y,t,\xi,\vartheta\right)}{\partial t}\right| \leq C\left(t-\vartheta\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{8}c_0 \frac{\left(\kappa-y\right)^2+\left(\kappa-\xi\right)^2}{t-\vartheta}\right).$$

Proof. This follows from the bounds given in Lemma 5.1.6 and Lemma 5.1.7 and similar calculations as in the proof of Lemma 4.1.13. □

Definition 5.1.9. *For $y, \xi > 0$ and $0 \le \vartheta < t \le 1$ let*

$$G_A\left(y,t,\xi,\vartheta\right) := G_{L_\lambda}\left(y,t,\xi,\vartheta\right) + G_{I_\lambda}\left(y,t,\xi,\vartheta\right)$$

and let

$$\psi_3(y,t) = \int_0^t \int_0^\infty G_A\left(y,t,\xi,\vartheta\right) H_2\left(\xi,\vartheta\right) d\xi d\vartheta.$$

Theorem 5.1.2. *Assume that $\sigma_R = r = 0$ and that the bound (4.0.38) on the tail distribution function $\bar{F}$ holds.*

(i) *Then $\psi_3(y,t)$ is a classical solution of the PIDE (5.0.25). Furthermore, there exists a constant C_β, depending on β, such that for every $y > 0$ and $t \in (0,1]$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{2}} \left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t}\right| \le C_\beta t \left(1+y\right)^{-\beta},$$

$$|\psi_3(y,t)| \le C t^{\frac{7}{4}} \sqrt{y},$$

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{2}} \left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t}\right| \le C_\beta t \left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial^l \psi(y,t)}{\partial y^l}\right| \le C\left(t^{-\frac{l}{2}} \exp\left(-\frac{1}{4}c_0 T \frac{y^2}{t}\right) + t^{\frac{2-l}{2}} C_\beta \left(1+y\right)^{-\beta}\right) \quad \text{and}$$

$$\left|\frac{\partial \psi(y,t)}{\partial t}\right| \le C\left(t^{-1} \exp\left(-\frac{1}{4}c_0 T \frac{y^2}{t}\right) + C_\beta \left(1+y\right)^{-\beta}\right).$$

(ii) *There exists a constant C_β, depending on β, such that, for every $y \in \left(0, \frac{1}{2}\kappa\right)$, $t \in (0,1]$ and $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l} - \frac{\partial^l \psi_{3,\kappa}}{\partial y^l}(y,t)\right| \le C_\beta t^{\frac{4-l}{2}} \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{128}c_0 \frac{\left(\kappa-y\right)^2}{t}\right),$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t} - \frac{\partial \psi_{3,\kappa}}{\partial t}(y,t)\right| \le C_\beta t \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{128}c_0 \frac{\left(\kappa-y\right)^2}{t}\right).$$

Proof. This follows from the identities and bounds given in Lemma 5.1.5 and Lemma 4.1.13, the bounds obeyed by $H_1\left(y,t\right)$ given in Proposition 5.1.2 and similar calculations as in the proof of Theorem 4.1.2. For part (ii) it is helpful to consider separately the cases $\xi \le \frac{\kappa+y}{2}$ and $\xi > \frac{\kappa+y}{2}$. □

5.2 Unbounded coefficients

In this section we will prove the existence of a classical solution of the PDE (5.0.25) and the PIDE (5.0.26) under the assumption that $\sigma_R > 0$. Quite similar to what we did in Section 4.2, the main idea is to consider a transformed equation of equation (5.0.25), using the change of variables $x = \ln(1+y)$, and look for a solution $\hat{\psi}_2(x,t)$ of the equations

$$\begin{cases} \hat{\psi}_2(x,0) & = 0, \quad x > 0 \\ \lim_{x\to\infty} \hat{\psi}_2(x,t) & = 0, \\ \hat{\psi}_2(0,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_2(x,t)}{\partial t} - \hat{L}\hat{\psi}_2(x,t) & = H_1(e^x - 1, t) \quad x > 0, t \in (0,1], \end{cases} \tag{5.2.1}$$

and

$$\begin{cases} \hat{\psi}_3(x,0) & = 0, \quad x > 0, \\ \hat{\psi}_3(0,t) & = 0, \quad t \in [0,1], \\ \lim_{x\to\infty} \hat{\psi}_3(x,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_3(x,t)}{\partial t} - \hat{A}\hat{\psi}_3(x,t) & = H_2(e^x - 1, t), \quad x > 0, t \in (0,1], \end{cases} \tag{5.2.2}$$

where $H_2(y,t)$ is defined in Definition 5.1.5. As we did in Section 5.1 we will also consider the convergence of the solutions as $\gamma \to \infty$. For the PDE (5.2.1) the first step is to establish bounds on a Green function $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ that is very similar to the auxiliary Green function $\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)$ except that, instead of satisfying

$$\lim_{x\to\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta) = 0,$$

$\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ will satisfy the asymptotic condition

$$\lim_{x\to\infty} \hat{G}_{\hat{L}}(x,t,\xi,\vartheta) = 0.$$

Definition 5.2.1. *For $\xi > 0$ and $t \in (0,1]$ let $\hat{P}^{(2)}_g$ be the operator defined in Definition 4.2.5. Let*

$$\hat{V}_{\xi,0}(t) := -2\hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0).$$

For $n \in 1,2,\ldots,$ define $\hat{V}_{\xi,n}(t)$ recursively as

$$\hat{V}_{\xi,n}(t) = -2\hat{P}^{(2)}_{\hat{V}_{\xi,n-1}}(t),$$

and let

$$\hat{U}_\xi(t) := \sum_{n=0}^{\infty} \hat{V}_{\xi,n}(t).$$

Lemma 5.2.1. *Assume that $\sigma_R > 0$.*

(i) *There exists a constant C such that for every $t \in (0,1]$ and every $\xi \in (0,\hat{\kappa})$*

$$\left|\hat{U}_\xi(t) - \hat{U}^{(2)}_{\xi,\hat{\kappa}}(t)\right| \le C \exp\left(-\frac{1}{4}\hat{c}_0\frac{\hat{\kappa}^2}{t}\right),$$

and

$$\left|\hat{U}'_\xi(t) - \hat{U}^{(2)\prime}_{\xi,\hat{\kappa}}(t)\right| \le C \exp\left(-\frac{1}{4}\hat{c}_0\frac{\hat{\kappa}^2}{t}\right).$$

(ii) *$\hat{U}_\xi(t)$ is differentiable on $(0,1]$. $\hat{U}_\xi(t)$ is a solution of the integral equation*

$$-\frac{1}{2}\hat{U}_\xi(t) - \hat{P}^{(2)}_{U_{\xi,n}} = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0).$$

Let

$$\hat{g}^*_{\hat{L}_0}(x,t,\xi) := -P^{(2)}_{\hat{U}_\xi}(x,t).$$

Then $\hat{g}^*_{\hat{L}_0}(x,t,\xi)$ is a classical solution of the PDE

$$\begin{cases} \hat{g}^*_{\hat{L}_0}(x,0,\xi) & = 0, \quad x>0, \\ \hat{g}^*_{\hat{L}_0}(0,t,\xi) & = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0), \quad t\in(0,1], \\ \lim_{x\to\infty}\hat{g}^*_{\hat{L}_0}(x,t,\xi) & = 0, \quad t\in(0,1], \\ \frac{\partial \hat{g}^*_{\hat{L}_0}(x,t,\xi)}{\partial t} & = \hat{a}_{1,1}(x)\frac{\partial^2 \hat{g}^*_{\hat{L}_0}(x,t,\xi)}{\partial x^2}, \quad (x,t)\in(0,\infty)\times(0,1]. \end{cases}$$

Proof. This follows from similar reasoning and calculation also making use of the bounds given in Lemma 5.2.2, as in the proof of Lemma 5.1.1) below, . □

Definition 5.2.2. *Define*

$$\hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta) := \hat{g}^*_{\hat{L}_0}(x,t-\vartheta,\xi).$$

Lemma 5.2.2. *Assume that $\sigma_R > 0$.*

(i) *There exists a constant C such that for every $x,\xi>0$ and $0\le\vartheta<t\le 1$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)$$

and

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right).$$

(ii) *There exists a constant C such that for every $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} + \frac{\partial^l}{\partial x^l}\hat{P}^{(2)}_{\hat{U}^{(2)}_{\xi,\hat{\kappa}}}(x,t-\vartheta)\right| \le C\exp\left(-\frac{1}{8}\hat{c}_0\frac{\hat{\kappa}^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} + \frac{\partial}{\partial t}\hat{P}^{(2)}_{\hat{U}^{(2)}_{\xi,\hat{\kappa}}}(x,t-\vartheta)\right| \le C\exp\left(-\frac{1}{8}\hat{c}_0\frac{\hat{\kappa}^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}\right|$$
$$\le Ct^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}\hat{c}_0\frac{(\hat{\kappa}-y)^2+\frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} - \frac{\partial \hat{g}_{\hat{L}_0,\kappa}(x,t,\xi,\vartheta)}{\partial t}\right|$$
$$\leq Ct^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + \frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $y, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ the following inequalities are all valid:*

$$\int_0^t \int_{\hat{\kappa}}^{\infty} \left|\frac{\partial^2 \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial y^2}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\hat{\kappa}^2}{t}\right),$$

and

$$\int_0^t \int_{\hat{\kappa}}^{\infty} \left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\hat{\kappa}^2}{t}\right).$$

(iv) *There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ and $l \in \{0,1,2\}$*

$$\left|\int_0^t \int_0^{\infty} \left(\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}\right) d\xi d\vartheta\right|$$
$$\leq Ct^{\frac{2-l}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2}{t}\right),$$

and such that

$$\left|\int_0^t \left(\int_0^{\infty} \frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} d\xi - \int_0^{\hat{\kappa}} \frac{\partial \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} d\xi\right) d\vartheta\right|$$
$$\leq C \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2}{t}\right).$$

Proof. Because of the bounds given in Lemma 5.2.1 this follows from similar reasoning and calculations as in the proof of Lemma 5.1.2 and the proof of Proposition 4.2.5. □

In the coming results we will establish the existence of a function $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ that has very similar properties on the entire unbounded domain $(y,t) \in (0,\infty) \times (0,1]$ as the Green function $\hat{G}_{\hat{L},\gamma}(x,t,\xi,\vartheta)$, does on the truncated domain. Moreover, we will show that the function $\hat{G}_{\hat{L},\gamma}(x,t,\xi,\vartheta)$ will converge to $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ if we let γ tend towards infinity.

Definition 5.2.3. *For $x, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ let*

$$\hat{G}_{\hat{L}_0}(x,t,\xi,\vartheta) := \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta),$$

let

$$\hat{Q}_0(x,t,\xi,\vartheta) := \hat{a}_1^*(x) \frac{\partial \hat{G}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}$$

and let the sequence of functions $\left\{\hat{Q}_n\right\}_{n=0}^{\infty}$ *be defined inductively for* $n \in 1, 2, \ldots,$ *by*

$$\hat{Q}_n(x,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\infty}\hat{Q}_0(x,t,z,s)\,\hat{Q}_{n-1}(z,s,\xi,\vartheta)\,dzds,$$

and let

$$\hat{Q}(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty}\hat{Q}_n(x,t,\xi,\vartheta).$$

Lemma 5.2.3. *Assume that* $\sigma_R > 0$ *and let* $\alpha \in (0,1)$.

(i) $\hat{Q}$ *solves the integral equation*

$$\hat{Q}(x,t,\xi,\vartheta) = \hat{Q}_0(x,t,\xi,\vartheta) + \int_{\vartheta}^{t}\int_{0}^{\infty}\hat{Q}_0(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds. \tag{5.2.3}$$

(ii) *There exists a constant* C *such that, for every* $x, x', \xi > 0$ *and every* $0 \leq \vartheta < t' < t \leq 1$ *the following identities and inequalities are all valid:*

$$\hat{Q}(x,t,\xi,\vartheta) = \hat{Q}(x,t-\vartheta,\xi,0),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x',t,\xi,\vartheta)\right| \leq C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x,t',\xi,\vartheta)\right| \leq C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant* C *such that, for every* $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\kappa}}$, *every* $x', \in (0,\hat{\kappa})$,*and every* $t' \in (0,t,)$ *the following identities and inequalities are all valid:*

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-1} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(\hat{\kappa}-x)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x',t,\xi,\vartheta) - \left(\hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\kappa}}(x',t,\xi,\vartheta)\right)\right|$$
$$\leq C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}(x,t,\xi,\vartheta)-\hat{Q}(x,t',\xi,\vartheta)-\left(\hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta)-\hat{Q}_{\hat{\kappa}}(x,t',\xi,\vartheta)\right)\right|$$
$$\leq C\left|t-t'\right|^{\frac{1}{4}}(t-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right).$$

Proof. For (i)-(ii): This follows from similar calculations and reasoning as in the proofs of Lemma 4.2.6 and Lemma 5.1.4.

For (iii): This follows from similar calculations as in the proof of Lemma (5.1.4) part (iii), and Lemma 5.1.7. □

Proposition 5.2.1. (i) *For every* $x,\xi>0$ *and* $0\leq\vartheta<t\leq 1$ *and* $l\in\{0,1,2\}$

$$\frac{\partial^l}{\partial x^l}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds$$
$$=\int_\vartheta^t\int_0^\infty \frac{\partial^l\hat{G}_{\hat{L}_0}(x,t,z,s)}{\partial x^l}\hat{Q}(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds=\hat{Q}(x,t,\xi,\vartheta)$$
$$+\int_\vartheta^t\int_0^\infty \frac{\partial\hat{G}_{\hat{L}_0}(x,t,z,s)}{\partial t}\hat{Q}(z,s,\xi,\vartheta)\,dzds.$$

(ii) *There exists a constant* C *such that for every* $x,\xi>0$, *every* $0\leq\vartheta<t\leq 1$ *and every and* $l\in\{0,1,2\}$

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right|$$
$$\leq C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right|$$
$$\leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}c_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant* C *such that for every* $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$ *and every* $l\in\{0,1,2\}$

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\left(\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right.\right.$$
$$\left.\left.-\int_0^{\hat{\kappa}}\hat{G}_{\hat{L}_0,\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\hat{\kappa}}(z,s,\xi,\vartheta)\,dz\right)\right|$$
$$\leq C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\left(\int_{0}^{\infty}\hat{G}_{\hat{L}_0}\left(x,t,z,s\right)\hat{Q}\left(z,s,\xi,\vartheta\right)dz\right.\right.$$
$$\left.\left.-\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\gamma}\left(x,t,z,s\right)\hat{Q}_{\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dz\right)ds\right|$$
$$\leq C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{8}\hat{c}_0\frac{\left(\hat{\kappa}-x\right)^2+\left(\hat{\kappa}-\xi\right)^2}{t-\vartheta}\right).$$

Proof. For (i)-(ii): Because of the regularity bounds obeyed by $\hat{Q}\left(x,t,\xi,\vartheta\right)$, given in Lemma 5.2.3, and the regularity bounds obeyed by $\hat{G}_{\hat{L}_0}$, this follows from similar calculations as in the proof of Lemma 4.1.5.

For part (iii): Because of the bounds given in Lemma 5.2.3 this follows from similar calculations as in the proof of Lemma (5.1.4) part (iii). □

Definition 5.2.4. *Let*

$$\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right):=\hat{G}_{\hat{L}_0}\left(x,t,\xi,\vartheta\right)$$
$$+\int_{\vartheta}^{t}\int_{0}^{\infty}\hat{G}_{\hat{L}_0}\left(x,t,z,s\right)\hat{Q}\left(z,s,\xi,\vartheta\right)dzds.$$

and let

$$\hat{\psi}_2(x,t):=\int_0^t\int_0^{\infty}\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right)H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta.$$

Theorem 5.2.1. *Assume that $\sigma_R>0$ and that the tail distribution satisfies the inequality* (4.0.38).

(i) *For every $x>0, t\in(0,1]$ and $l\in\{1,2\}$*

$$\frac{\partial^l\hat{\psi}_2(x,t)}{\partial x^l}=\int_0^t\int_0^{\infty}\frac{\partial^l\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right)}{\partial x^l}H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta,$$

and

$$\frac{\partial\hat{\psi}_2(x,t)}{\partial t}=H_{1,\gamma}\left(e^{x}-1,t\right)$$
$$+\int_0^t\int_0^{\infty}\frac{\partial\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right)}{\partial t}H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that for every $x>0, t\in(0,1]$ and every $l\in\{0,1\}$ the following bounds are valid:*

$$\left|\frac{\partial^l\hat{\psi}_2(x,t)}{\partial x^l}\right|\leq C_\beta t^{\frac{2-l}{2}}\exp\left(-\beta x\right),$$
$$\left|\frac{\partial^2\hat{\psi}_2(x,t)}{\partial x^2}\right|\leq C_\beta\exp\left(-\frac{1}{2}\beta x\right),$$
$$\left|\frac{\partial\hat{\psi}_2(x,t)}{\partial t}\right|\leq C_\beta\exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\hat{\psi}_2(x,t)\right| \le C_\beta t^{\frac{3}{4}} \min\left(x^{\frac{1}{2}}, t^{\frac{1}{4}} \exp\left(-\beta x\right)\right).$$

(iii) $\hat{\psi}_2(x,t)$ *is a classical solution of the PDE* (5.2.1) *and*

$$\psi_2(y,t) := \hat{\psi}_2(\ln(1+y), t)$$

is a classical solution of the PDE (5.0.25).

(iv) *For every* $x' > x, t' \in (0,t)$ *and* $\alpha \in \left(0, \frac{1}{2}\right)$ *the following bounds hold:*

$$\left|\hat{\psi}_2(x,t) - \hat{\psi}_2(x',t)\right| \le C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\beta x\right) \text{ and}$$
$$\left|\hat{\psi}_2(x,t) - \hat{\psi}_2(x,t')\right| \le C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\beta x\right).$$

(v) *There exists a constant* C_β*, depending on* β*, such that for every* $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ *and every* $l \in \{0,1\}$

$$\left|\frac{\partial^l \hat{\psi}_2(x,t)}{\partial x^l} - \frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{2-l}{2}} \exp\left(-\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$
$$\left|\frac{\partial^2 \hat{\psi}_2(x,t)}{\partial x^2} - \frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

and

$$\left|\frac{\partial \hat{\psi}_2(x,t)}{\partial t} - \frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right).$$

(vi) *There exists a constant* C_β*, depending on* β*, such that for every* $(y,t) \in (0,\gamma) \times (0,1]$ *and every* $l \in \{0,1\}$

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l} - \frac{\partial^l \psi_{2,\gamma}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\beta})$$
$$\times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right),$$
$$\left|\frac{\partial^2 \psi_2(y,t)}{\partial y^2} - \frac{\partial^2 \psi_{2,\gamma}(y,t)}{\partial y^2}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\frac{1}{2}\beta} \left(1+y\right)^{-2}$$
$$\times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right),$$

and

$$\left|\frac{\partial \psi_2(y,t)}{\partial t} - \frac{\partial \psi_{2,\gamma}(y,t)}{\partial t}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\frac{1}{2}\beta}$$
$$\times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right).$$

(vii) *For every* $0 < x < x' < \hat{\kappa}$, *every* $t' \in (0,t)$ *and* $\alpha \in (0, \frac{1}{2}]$ *the following bounds hold:*

$$\left|\left(\hat{\psi}_2(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x,t\right) - \left(\hat{\psi}_2(x',t) - \hat{\psi}_{2,\hat{\kappa}}(x',t)\right)\right|$$
$$\leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x'\right)^2}{t}\right),$$
$$\left|\left(\hat{\psi}_2(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x,t\right) - \left(\hat{\psi}_2(x,t') - \hat{\psi}_{2,\hat{\kappa}}(x,t')\right)\right|$$
$$\leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x'\right)^2}{t}\right).$$

Proof. This follows from similar considerations and calculations as in the proof of Lemma 4.1.5 using the bound given in Lemma 4.2.8, Theorem 5.1.1 and Proposition 5.2.1. We also need to use the chain rule and consider the change of variable

$$y = e^x - 1.$$

The bounds given in part (vii) follow from the bounds given in part (v), Proposition 3.0.3, Proposition 3.0.4 and considering the function

$$\Delta\psi_{2,\gamma}(x,t) = \psi_2(x,t) - \psi_{2,\gamma}(x,t).$$

□

Lemma 5.2.4. *Assume that* $\sigma_R > 0$ *and that the tail distribution satisfies the bound* (4.0.38). *Let the function* H_2 *be as in Definition 5.1.5.*

(i) *There exists a constant* C_β, *depending on* β, *such that the bounds stated below all hold for every* $x' > x > 0$, *and every* $0 \leq t' < t \leq 1$ *and every* $\alpha \in (0, \frac{\min(\beta,1)}{2}]$

$$\left|H_2\left(e^x-1,t\right)\right| \leq C_\beta t \exp\left(-\beta x\right),$$
$$\left|H_2\left(e^x-1,t\right) - H_2\left(e^x-1,t'\right)\right| \leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\beta x\right),$$

and

$$\left|H_2\left(e^x-1,t\right) - H_2\left(e^{x'}-1,t\right)\right| \leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\left(\beta-\alpha\right)x\right).$$

(ii) *Let* $\gamma = e^{\hat{\kappa}} - 1$. *There exists a constant* C_β, *depending on* β, *such that the bounds stated below all hold for every* $0 < x < x' < \hat{\kappa}$, *and every* $0 \leq t' < t \leq 1$ *and every* $\alpha \in (0, \frac{\min(\beta,1)}{2}]$

$$\left|H_2\left(e^x-1,t\right) - H_{2,\gamma}\left(e^x-1,t\right)\right| \leq C_\beta t \gamma^{-\beta} \exp\left(-\beta\hat{\kappa}\right)$$
$$\times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

$$\left|\left(H_2\left(e^x-1,t\right) - H_{2,\gamma}\left(e^x-1,t'\right)\right) - \left(H_2\left(e^x-1,t\right) - H_{2,\gamma}\left(e^x-1,t'\right)\right)\right|$$
$$\leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

and

$$\left|\left(H_2\left(e^x-1,t\right)-H_2\left(e^x-1,t'\right)\right)-\left(H_{2,\gamma}\left(e^x-1,t\right)-H_{2,\gamma}\left(e^x-1,t'\right)\right)\right|$$
$$\leq C_\beta\left(t-t'\right)^\alpha t^{1-\alpha}\exp\left(-\frac{1}{2}\beta\hat{\kappa}\right)\exp\left(-\frac{1}{8}\hat{c}_0\frac{\left(\hat{\kappa}-x\right)^2}{t}\right).$$

Proof. This follows from the bounds given in Theorem 5.2.1 and similar calculations as in Lemma 4.2.9. □

The last part of this article will be a discussion on the PIDE (5.0.26) (transformed to the PIDE (5.2.2)) for the case $\sigma_R > 0$. Most of this discussion will be analogous to the discussions in Section 4.2.2.

Definition 5.2.5. *For $x, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ let*

$$\hat{Q}_{\lambda,0}\left(x,t,\xi,\vartheta\right)=-\lambda\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right),$$

and let the sequence of functions $\left\{\hat{Q}_{\lambda,n}\right\}_{n=0}^{\infty}$ be defined inductively for $n \in 1,2,\dots,$ by

$$\hat{Q}_{\lambda,n}\left(x,t,\xi,\vartheta\right)=\int_\vartheta^t\int_0^\infty\hat{Q}_{\lambda,0}\left(x,t,z,s\right)\hat{Q}_{\lambda,n-1}\left(z,s,\xi,\vartheta\right)dzds,$$

and let

$$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)=\sum_{n=0}^{\infty}\hat{Q}_{\lambda,n}\left(x,t,\xi,\vartheta\right).$$

Lemma 5.2.5. *Assume that $\sigma_R > 0$. Let $\alpha \in (0,1)$.*

(i) *$\hat{Q}_\lambda$ is a solution of the integral equation*

$$\begin{aligned}\hat{Q}_\lambda\left(x,t,z,\vartheta\right)=&-\lambda\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right)\\&-\lambda\int_\vartheta^t\int_0^\infty\hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dzds.\end{aligned}\tag{5.2.4}$$

(ii) *$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)$ is differentiable with respect to all four variables. Furthermore, there exists a constant C, such that, for every $x, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ the following identities and inequalities are all valid:*

$$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)=\hat{Q}_\lambda\left(x,t-\vartheta,\xi,0\right),$$

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial x}\left(x,t,\xi,\vartheta\right)\right|\leq C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial \xi}\left(x,t,\xi,\vartheta\right)\right|\leq C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial t}\left(x,t,\xi,\vartheta\right)\right|\leq C\left(t-\vartheta\right)^{-\frac{2}{3}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant* C *such that for every* $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ *and every* $l \in \{0,1\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{Q}_\lambda (x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}} (x,t,\xi,\vartheta)}{\partial x^l}\right| \leq C (t-\vartheta)^{-\frac{1+l}{3}} \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t-\vartheta}\right) \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.5}$$

and

$$\left|\frac{\partial \hat{Q}_\lambda (x,t,\xi,\vartheta)}{\partial t} - \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}} (x,t,\xi,\vartheta)}{\partial t}\right| \leq C (t-\vartheta)^{-\frac{2}{3}} \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t-\vartheta}\right) \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-\xi)^2}{t-\vartheta}\right). \tag{5.2.6}$$

Proof. For part (5.2.4) and part (ii): This follows from similar calculations as in the proof of Lemma 5.2.5.

For part (iii): This follows from similar calculations as in the proof of Lemma 5.1.4. □

Definition 5.2.6. *For* $x,\xi > 0$ *and* $0 \leq \vartheta < t \leq 1$ *let*

$$\hat{G}_{\hat{L}_\lambda} (x,t,\xi,\vartheta) := \hat{G}_{\hat{L}} (x,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta)\, dz ds.$$

Lemma 5.2.6. *Assume that* $\sigma_R > 0$.

(i) *For every* $x,\xi > 0$ *and* $0 \leq \vartheta < t \leq 1$ *and* $l \in \{1,2\}$

$$\frac{\partial^l}{\partial x^l} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta)\, dz ds = \int_\vartheta^t \int_0^\infty \frac{\partial^l \hat{G}_{\hat{L}} (x,t,z,s)}{\partial x^l} \hat{Q}_\lambda (z,s,\xi,\vartheta)\, dz ds,$$

and

$$\frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta)\, dz ds = \hat{Q}_\lambda (x,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial \hat{G}_{\hat{L}} (x,t,z,s)}{\partial t} \hat{Q}_\lambda (z,s,\xi,\vartheta)\, dz ds.$$

(ii) *There exists a constant C such that, for every $x, \xi > 0$ and $0 \le \vartheta < t \le 1$ the following inequalities are all valid:*

$$\left| \frac{\partial^l}{\partial x^l} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dzds \right| \le C(t-\vartheta)^{\frac{1-l}{3}} \exp\left(-\frac{1}{4} \hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta} \right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dzds \right| \le C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{4} \hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta} \right).$$

(iii) *There exists a constant C such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left| \frac{\partial^l}{\partial x^l} \int_\vartheta^t \left(\int_0^\infty \hat{G}_{\hat{L}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dz - \int_0^\infty \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dz \right) ds \right| \le C(t-\vartheta)^{\frac{1-l}{3}} \exp\left(-\frac{1}{2} \hat{c}_0 \frac{(\hat{\kappa}-x)^2 + \frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta} \right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \left(\int_0^\infty \hat{G}_{\hat{L}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dz - \int_0^\infty \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\, \hat{Q}_\lambda(z,s,\xi,\vartheta)\, dz \right) ds \right| \le C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \hat{c}_0 \frac{(\hat{\kappa}-x)^2 + \frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta} \right).$$

Proof. Because of the bounds on $\hat{G}_{\hat{L}}$ and on $\hat{G}_{\hat{L}} - \hat{G}_{\hat{L},\hat{\kappa}}$ given in Lemma 5.2.5, this follows from similar calculations as in the proof of Lemma 5.1.5. □

Definition 5.2.7. *For $x, \xi > 0$ and $0 \le \vartheta < t \le 1$ let*

$$\hat{Q}_{I,0}(x,t,\xi,\vartheta) = \lambda \int_0^{e^x - 1} \hat{G}_{\hat{L}_\lambda}(\ln(e^x - \zeta), t, \xi, \vartheta)\, dF(\zeta).$$

Let the sequence of functions

$$\left\{ \hat{Q}_{I,n} \right\}_{n=0}^\infty$$

be defined inductively by

$$\hat{Q}_{I,n}(x,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\infty \hat{Q}_{I,0}(x,t,z,s)\, \hat{Q}_{I,n-1}(z,s,\xi,\vartheta)\, dzds, \quad n \in 1,2,\ldots.$$

Let

$$\hat{Q}_I(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{I,n}(x,t,\xi,\vartheta).$$

Let

$$\hat{G}_I(x,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\infty} \hat{G}_{\hat{L}_\lambda}(x,t,z,s)\,\hat{Q}_I(z,s,\xi,\vartheta)\,dzds. \tag{5.2.7}$$

Lemma 5.2.7. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ obeys the bound* (4.0.38)*.*

(i) *There exists a constant C_β, depending on β such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $(x',t) \in (x,\infty) \times (\vartheta,t)$*

$$\left|\hat{Q}_I(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \exp(-\beta|x-\xi|), \tag{5.2.8}$$

$$\left|\hat{Q}_I(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \times \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big), \tag{5.2.9}$$

$$\begin{aligned}\left|\hat{Q}_I(x,t,\xi,\vartheta) - \hat{Q}_I(x,t',\xi,\vartheta)\right| \le{}& C_\beta (t-t')^{\frac{1}{4}} (t'-\vartheta)^{-\frac{3}{4}} \\ &\times \Big(\exp(-2\beta|x-\xi|) \\ &+ \exp(-\beta x)\Big),\end{aligned} \tag{5.2.10}$$

and

$$\begin{aligned}\left|\hat{Q}_I(x,t,\xi,\vartheta) - \hat{Q}_I(x',t,\xi,\vartheta)\right| \le{}& C_\beta |x-x'|^{\alpha} (t-\vartheta)^{-\frac{1+\alpha}{3}} \\ &\times \Big(\exp(-2\beta|x-\xi|) \\ &+ \exp(-2\beta|x'-\xi|) \\ &+ \exp(-(\beta-\alpha)x)\Big).\end{aligned} \tag{5.2.11}$$

(ii) *$\hat{Q}_I$ is a solution of the integral equation*

$$\begin{aligned}\hat{Q}_I(x,t,\xi,\vartheta) ={}& \lambda \int_0^{e^x-1} \hat{G}_{\hat{L}_\lambda}(\ln(e^x-\zeta),t,\xi,\vartheta)\,dF(\zeta) \\ &+ \lambda \int_0^{e^x-1} \hat{G}_I(\ln(e^x-\zeta),t,\xi,\vartheta)\,dF(\zeta).\end{aligned} \tag{5.2.12}$$

(iii) *For every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1,2\}$ the following identities are all valid:*

$$\frac{\partial^l \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^l} = \int_{\vartheta}\int_0^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L}_\lambda}(x,t,z,\vartheta)}{\partial x^l} \hat{Q}_I(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial \hat{G}_I(x,t,\xi,\vartheta)}{\partial t} = \hat{Q}_I(x,t,\xi,\vartheta) + \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L}_\lambda}(x,t,z,\vartheta)}{\partial t} \hat{Q}_I(z,s,\xi,\vartheta)\,dzds.$$

(iv) *There exists a constant C_β depending on β such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^l}\right| \leq C_\beta (t-\vartheta)^{\frac{1-l}{3}} \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big),$$

$$\left|\frac{\partial^2 \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^2}\right| \leq C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right),$$

$$\left|\frac{\partial \hat{G}_I(x,t,\xi,\vartheta)}{\partial t}\right| \leq C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right),$$

and

$$\left|\hat{G}_I(x,t,\xi,\vartheta)\right| \leq C_\beta \min(x,1)^{\frac{1}{4}} (t-\vartheta)^{\frac{1}{4}} \times \left(\exp\left(-\frac{1}{2}\beta|x-\xi|\right) + \exp\left(-\frac{1}{2}\beta\xi\right)\right).$$

Proof. This follows from similar calculations as in the proof of Lemma 4.2.13. □

Lemma 5.2.8. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies the bound (4.0.38). There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$, and every $(x',t') \in (x,\hat{\kappa}) \times (0,t)$ the following inequalities are all valid:*

$$|Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-\frac{1}{2}} \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.13}$$

$$|(Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)) - (Q_I(x',t,\xi,\vartheta) - Q_{I,\kappa}(x',t,\xi,\vartheta))| \leq C(y'-y)^{\frac{1}{2}} (t-\vartheta)^{-\frac{3}{4}} \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y')^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.14}$$

and

$$|(Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)) - (Q_I(x,t',\xi,\vartheta) - Q_{I,\kappa}(x,t',\xi,\vartheta))| \leq C(t-t')^{\frac{1}{4}} t'^{-\frac{3}{4}} \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.15}$$

Proof. Because of the bounds given in Lemma 5.2.7 this follows from similar calculations as in the proof of Lemma 5.1.7 □

Definition 5.2.8. *For $x, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ define*

$$\hat{G}_{\hat{A}}\left(x, t, \xi, \vartheta\right) := G_{\hat{L}_\lambda}\left(x, t, \xi, \vartheta\right) + \hat{G}_I\left(x, t, \xi, \vartheta\right),$$

define

$$\hat{\psi}_3(x, t) = \int_0^t \int_0^\infty \hat{G}_{\hat{A}}\left(x, t, \xi, \vartheta\right) H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta,$$

and for $y \geq 0$ define

$$\psi_3(y, t) = \hat{\psi}_3\left(\ln(1+y), t\right).$$

Lemma 5.2.9. (i) *For every $x > 0$ and $t \in (0, 1]$ and $l \in \{1, 2\}$*

$$\frac{\partial^l \hat{\psi}_3(x, t)}{\partial x^l} = \int_0^t \int_0^\infty \frac{\partial^l \hat{G}_{\hat{A}}\left(x, t, \xi, \vartheta\right)}{\partial x^l} H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta$$

and

$$\frac{\partial \hat{\psi}_3(x, t)}{\partial t} = H_2\left(e^x - 1, \vartheta\right) + \int_0^t \int_0^\infty \frac{\partial \hat{G}_{\hat{A}}\left(x, t, \xi, \vartheta\right)}{\partial t} H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $x > 0$ and $t \in (0, 1]$ and every $l \in \{0, 1\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{\psi}_3(x, t)}{\partial x^l}\right| \leq C_\beta t^{\frac{4-l}{3}} \exp\left(-\beta x\right),$$

$$\left|\frac{\partial^2 \hat{\psi}_3(x, t)}{\partial x^2}\right| \leq C_\beta t \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial \hat{\psi}_3(x, t)}{\partial t}\right| \leq C_\beta t \exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\hat{\psi}_3(x, t)\right| \leq C_\beta t \min\left(x, 1\right).$$

(iii) *There exists a constant C_β, depending on β, such that, for every $y > 0$, every $t \in (0, 1]$ and every $l \in \{0, 1\}$*

$$\left|\frac{\partial^l \psi_3\left(y, t\right)}{\partial y^l}\right| \leq C_\beta t^{\frac{4-l}{3}} \left(1+y\right)^{-(\beta+l)},$$

$$\left|\frac{\partial^2 \psi_3\left(y, t\right)}{\partial y^2}\right| \leq C_\beta t \left(1+y\right)^{-\left(\frac{1}{2}\beta+2\right)},$$

$$\left|\frac{\partial \psi_3\left(y, t\right)}{\partial t}\right| \leq C_\beta t \left(1+y\right)^{-\frac{1}{2}\beta}$$

and

$$\left|\psi_3\left(x, t\right)\right| \leq C_\beta t \min\left(y, \kappa - y\right).$$

(iv) *There exists a constant C and a constant C_β, depending on β, such that, for every $(y,t,) \in (0,\kappa) \times (0,1]$*

$$\left|\frac{\partial^l \psi(y,t)}{\partial y^l}\right| \le Ct^{-\frac{l}{3}} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{y^2}{t}\right) + C_\beta t^{\frac{2-l}{3}} (1+y)^{-(\beta+l)},$$

$$\left|\frac{\partial^2 \psi(y,t)}{\partial y^2}\right| \le Ct^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{y^2}{t}\right) + C_\beta (1+y)^{-\left(\frac{1}{2}\beta+2\right)},$$

and

$$\left|\frac{\partial \psi(y,t)}{\partial t}\right| \le Ct^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{y^2}{t}\right) + C_\beta (1+y)^{-\frac{1}{2}\beta}.$$

(v) *There exists a constant C_β, depending on β, such that for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \hat{\psi}_3(x,t)}{\partial x^l} - \frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{2-l}{3}} \exp(-\beta\hat{\kappa}) \exp\left(-\frac{1}{128}\hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t}\right),$$

$$\left|\frac{\partial^2 \hat{\psi}_3(x,t)}{\partial x^2} - \frac{\partial^2 \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{128}\hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t}\right)$$

and

$$\left|\frac{\partial \hat{\psi}_3(x,t)}{\partial t} - \frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{128}\hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t}\right).$$

(vi) *There exists a constant C_β, depending on β, such that for every $(y,t) \in (0,\kappa) \times (0,1]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l} - \frac{\partial^l \psi_{3,\kappa}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\beta}) \times \exp\left(-\frac{1}{128}\hat{c}_0 \left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right),$$

$$\left|\frac{\partial^2 \psi_3(y,t)}{\partial y^2} - \frac{\partial^2 \psi_{3,\kappa}(y,t)}{\partial y^2}\right| \le C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\frac{1}{2}\beta} (1+y)^{-2} \times \exp\left(-\frac{1}{128}\hat{c}_0 \left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right)$$

and

$$\left|\frac{\partial \psi_3(y,t)}{\partial t} - \frac{\partial \psi_{3,\kappa}(y,t)}{\partial t}\right| \le C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\frac{1}{2}\beta} \times \exp\left(-\frac{1}{8}\hat{c}_0 \left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right).$$

Proof. This follows from similar calculations as in the proof of Theorem 4.2.3. □

We are now finally in position to establish existence on unbounded domain for the main case $\sigma_R > 0$.

Theorem 5.2.2. *Assume that* $\sigma_R > 0$ *and that the tail distribution* $\bar{F}$ *satisfies the bound* (4.0.38)*.* $\hat{\psi}_3(x,t)$ *is a classical solution of the PIDE* (5.2.2) *and* $\psi_3(y,t)$ *is a classical solution of the PIDE* (5.0.26)*.*

Proof. It follows from the identities given in Lemma 5.2.6 and Lemma 5.2.9 that $\hat{\psi}_3(x,t)$ satisfies the PIDE (5.2.2) on the inner domain, i.e for $y > 0$ and $t \in (0,1]$. Similar arguments as in the proof of Lemma 5.1.1 yield that $\hat{\psi}_3(x,t)$ satisfies the initial condition and the boundary conditions. Since $\hat{\psi}_3(x,t)$ is a classical solution of the PIDE (5.2.2) it follows from the chain rule that $\psi_3(y,t) = \hat{\psi}_3(\ln(1+y), t)$ is a solution of the PIDE (5.0.26). □

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