A numerical approach to ruin probability in finite time for fitted models with investment

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Abstract

In this paper we present a numerical method for solving a partial integro-differential equation (PIDE) associated with ruin probability, when the surplus is continuously invested in stochastic assets. The method uses precalculated Gaussian quadrature rules for the numerical integration. Except for the numerical integration part, the method is based largely on the finite differences method used in Halluin et al. (2005) for a PIDE associated with a more general option pricing problem. In our numerical examples we use historical data for inflation and returns on U.S. Treasury bills, U.S. Treasury bonds and American stocks. The log-returns of the investments are adjusted for an assumed constant force of inflation. We consider four different strategies for continuous investment: (a) U.S. Treasury bills with a constant maturity of 3 months, (b) U.S. Treasury bonds with a constant maturity of 10 years, and (c) the Standard and Poor 500 index and (d) another index of American stocks. For each of these strategies a geometric Brownian motion process is fitted to the aforementioned historical data. The results suggest that the ruin probabilities obtained can vary substantially, depending on whether the models are fitted to data for the last decade or for a longer time period. We also discuss numerical solution of investment models with jumps.

1 Introduction

In the classical Cramér-Lundberg model the risk process of an insurance company at time t is assumed to be of the form

$$Y_t = y + pt - \sum_{n=1}^{N_t} S_n.$$

Here $y > 0$ is the initial capital, pt is the accumulated premium income up to time t, coming at a constant rate p. The sum $\sum_{n=1}^{N_t} S_n$ is a compound Poisson process with only non-negative jumps and whose counting process N has a constant intensity λ. In the following we will follow the convention that $\sum_{n=1}^{0} = 0$ and that $\Pi_{n=1}^{0} = 1$.

In Paulsen and Gjessing (1997) the classical model is generalized to possibly include a scaled Brownian motion $\sigma_P W_P$, where $\sigma_P \geq 0$. In addition it is assumed that the surplus generated from the basic process

$$P_t = pt + \sigma_P W_{P,t} - \sum_{n=1}^{N_t} S_n, \qquad t \geq 0, \tag{1}$$

is continuously invested in risky assets that follow a jump-diffusion process

$$R_t = rt + \sigma_R W_{R,t} - \sum_{n=1}^{N_{R,t}} S_{R,n}, \qquad t \geq 0.$$

In the above $\sigma_R \geq 0$, $r \in \mathbb{R}$, W_R is a Brownian motion, and the sum $\sum_{n=1}^{N_{R,t}} S_{R,n}$ is a compound Poisson process whose counting process $N_{R,t}$ has a constant intensity λ_R and a common jump size distribution F_R. With these assumptions and y as the initial capital, the risk process becomes

$$Y_t = y + P_t + \int_0^t Y_{s-} dR_s, \qquad t \geq 0. \tag{2}$$

It is shown in Paulsen (1998) that the solution of this equation is

$$Y_t = \bar{R}_t \left(y + \int_0^t \bar{R}_s^{-1} dP_s \right), \tag{3}$$

where $\bar{R}_t = \exp\left\{ \left(r - \frac{1}{2}\sigma_R^2\right) t + \sigma_R W_{R,t} \right\} \Pi_{n=1}^{N_{R,t}} (1 + S_{R,n})$.

In Paulsen (1993) a third process I, representing inflation, is included in the model. In this model inflation is assumed to have the same effect on both the premium income and the insurance claim sizes. It is shown in Paulsen (1993) that if inflation is a deterministic process then the effect on the risk process is the same as if we substituted R with $R - I$. We will assume that there is such an inflation process, with a constant force $\bar{\imath}$, i.e. at time t

$$I_t = \bar{\imath} t.$$

Let the R process be an *inflation-adjusted* return on investment process. This corresponds to replacing the parameter r with $\bar{r} = r - \bar{\imath}$. In this context inflation refers to geometric growth of both insurance claim sizes and premium rates. In the numerical examples we let $\bar{\imath}$ be the geometrical mean of the inflation for the corresponding time periods. The data for annualized inflation are taken from inflationdata.com (2012).

For a risk process like the one defined above, the time of ruin is defined as $\tau := \inf \{t : Y_t < 0\}$ and the probability of ruin in finite time is defined as

$$\psi(y, t) := P(\tau \leq t | Y_0 = y). \tag{4}$$

In this paper we will discuss a method for numerical computation of ruin probability in finite time for these models, based on solving an associated partial integro-differential equation (PIDE) using finite differences. In our numerical examples in Section 4 we consider two different claim size distributions. In the first example the claims follow a light-tailed standard exponential distribution, while in the second they follow a mixture of a standard exponential distribution and a heavy-tailed standardized Pareto distribution with expectation 1. For the standardized Pareto distribution part of the mixed distribution we choose a parameter value based on the fitting discussed in chapter 6 in Embrechts et al. (1997) of a Pareto distribution to data for Danish fire insurance claims.

We consider four different strategies for continuous investment: (a) U.S. Treasury bills with a constant maturity of 3 months, (b) U.S. Treasury bonds with a constant maturity of 10 years, (c) the Standard and Poor 500 index and (d) another index of American stocks. We fit a geometric Brownian motion (GBM) to data for annual return of bonds and stocks for the period 1928-2011, taken from Damodaran (2012). In one example we use data for the entire time period. In another example we only use data for 2000-2011. We also calculate ruin probabilities based on data fittings of GBM models, and some jump-diffusion models in Damodaran (2012), to the SP 500 index for the period 1962-2003.

2 Integro-differential equations for the ruin probability

In Paulsen (2008) a partial integro-differential equation (PIDE) is stated for the survival probability $\phi(y,t) = 1 - \psi(y,t)$. First let L be the integro-differential operator

$$\begin{aligned} Lh(y) = {} & \frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2)h''(y) + (p + \bar{r}y)h'(y) \\ & + \lambda \int_0^y (h(y-x) - h(x))\, dF(x) \\ & + \lambda_R \int_{-1}^{\infty} (h\,(y\,(1+x)) - h(y))\, dF_R(x), \end{aligned} \tag{5}$$

where L is acting on the variable y, y and t are assumed non-negative, $\bar{r} \in \mathbb{R}$ and $\sigma_P, \sigma_R, p, \lambda$ and λ_R are assumed to be nonnegative. Then the PIDE is given as

$$\frac{\partial}{\partial t}\phi(y,t) = L\phi(y,t). \tag{6}$$

The initial condition is $\phi(y,0) = 1$ for every $y > 0$. Asymptotically the solution must satisfy the condition $\lim_{y\to\infty} \phi(y,0) = 1$. When $\sigma_P > 0$ the infinite variation of the Brownian motion W_P implies that

$$\inf\{t : Y_t < 0\} = \inf\{t : Y_t = 0\}.$$

Hence in this case the survival probability must satisfy $\phi(0,t) = 0$.

2.1 Regularity of solution

Consider the case when $\lambda_R = 0, \sigma_P > 0$, and either $\sigma_R = \bar{r} = 0$ or $\sigma_R > 0$. If an additional weak condition on the probability measure F also holds it is shown in Paper D that the integro-differential equation (6) has a classical solution except at the origin. That is, a solution which is differentiable with respect to

t, twice differentiable with respect to y on the inner domain, and continuous at every point of the boundary except for the origin. It is also known that a classical solution exists when the investment earns an interest with a constant force, i.e., if $\sigma_P = \sigma_R = \lambda_R = 0$ (see Pervozvansky Jr. (1998); Paulsen (2008)). To the author's knowledge there are no known regularity results for other cases, when $\sigma_P^2 + \sigma_R^2 > 0$. However, the behavior of the numerical solution in our experiments suggests that letting $\sigma_P = 0$ or letting $\lambda_R > 0$ (adding the last integral term in (5)) does not negatively affect the smoothness of the solution, at least as long as the distribution functions $F(x)$ and $F_R(x)$ are smooth.

2.2 Localization to a bounded domain and choice of coordinates

The domain of equation (6) is unbounded in the space dimension, which of course is not computationally feasible. Instead we introduce an *artificial* boundary condition (see Section 12.4.1 in Cont and Tankov (2004)), namely that $\phi(y,t) = 1$ for every $y \geq \kappa$. The introduction of an artificial boundary condition leads to an error generally referred to as a *localization error*. Let ϵ_κ be this localization error and let $\bar{\ }(x) = 1 - F(x)$ be the tail distribution. In Paper D it is shown that if $\sigma_P, \sigma_R > 0$, $\lambda_R = 0$, and for some $c > 0$

$$\sup_{x>0} x^c \bar{F}(x) < \infty,$$

then for some constant C

$$|\epsilon_\kappa| < C (1+\kappa)^{-c}$$

for any $\kappa > 0$.

In our numerical experiments we found it more numerically efficient (leading to better accuracy) to make the change of variable $z = \ln(1+y)$. In the following we rewrite the above integro-differential operator L in terms of the new variable z. We also denote the finite time horizon by T. Since $y = e^z - 1$, first let

$$\rho(z,t) := \phi(e^z - 1, t), \quad (z,t) \in [0, \ln(1+\kappa)] \times [0,T].$$

For $z \in [0, \ln(1+\kappa)]$ let

$$\begin{aligned} a_2(z) &:= \frac{1}{2}\left(\sigma_P^2 e^{-2z} + \sigma_R^2 \left(1 - e^{-z}\right)^2\right), \quad \text{and} \\ a_1(z) &:= p e^{-z} + \bar{r}\left(1 - e^{-z}\right) - a_2(z). \end{aligned} \tag{7}$$

Now the operator L becomes

$$\begin{aligned} L_z g(z) = {} & a_2(z) g''(z) + a_1(z) g'(z) \\ & + \lambda \int_0^{e^z - 1} \left(g\left(\ln\left(e^z - x\right)\right) - g(z)\right) dF(x) \\ & + \lambda_R \int_{-1}^{\infty} \left(g\left(\ln\left(1 + \left(e^z - 1\right)(1+x)\right)\right) - g(z)\right) dF_R(x). \end{aligned}$$

Making this change of variables and including the artificial boundary condition gives the equation

$$\begin{cases} \rho(z,0) = 1, \quad z \in (0, \ln(1+\kappa)) \,. \\ \rho(\ln(1+\kappa), t) = 1, \quad t \in (0,T]. \\ \frac{\partial \rho(z,t)}{\partial t} = L_z \rho(z,t) \qquad \text{on } (z,t) \in (0, \ln(1+\kappa)) \times (0,T]. \end{cases} \tag{8}$$

Here L_z is acting on the variable z. When $\sigma_P > 0$ we have the extra boundary condition

$$\rho(0,t) = 0, \quad t \in (0,T].$$

In the following we will also define that $\rho(z,t) = 1$ for every $z \geq \ln(1+\kappa)$ and $t \in [0,T]$.

The rest of this paper is a discussion of numerical finite-difference methods for solving (8), with some numerical examples for fitted models with investment in U.S. Treasury bills, U.S. Treasury bonds and American stocks. In all our examples the space grid will be equally spaced on $[0, \ln(1+\kappa)]$. An advantage with this grid, compared with an equally spaced grid in the original coordinate system, is that it gives a more numerically efficient distribution of grid points. This is especially true for the case when $\sigma_P > 0$, since in this case the solution is discontinuous at the origin. Having many grid points near the bottom of the domain seems to give higher accuracy.

3 Numerical algorithm

The finite difference schemes discussed in this paper are adaptations of the schemes developed in Halluin et al. (2005) to fit the problem (8). The basic idea is to solve (8) using Crank-Nicolson time integration on an equally spaced two-dimensional grid. To ensure numerical stability we follow the recommendation in Giles and Carter (2005) and replace the first Crank-Nicolson step with four quarter-timesteps of Backward Euler time integration. After explaining how we do the numerical integration we discuss the difference equations associated with these finite difference schemes.

3.1 Evaluation of the integrals

In the following we assume that both the claim size distribution and the distribution of the jumps of the R-process are smooth. We denote their respective densities as f and f_R. In what follows let m be the grid size and $h = \frac{\ln(1+\kappa)}{m}$ be the step size in the z grid. Thus the nodes in the z grid are $z_i = ih$ for $i \in 0,1,\ldots m$. Let the nodes in the time grid be $t_0 = 0, t_1, \ldots, t_n$. Since $y = e^x - 1$ let $y_i = e^{z_i} - 1$, for $i \in 0,1,\ldots,m-1$. Let

$$\begin{aligned} \rho_i^k &= \rho(ih, t_k), \qquad i \in 0,1,\ldots,m, k \in 0,1,\ldots,n, \\ I_i^k &= \int_0^{y_i} \rho(\ln(1+y_i-x), t_k)\, dF(x), \qquad i \in 1,2,\ldots,m-1, \\ J_i^k &= \int_{-1}^{\infty} \rho(\ln(1+y_i(1+x)), t_k)\, dF_R(x), \qquad i \in 1,2,\ldots,m-1, \end{aligned} \tag{9}$$

and

$$\tilde{I}_i^k = \int_{y_{i-1}}^{y_i} \rho\left(\ln\left(1+x\right), t_k\right) f_i(x) dx, \qquad i \in 1, 2, \ldots, m-1.$$

where $f_i(x) = f\left(y_i - x\right)$.

The sequence $\left\{I_i^k\right\}$ defined above is a semi-discretization of the insurance claim integrals

$$I\left(y,t\right) = \int_0^y \left(\rho\left(\ln\left(1+y-x\right), t\right)\right) dF(x) \qquad \text{on } (y,t) \in (0,\kappa) \times (0,T].$$

Similarly, when $\lambda_R > 0$, the sequence $\left\{J_i^k\right\}$ is a discretization of the investment integrals

$$J\left(y,t\right) = \int_{-1}^{\infty} \rho\left(\ln\left(1+y\left(1+x\right)\right), t\right) dF_R(x).$$

In Section 4.5 we discuss some examples with jumps in the investment process. In these examples the J_i^k are calculated as

$$\int_{-\infty}^{\infty} \rho\left(\ln\left(1+y_i e^x\right), t_k\right) f_{\tilde{R}}(x) dx, \tag{10}$$

where

$$f_{\tilde{R}}(x) = e^x f_R\left(e^x - 1\right).$$

As we will see in Section 3.4, for each time step each integral in the sequence $\left\{I_i^k\right\}_{i=1}^{m-1}$ must be computed more than once for every time step, as part of an iteration method. When $\lambda_R > 0$ this also has to be done for each integral in the sequence $\left\{J_i^k\right\}_{i=1}^{m-1}$. Moreover, the integrands in the sequences $\left\{I_i^k\right\}$ and $\left\{J_i^k\right\}$ depend on i. This means that the numerical complexity for numerical integration based on such Newton-Coates quadrature methods as Simpson's rule would be $O(m^2)$ for just one calculation of $\left\{I_i^k\right\}_{i=1}^{m-1}$. Fortunately there are ways of avoiding this, as discussed below.

A popular model is to let the jump sizes be exponential distributed. Below we first show how for this model it is relatively simple to compute the integrals efficiently. We then return to general claim size distributions in 3.1.2.

3.1.1 Exponentially distributed jumps

For $\alpha > 0$ let

$$f_i(x) = \alpha e^{-\alpha(y_i - x)}, \qquad i \in 1, 2, \ldots, m-1.$$

In the special case of exponentially distributed claim sizes with parameter α we observe that

$$\int_0^y \rho\left(\ln\left(1+y-x\right), t\right) f\left(x\right) dx = e^{-\alpha y} \int_0^y \rho\left(\ln\left(1+x\right), t\right) \alpha e^{\alpha x} dx.$$

Thus in this case the insurance claim integrals are dependent on y only through the upper limit and a factor that can be taken outside the integral. Moreover, we have the recursive relation

$$I_{i+1}^k = \exp\left(-\alpha\left(y_{i+1} - y_i\right)\right) I_i^k + \tilde{I}_{i+1}^k. \tag{11}$$

Here we are indebted to the discussion in Toivanen (2008). Due to (11) fast evaluation of the sequence $I_1^k, \ldots, I_{m-1}^k$ is much simpler when the claims are exponentially distributed than in the general case.

As in Toivanen (2008) we approximate the integrand $\rho\left(\frac{x}{1+x}, t_k\right)$ in $\tilde{I}_i^k$ by linear interpolation. This gives the approximation

$$\tilde{I}_i^k \approx \tilde{a}_i^k \int_{y_{i-1}}^{y_i} f_i(x)dx + \tilde{b}_i^k \int_{y_{i-1}}^{y_i} x f_i(x)dx, \tag{12}$$

where

$$\tilde{b}_i^k = \frac{\rho_i^k - \rho_{i-1}^k}{y_i - y_{i-1}}$$

and

$$\tilde{a}_i^k = \rho_i^k - \tilde{b}_i^k y_i.$$

Lastly, we have that

$$\int_{y_{i-1}}^{y_i} f_i(x)dx = 1 - \exp\left(-\alpha\left(y_i - y_{i-1}\right)\right)$$

and that

$$\int_{y_{i-1}}^{y_i} x f_i(x)dx = \left(y_i - \frac{1}{\alpha}\right)\left(1 - \exp\left(-\alpha\left(y_i - y_{i-1}\right)\right)\right) + \left(y_i - y_{i-1}\right)\exp\left(-\alpha\left(y_i - y_{i-1}\right)\right).$$

If the return on investment process R is like that in the Kou model (see Kou (2002)), the jumps of the log-returns follow an asymmetric exponential distribution. That is, for some parameters $\eta_1, \eta_2 > 0$ and a weight $q \in [0,1]$, the probability density $f_{\tilde{R}}(x)$ of the jumps of the log-returns is

$$f_{\tilde{R}}(x) = q 1_{x>0} \eta_1 \exp\left(-\eta_1 x\right) + \left(1-q\right) 1_{x<0} \eta_2 \exp\left(-\eta_2 \left|x\right|\right).$$

In our context this corresponds to letting the investment jump integral in (8) be of the form

$$\int_{-1}^{\infty} \rho\left(\ln\left(1 + y\left(1+x\right)\right), t\right) dF_R(x) = q J_1 + \left(1-q\right) J_2,$$

where

$$J_1 = \int_0^{\infty} \rho\left(\ln\left(1 + y e^v\right), t\right) \eta_1 \exp\left(-\eta_1 v\right) dv$$

and

$$J_2 = \int_{-\infty}^{0} \rho\left(\ln\left(1 + y e^v\right), t\right) \eta_2 \exp\left(\eta_2 v\right) dv.$$

Making the substitution $w = v + \ln\left(y\right)$ gives

$$J_1 = y^{\eta_1} \int_{\ln y}^{\infty} \rho\left(\ln\left(1 + e^w\right), t\right) \eta_1 \exp\left(-\eta_1 w\right) dw,$$

and

$$J_2 = y^{-\eta_2} \int_{-\infty}^{\ln y} \rho\left(\ln\left(1+e^{w}\right), t\right) \eta_2 \exp\left(\eta_2 w\right) dw.$$

From these formulas one can derive a recursive relation given in Toivanen (2008) and similar to (11). In this model the investment integrals can be evaluated in a way similar to the method described above for the insurance claim integrals.

3.1.2 Computation of Gaussian quadrature rules

Returning to general smooth claim size distributions, we can evaluate the integrals in (9) using Gaussian quadrature methods. The main idea of an l-point Gaussian quadrature rule is to find abscissas $x_1, \ldots, x_l$ and corresponding weights $w_1, \ldots, w_l$ such that, for a known function $\omega(x) : [-1, 1] \to \mathbb{R}$, and given function values of a continuous function $g : [-1, 1] \to \mathbb{R}$,

$$\int_{-1}^{1} g(x)\omega(x)dx \approx \sum_{i=1}^{l} w_i g(x_i). \tag{13}$$

In our numerical method these rules are calculated using the subroutines 'dlancz' and 'dgauss' from the Netlib package 726 'ORTHPOL', developed by Walter Gautschi. The package is an implementation of a Golub-Welsch algorithm. For the integral (13) a Golub-Welsch algorithm (see Golub and Welsch (1969)) involves finding the roots of a sequence of polynomials $p_0(x), \ldots, p_l(x)$. The polynomials in this sequence are required to be orthogonal in the following inner product space, defined by

$$\langle q_1, q_2 \rangle = \int_{-1}^{1} q_1(x) q_2(x) \omega(x) dx,$$

where q_1, q_2 are continuous functions.

Following this procedure it can be shown that the resulting Gaussian quadrature rule is exact for polynomials of degree at most $2l - 1$ (see Theorem 4.7.7 in Cheney (2001)). In order to apply a quadrature rule it is necessary to evaluate the solution at points that are not on the z-grid. We do this by means of linear interpolation.

In our numerical method, $m - 1$ Gaussian quadrature rules are precalculated for each $I_1^k, \ldots, I_{m-1}^k$ before the actual finite differences method begins. The obvious choice of weighting function for these rules is the density $f(x)$. While the weighting function is the same for every I_i^k, the integrals have upper limits that increase with i. This makes it necessary to calculate a separate Gaussian quadrature rule for each I_i^k. However, since the weighting function is the same, we found that the rules were more rapidly and more accurately calculated when a rule calculated for I_k^k is used in the calculation of a rule for the next integral I_{k+1}^k. We also found that when the claim size distribution has a heavy tail it has a positive effect on the accuracy to make the substitution $v = \ln(1 + x)$, and calculate I_i^k as

$$\int_{0}^{\ln(1+y_i)} \rho\left(\ln\left(1 + y_u - e^{v}\right), t\right) e^{v} f\left(e^{v} - 1\right) dv.$$

As is normally the case in numerical problems there is a trade-off between the numerical complexity of the Golub-Welsch algorithm and the accuracy of the results. To control the accuracy of the weights and abscissas, our method first applies the routines ‘dlancz’ and ‘dgauss’ with a relatively low complexity. Then the subroutines are called again with increasing resolutions until the differences between succeeding weights and succeeding abscissas are small. In the numerical integration of the $\left\{J_i^k\right\}_{i=1}^{m-1}$ integrals, only one quadrature rule needs to be calculated with the Golub-Welsch algorithm. Denoting the weights of this quadrature rule by $w_{J,1}, \ldots, w_{J,m}$, and denoting the abscissa points by $x_{J,1}, \ldots, x_{J,m_J}$, these integrals are calculated as

$$J_i^k \approx \sum_{j=1}^{m_J} w_{J,j} \rho\left(\ln\left(1 + y_i e^{x_{J,j}}\right), t_k\right).$$

In the special case of the Merton model the required quadrature rule corresponds to Gauss-Hermite quadrature. Calculation of these rules is implemented in the subroutine ‘gaussq’, also in the ‘ORTHPOL’ package.

3.2 Backward Euler time integration

As mentioned above we follow a suggestion in Giles and Carter (2005), the numerical differentiation part of our method consists of computing the first four time steps with backward Euler time integration, where each time step is of length Δt. The subsequent time steps are of length $4\Delta t$ and are computed using Crank-Nicolson time integration.

Now let us look at the inner z-grid and time grid points. Here we discretize the time derivative with backward Euler finite differences. In the z variable we discretize both the first and second derivatives by means of central differences. This yields the following set of difference equations, where as before $\rho_i^k = \rho\left(ih, t_k\right)$.

$$\begin{aligned}\frac{\rho_i^{k+1} - \rho_i^k}{\Delta t} = a_2\left(z_i\right) \frac{\rho_{i+1}^{k+1} - 2\rho_i^{k+1} + \rho_{i-1}^{k+1}}{h^2} &+ a_1\left(z_i\right)\left[\frac{\rho_{i+1}^{k+1} - \rho_{i-1}^{k+1}}{2h}\right] \\ &- \lambda \rho_i^{k+1} + \lambda \sum_{j=0}^{m} c_{i,j} \rho_j^{k+1} \\ &- \lambda_R \rho_i^{k+1} + \lambda_R \sum_{j=0}^{m} d_{i,j} \rho_j^{k+1}.\end{aligned}$$

In the above a_1 and a_2 are defined in (7). The sum $\sum_{j=0}^{m} c_{i,j} \rho_j^{k+1}$ is related to the evaluation of the integral I_i^k, while the sum $\sum_{j=0}^{m} d_{i,j} \rho_j^{k+1}$ is related to the evaluation of the integral J_i^k. Since the $c_{i,j}$’s and $d_{i,j}$’s are integral weights they are non-negative constants.

If we let

$$\begin{aligned}\hat{\lambda} &= \lambda + \lambda_R, \\ \hat{c}_{i,j} &= \frac{\lambda}{\lambda + \lambda_R} c_{i,j} + \frac{\lambda_R}{\lambda + \lambda_R} d_{i,j},\end{aligned} \tag{14}$$

$$\alpha_i = \frac{a_2(z_i)}{h^2} - \frac{a_1(z_i)}{2h}, \tag{15}$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2} + \frac{a_1(z_i)}{2h}, \tag{16}$$

then the difference equation above can be rearranged as

$$\rho_i^{k+1}\left[1+\left(\alpha_i+\beta_i+\hat{\lambda}\right)\Delta t\right] - \Delta t\beta_i\rho_{i+1}^{k+1} - \Delta t\alpha_i\rho_{i+1}^{k+1} - \hat{\lambda}\Delta t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^{k+1} = \rho_i^k. \tag{17}$$

Now let us see what happens if we change definitions (15) and (16) a little. Let

$$\alpha_i = \frac{a_2(z_i)}{h^2}, \tag{18}$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2} + \frac{a_1(z_i)}{h}. \tag{19}$$

Then (17) corresponds to discretizing the first space derivative using forward differences.

Another alternative is to discretize the first space derivative using backward differences. This gives

$$\alpha_i = \frac{a_2(z_i)}{h^2} - \frac{a_1(z_i)}{h},$$

and

$$\beta_i = \frac{a_2(z_i)}{h^2}.$$

Theorem 1. *Assume that, for every $i \in 1, \ldots, m-1$, $\alpha_i \geq 0$, $\beta_i \geq 0$ and*

$$\sum_{j=0}^{m}\hat{c}_{i,j} \leq 1.$$

Then the backward Euler scheme given in (17) is unconditionally stable in the max norm. Moreover, for any given index i, at least one of the options for discretizing $\frac{\rho(z,t)}{\partial z}$ given above, i.e, central differences, forward differences and backward differences, gives $\min(\alpha_i, \beta_i) \geq 0$.

Proof. This follows from Theorem 3.1 in Halluin et al. (2005). □

In the rest of the paper we will assume that $\alpha_i, \beta_i \geq 0$ for every $i \in 1, \ldots, m-1$. Since discretizing the first space derivative with central differences gives a second order convergence rate, whereas forward and backward differences give only first order convergence, we choose central differences for those nodes where this does not lead to negative α_i or β_i.

3.3 Crank-Nicolson time integration

While the fully implicit scheme given in (17) is unconditionally stable, it has the disadvantage of being only first order convergent in the time variable. An alternative, suggested in Giles and Carter (2005) and mentioned above, is to use backward Euler time integration only for the initial four quarter-steps, each with length Δt, and then continue with Crank-Nicolson time integration with time steps of length $\hat{\Delta} t = 4\Delta t$. This approach results in the following set of discrete equations for the Crank-Nicolson part:

$$\begin{aligned}\rho_i^{k+1}\left[1+\left(\alpha_i+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}\right]-\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^{k+1}-\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^{k+1}\\ =\rho_i^{k}\left[1-\left(\alpha_i+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}\right]+\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^{k}+\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^{k}\\ +\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{i}\hat{c}_{i,j}\rho_j^{k+1}+\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^{k}.\end{aligned}$$

Let

$$\rho^k := \left(\rho_0^k, \rho_1^k, \ldots, \rho_m^k\right)'$$

and define the matrix M such that

$$-\left[M\rho_i^k\right]_i=\rho_i^k\left(\alpha+\beta_i+\hat{\lambda}\right)\frac{\hat{\Delta} t}{2}-\frac{\hat{\Delta} t}{2}\beta_i\rho_{i+1}^k-\frac{\hat{\Delta} t}{2}\alpha_i\rho_{i-1}^k-\frac{1}{2}\hat{\lambda}\hat{\Delta} t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k. \tag{20}$$

Also let

$$B=[I-M]^{-1}[I+M].$$

Then (20) can be written either as

$$[I-M]\rho^{k+1}=[I+M]\rho^k, \tag{21}$$

or as

$$\rho^k=(B)^k\rho^0.$$

Theorem 2. *Assume that for every $i \in 1,\ldots,m-1$, $\beta_1 \geq 0$, $\alpha_i \geq 0$ and*

$$\sum_{j=0}^{m}\hat{c}_{i,j}<1.$$

Then the Crank-Nicolson discretization (20) *is algebraically stable in the sense that there exists a C such that for every n and every grid size*

$$\|(B)^n\|_\infty \leq Cn^{\frac{1}{2}}. \tag{22}$$

The norm used above is the l_∞ norm.

Proof. This follows from Theorem 4.1 in Halluin et al. (2005). □

In contrast to (22), the Lax-Meyer theorem states that strong stability, i.e.

$$\|(B)^n\|_\infty \leq C,$$

for some C independent of n, is a necessary condition for convergence for all initial data. As noted in Halluin et al. (2005), the form of stability given in (22) is clearly weaker than strong stability, and hence yields convergence only for certain initial data. Some caution is thus in order, in particular for the case $\sigma_P > 0$, where the exact solution is discontinuous at the origin. This is why our method uses four quarter-time steps of backward Euler time integration for the first time step, instead of using the Crank-Nicolson method there.

3.4 Fixed-point iteration method

As noted in Halluin et al. (2005), it is computationally very expensive to solve the full linear system of the form (20) or (17), since this means solving a system of linear equations whose numerical complexity grows as $O(m^2)$. Instead we will follow Halluin et al. (2005) and solve the system using the fixed-point iteration method described below. The main advantage with this iteration scheme is that the integrals can be calculated using only the results from the previous time step and the previous iteration. Hence, for a given iteration, the evaluation of the integrals can be considered to be explicit. Thus we define the matrix $\hat{M}$ such that

$$-\left[\hat{M}\rho^k\right]_i = \rho_i^k\left(\alpha_i + \beta_i + r + \hat{\lambda}\right)\hat{\Delta}t - \hat{\Delta}t\beta_i\rho_{i+1}^k - \hat{\Delta}t\alpha_i\rho_{i-1}^k.$$

The only difference between $\hat{M}$ and M is that $\hat{M}$ does not include the integral terms. From the representation (21) it follows that the Crank-Nicolson discretization (20) can be written as follows:

$$\left[I - \frac{1}{2}\hat{M}\right]\rho^{k+1} = \left[I + \frac{1}{2}\hat{M}\right]\rho^k + \frac{1}{2}\hat{\lambda}\Delta t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^{k+1} + \frac{1}{2}\hat{\lambda}\Delta t\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k. \quad (23)$$

Using this notation the fixed-point iteration method in Halluin et al. (2005) is described as follows:

Let $\left(\rho^{k+1}\right)^0 = \rho^k$.
Let $\hat{\rho}^j = \left(\rho^{k+1}\right)^j$.
For $j = 0, 1, 2, \ldots$ until convergence
    Solve $\left[I - \frac{1}{2}\hat{M}\right]\hat{\rho}^{j+1} = \left[I + \frac{1}{2}\hat{M}\right]\rho^k$
        $+\frac{1}{2}\hat{\lambda}\hat{\Delta}\sum_{j=0}^{m}\hat{c}_{i,j}\hat{\rho}_j^j + \frac{1}{2}\hat{\lambda}\hat{\Delta}\sum_{j=0}^{m}\hat{c}_{i,j}\rho_j^k$.
    If $\max_i\left|\hat{\rho}_i^{j+1} - \hat{\rho}_i^j\right| < tolerance$, then quit.
EndFor

In Theorem 5.1 in Halluin et al. (2005) it is proven not only that the iteration scheme above converges, but that the error $e^j = \rho^{k+1} - \hat{\rho}^j$ has an upper bound

$$\| e^{j+1} \|_\infty \leq \| e^j \|_\infty \frac{\frac{1}{2}\hat{\lambda}\hat{\Delta}t}{1 + \frac{1}{2}\hat{\lambda}\hat{\Delta}t}. \quad (24)$$

We used an integration algorithm very similar to the above algorithm for the initial backward Euler timesteps. In our implementation the iteration is set to terminate when the maximal absolute difference between ρ_i^{k+1}-values of consecutive iterations is less than 10^{-8}. We found that for good convergence of this iteration scheme it was advantageous to choose time steps $\hat{\Delta}t$ smaller than $\frac{1}{\lambda+\lambda_R}$.

4 Experimental results

In this section we will discuss numerical examples, where we first fit parameter values for the risk models discussed in the introduction, and then calculate the corresponding ruin probabilities by solving the PIDE (6). We will first consider the case when the claim sizes follow the standard exponential distribution. Then we let the claim sizes follow a mixture of a standard exponential distribution and a Pareto distribution, standardized to have expectation 1. For both claims processes we choose a value for the intensity λ based on data for inflation-adjusted Danish insurance claims. In the examples where the claim distribution is a mixture of a Pareto distribution and an exponential distribution, we let the tail index of the Pareto distribution be the same as the fitted value in Embrechts et al. (1997).

The Danish fire insurance data set consists of 2167 claims over a period of 11 years. We choose a year as the time unit, which gives a maximum likelihood estimate for λ of 197 with a standard error of 4.26. In all our examples we let $\lambda = 197$, let the claims have expectation value 1, and let $p = 216.7$. This corresponds to letting the premium be decided by the expected value principle, with safety loading of 0.1. As already mentioned we adjust the returns of the investments for a constant force of inflation $\bar{\imath}$. We use inflation data from inflationdata.com (2012) to choose an $\bar{\imath}$ for each time period that we consider. These values are given in Table 1.

For the investment return process we consider three different strategies. The first strategy is to continuously invest in Treasury bills with a 3-month rate, the second strategy is to continuously invest in 10-year Treasury bonds that also earn coupons and price appreciation. The last strategy is to invest in American stocks. We use a dataset from Damodaran (2012), which covers annual returns on U.S. Treasury bills, U.S. Treasury bonds and American stocks for the period from 1928 to 2011. For the S&P 500 data for 1962-2003, we use parameter estimates from Ramezani and Zeng (2007) for a geometric Brownian motion model, a Merton model and a Kou model.

4.1 Fitting of geometric Brownian motion to data

In a geometric Brownian motion investment model with drift parameter r and diffusion parameter σ, the log-returns (log-differences) are normally distributed with variance $\sigma^2 t$ and expectation $\left(r - \frac{1}{2}\sigma^2\right) t$. Let $X_0, X_1, \ldots, X_l$ be $l + 1$ observations of the index values at equally spaced times $t_0 = 0, t_1, \ldots, t_l = lt_1$, with one year as the unit of time. Let $Z_1 = \ln\left(\frac{X_1}{X_0}\right), \ldots, Z_l = \ln\left(\frac{X_l}{X_{l-1}}\right)$ be the log-returns.

Period	1928 -2011	1963-2003	2000-2011
Force of inflation	0.03058	0.04406	0.02506

Table 1: The assumed constant force of inflation $\bar{\imath}$ fitted to different time periods.

Let $\bar{Z}$ be the sample mean and let S^2 be the sample variance of the Z_i's. Since the log-returns are i.i.d. normal distributed $\mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)t_1, \sigma^2 t_1\right)$, the method of moment estimator for σ is $\sqrt{\frac{1}{t_1}S^2}$. We thus use $\sqrt{\frac{1}{t_1}S^2}$ as our statistic for σ_R. The method of moment estimator for r is $\frac{1}{t_1}\left(\bar{Z} + \frac{1}{2}S^2\right)$. Since we are adjusting the log-returns for an assumed constant force of inflation $\bar{\imath}$, we use $\frac{1}{t_1}\left(\bar{Z} + \frac{1}{2}S^2\right) - \bar{\imath}$ as the statistic for $\bar{r}$. The resulting estimated parameter values for $\bar{r}$ and σ_R that we use in the geometric Brownian models are given in Table 2. The confidence intervals for σ_R and the standard errors for $\bar{r}$ are based on the fact that $\frac{l-1}{\sigma_R^2}S^2$ is χ^2_{l-1}-distributed and that the sample mean and sample variance of normal random variables are independent. The latter property leads to a standard error for $\bar{r}$ of $\sqrt{S^2\left(\frac{1}{l} + \frac{1}{2}\frac{S^2}{l-1}\right)}$. The standard error for the $\bar{r}$ parameter based on the daily S&P 500 data for 7/1962-12/2003 is the same as the standard error given in Ramezani and Zeng (2007) multiplied with 252. This last multiplication is due to the standardization of the time dimension. Our estimates for U.S. Treasury bills, U.S. bonds and American stocks are based on data for annual returns for 1928-2001 (83 observations for each asset class) from Damodaran (2012). The estimates for S&P 500 are annualized and inflation-adjusted versions of the parameter estimates given in Ramezani and Zeng (2007). These estimates are based on 10446 dividend-adjusted daily observations covering the period 7/1962-12/2003. The estimates for S&P 500 1/2000-11/2011 are based on 3000 observations. Our estimates for U.S. Treasury bills, U.S. bonds and American stocks for the period 2000-2011 are based on just 12 observations. To determine the force of inflation $\bar{\imath}$ we used historical data from inflationdata.com (2012).

An alternative parameterization is to let $\tilde{r} = \bar{r} - \frac{1}{2}\sigma_R^2$. For this parameter the natural statistic (for both method of moments and maximum likelihood) is $\bar{Z} - \bar{\imath}$, where $\bar{Z}$ is the sample mean of the log-returns. The fact that $\frac{\bar{Z} - \bar{\imath} - \tilde{r}}{\sqrt{\frac{S^2}{l}}}$ is t-distributed can be used to construct confidence intervals. Estimates for $\tilde{r}$ as well as 95% confidence intervals are given in the rightmost column in Table 2.

4.2 About the implementation and execution

4.2.1 Software and hardware

We implemented the algorithms described in Section 3 using R software. This was augmented by some Fortran subroutines. In particular the Net lib 'ORTHPOL' package 726 by W. Gautschi was used to calculate the Gaussian quadrature rules.

parameter	$\bar{r}$	σ_R	$\tilde{r}$
U.S. T-bills 1928-2011	0.00534 (0.00316)	0.02900 (0.02518, 0.03419)	0.00492 (−0.00138, 0.01121)
U.S. T-bonds 1928-2011	0.02259 (0.0078)	0.07131 (0.06192, 0.08409)	0.02005 (0.00457, 0.03552)
U.S. Stocks 1928-2011	0.07815 (0.02165)	0.19648 (0.17060, 0.23169)	0.05885 (0.01621, 0.10149)
S&P 500 7/1962-12/2003	0.08194 (0.0252)	0.15081 (0.14879, 0.15288)	0.07057 n.a.
U.S. T-bills 2000-2011	−0.00233 (0.0056)	0.01941 (0.01375, 0.03296)	−0.00252 (−0.01485, 0.00982)
U.S. T-bonds 2000-2011	0.04811 (0.0242)	0.08367 (0.05927, 0.14206)	0.04461 (−0.00855, 0.09777)
U.S. Stocks 2000-2011	0.00129 (0.06001)	0.20551 (0.14558, 0.34894)	−0.01982 . (−0.15040, 0.11075)
S&P 500 1/2000-11/2011	−0.00125 (0.00394)	0.21331 (0.20804, 0.21884)	−0.02789 (−0.03553, −0.02026)

Table 2: Parameter estimates for the geometric Brownian motion investment model with normally distributed inflation-adjusted log-returns. $\bar{r}$ is $r - \bar{\imath}$ (nominal return subtracted with the inflation force $\bar{\imath}$), while $\tilde{r}$ is defined as $\bar{r} - \frac{1}{2}\sigma_R^2$. The drift term for nominal log-returns (r) can be obtained by adding the corresponding inflation forces in Table 1. All the asset returns except the S&P 500 returns for 1/2000-11/2011 include dividends or coupons. 95% confidence intervals for σ_R and $\tilde{r}$, and standard errors for $\bar{r}$ are given in parentheses.

4.2.2 Grid sizes and tolerance values

In the implementations we let $\kappa = 2000$. We let the artificial boundary condition be at $z = \ln(1+\kappa)$. Recall that $\hat{\Delta}t$ is the length of the Crank-Nicolson time steps. The error bound (24) and our experiments suggest that letting $(\lambda + \lambda_R)\,\hat{\Delta}t$ be large may lead to poor convergence in the integral iteration described in section 3.4. Our experience shows that to avoid excessive iterations $\hat{\Delta}t$ should be less than $\frac{1}{\lambda+\lambda_R}$. In the case of exponentially distributed jumps we calculated the integral terms as described in section 3.1.1. In the examples where we illustrate the convergence rate, we use the same spatial grid sizes as in Halluin et al. (2005). Unless denoted otherwise the space grid has $m = 2^K$ for $K \in 7, \ldots, 12$, for the results in the tables. As explained in Section 3.3, in the time domain the first four time-steps have length $\frac{1}{24m}$, while the rest of the steps have length $\frac{1}{6m}$. For the results in the tables the size of the space grid is mostly $m = 4096$, or 2^{12}. For the examples with Pareto-distributed claim sizes, Gaussian quadrature rules of length $2K$ are calculated before the finite-differences method begins. In the example with the Merton model, a Gauss-Hermite rule of length 24 is applied. As convergence criterion for the fixed-point iteration method we required the max norm difference between two iterations to be less than 10^{-8}. In the tables containing the experimental results, the abbreviation 'C.R' refers to the convergence ratio, defined in equation (8.2) in Halluin et al. (2005) as

$$C.R = \left| \frac{\rho_{\text{approx}}(h/2) - \rho_{\text{approx}}(h))}{\rho_{\text{approx}}(h/4) - \rho_{\text{approx}}(h/2)} \right|.$$

Here h is a given step size.

4.3 Exponentially distributed jumps

In the case of exponentially distributed jumps we can calculate the integrals as described in Section 3.1.1. For the Cramér-Lundberg model with exponentially distributed claim sizes, the ruin probability was calculated using an integral formula given in Chapter IV in Asmussen (2000). We used this solution to check the accuracy of the method. Table 3 shows the relative errors of the calculated ruin probabilities using the method described in 3.1.1, with parameter vales $p = 216.7, \lambda = 197$, $\alpha = 1$ (standard exponential) and $t = 1$ (1 year). We also used the case with exponentially distributed jumps to check the accuracy of using Gaussian quadrature rules to evaluate the integrals. The results from this test suggest that the errors from the numerical integration are small in comparison to the errors from the numerical differentiation.

In order to avoid oscillations, α_i and β_i in (20) should be non-negative. For this to be satisfied the derivative terms in the space variable have to be discretized using forward differences. The drawback of forward differences is that it gives only first order convergence (i.e. consistency error $O(h)$) as opposed to the second order convergence (i.e. consistency error $O(h^2)$) we get with using the central differences as in (15). So although the convergence of Crank-Nicolson time integration itself is of second order, the overall convergence is only first order. For models where σ_R^2 is not very small we can use central differences on most of the domain without violating the conditions in Theorem 1 and Theo-

m		128	512	2048	4096
n		105	408	1620	3237
y	Exact solution	Relative error	Relative error	Relative error	Relative error
0	0.90080	0.00018	0.00044	0.00014	0.00007
7	0.43782	0.07168	0.01502	0.00354	0.00175
15	0.18097	0.30361	0.06881	0.01667	0.00829
31	0.02474	1.67517	0.32926	0.07630	0.03766
63	0.00017	35.45299	2.40341	0.39177	0.18208

Table 3: Relative errors for the Cramér-Lundberg model with various values of m. The parameter values are $p = 216.7$, $\lambda = 197$, and the claim sizes are standard exponentially distributed. The units for y and p are the expected value of a claim. λ corresponds to the expected number of claims per year. Hence $y = 63$ corresponds to a starting capital equal to 32% of the expected annual cost of claims.

rem 2. Thus for these models it is important to have an accurate method of integration in order to utilize the accuracy of the differential terms.

For Treasury bonds the variance in the log-returns is not large enough to allow use of central differences at more than a small minority of the grid points. There the convergence rate seems to be very similar to what is was with the Cramér-Lundberg model. For the log-return on stocks the volatility is higher, and for the grids with space grid size $m > 1000$ central differences can be used at a majority of the grid points. As seen in Table 4 this gives improved convergence. Since convergence is slower when the diffusion term (in effect ellipticity) is small, we used an even finer space grid ($m = 8192$) for the Treasury bond data.

Regarding ruin probabilities for models fitted to long term trends, we see in Table 5 that in the case of U.S. Treasury bills the difference between the classical Cramer-Lundberg model and the investment model fitted to annual returns is very small. For the investment model fitted to annual returns of U.S. Treasury bonds and the model fitted to daily returns of the SP 500 index, the ruin probabilities are slightly lower. For the model fitted to annual returns of American stocks the ruin probabilities are slightly higher than in the classical model, especially for the highest intial capital ($y = 63$). Lastly, we note that the increases in ruin probabilities flatten out after 5 years.

For years after 2000 the results are very different. In particular, at the highest intial capital ($y = 63$) the ruin probabilities for models with stocks are $2 - 3$ times higher than for models with bonds or with the classical model. Again the increases in ruin probability flatten out after 5 years.

4.4 Heavy Tail Models

For regularly varying claim size distributions (defined below) we have the following asymptotic result, based on Theorem 2 and Example 1 in Hult and Lindskog

m	128	512	2048	4096
n	105	408	1620	3237
	y=31			
Value	0.06414	0.03222	0.02488	0.02444
C.R	n.a.	2.44	1.36	7.06
	y=45			
Value	0.02261	0.00631	0.00367	0.00359
C.R	n.a.	3.07	2.44	9.63
	y=63			
Value	0.00628	0.00071	0.00026	0.00025
C.R	n.a.	4.54	6.29	8.86

Table 4: For standard exponentially distributed claim sizes, experimental results showing convergence as the number of grid points increases. The model includes a return on investment process, which is a geometric Brownian motion process. The parameter values are $\sigma_R = 0.19648$, $p = 216.7$ (premium rate corresponding to a safety loading of 0.1), $\bar{r} = 0.07815$ (real rate of interest) and $\lambda = 197$. The data used is return on American stocks for the period 1928-2011.

Model	Cramér-Lundberg	Geometric Brownian Motion			
Data Period	n.a. n.a.	U.S. T-bills 1928-2011 (annual)	U.S. T-bonds 1928-2011 (annual)	U.S. Stocks 1928-2011 (annual)	S&P 500 7/1962-12/2003 (daily)
$T = 1$					
$y = 31$	0.02468	0.02503	0.02458	0.02444	0.02312
$y = 45$	0.00330	0.00342	0.00331	0.00359	0.00313
$y = 63$	0.00017	0.00018	0.00017	0.00025	0.00018
$T = 2$					
$y = 31$	0.04188	0.04228	0.04110	0.04034	0.03765
$y = 45$	0.00897	0.00920	0.00874	0.00918	0.00787
$y = 63$	0.00103	0.00110	0.00101	0.00132	0.00095
$T = 5$					
$y = 31$	0.05294	0.05330	0.05116	0.04989	0.04562
$y = 45$	0.01437	0.01465	0.01359	0.01407	0.01164
$y = 63$	0.00260	0.00273	0.00241	0.00301	0.00206
$T = 10$					
$y = 31$	0.05423	0.05458	0.05220	0.05094	0.04627
$y = 45$	0.01516	0.01544	0.01422	0.01475	0.01202
$y = 63$	0.00294	0.00307	0.00267	0.00336	0.00222

Table 5: Ruin probabilities for the Cramér-Lundberg model and four fitted models, with investment following geometric Brownian motion (GBM) fitted to long term trends. The premium rate p is 216.7, $\lambda = 197$, and claim sizes are assumed to be standard exponentially distributed. Again, y = 63 corresponds to an initial capital equal to 32% of the expected annual claim cost. Note that the increases in ruin probability flatten out after T = 5 years.

Data Period	U.S. T-bills 2000-2011 (annual)	U.S. T-bonds 2000-2011 (annual)	U.S. Stocks 2000-2011 (annual)	S&P 500 1/2000-11/2011 (daily)
		$T = 1$		
$y = 31$	0.02534	0.02345	0.02830	0.02871
$y = 45$	0.00349	0.00306	0.00463	0.00478
$y = 63$	0.00019	0.00016	0.00038	0.00041
		$T = 2$		
$y = 31$	0.04300	0.03864	0.04898	0.04984
$y = 45$	0.00946	0.00790	0.01278	0.01323
$y = 63$	0.00115	0.00087	0.00225	0.00241
		$T = 5$		
$y = 31$	0.05453	0.04723	0.06490	0.06644
$y = 45$	0.01523	0.01188	0.02216	0.02316
$y = 63$	0.00291	0.00195	0.00628	0.00679
		$T = 10$		
$y = 31$	0.05595	0.04796	0.06832	0.07023
$y = 45$	0.01613	0.01230	0.02476	0.02608
$y = 63$	0.00330	0.00212	0.00793	0.00869

Table 6: Ruin probabilities for four fitted models, with investment following geometric Brownian motion (GBM) fitted to data for 2000-2011. The premium rate p is 216.7, $\lambda = 197$, and claim sizes are assumed to be standard exponentially distributed. Note that the increases in ruin probability flatten out after T = 5 years.

(2011).

Definition 1. *A function* $L(x)$ *is said to be* slowly varying *if*

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1, \quad \text{for all } c > 0.$$

A positive random variable S *and its distribution are said to be regularly varying with (tail) index* $\hat{\alpha}$ *if for some* $\hat{\alpha} \geq 0$*, the right tail of the distribution has the representation*

$$P(S > x) = L(x)x^{-\hat{\alpha}},$$

where L *is a slowly varying function.*

Theorem 3. *Assume that the claim size distribution function* $F(x)$ *is regularly varying with index* α*.*

(a) *In the case of the Cramér-Lundberg model the probability of ruin before time* t *is asymptotically given by*

$$\psi(y,t) \sim \lambda t \bar{F}(y),$$

where $\bar{F}(x) = 1 - F(x)$ *is the tail distribution.*

(b) *Consider a risk process of the form given in* (2)*, with investment. Let*

$$\theta = \frac{1}{2}\sigma_R^2\alpha^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(E(1+S_R)^{-\alpha} - 1\right)$$

and make the following additional assumptions:

(i) *Either* $\lambda_R = 0$ *or, for some* $\delta > 0$*,* $E(1+S_R)^{-(\alpha+\delta)} < \infty$ *.*

(ii)

$$\theta \neq 0.$$

Then the probability of ruin before time t *is asymptotically given by*

$$\psi(y,t) \sim \frac{1}{\theta}\left(e^{\theta t} - 1\right)\lambda\bar{F}(y). \tag{25}$$

Proof. We first consider the case when $T = 1$. As discussed in Section 1 the inflation-adjusted risk process Y at time t is given as

$$Y_t = \bar{R}_t\left(y + \int_0^t \bar{R}_s^{-1} dP_s\right), \tag{26}$$

where $\bar{R}_t = \exp\left\{\left(\bar{r} - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}\right\}\Pi_{n=1}^{N_{R,t}}(1 + S_{R,n})$. At $t = 1$ the Lévy process P_t, defined in (1) as a jump diffusion process with negative jumps $-\sum_{i=1}^{N_t} S_i$, has Lévy measure $\nu(-\infty, -u) = \lambda\bar{F}(u)$. Consequently Theorem 4.1 in Hult and Lindskog (2011) can be applied, with P playing the role of their Lévy process Y and the process $\left\{\bar{R}_t^{-1}\right\}_{t\geq 0}$ playing the role of their cáglad strictly positive process A. With these adaptations it follows from Theorem 4.1 and Example 3.5 in Hult and Lindskog (2011) that the stated results are valid for $T = 1$.

Assume that $T \neq 1$. Since P and R are Lévy processes, changing the parameters from $p, \sigma_P, \lambda, \bar{r}, \sigma_R, \lambda_R$ to $pt, \sigma_P\sqrt{t}, \lambda t, \bar{r}t, \sigma_R\sqrt{t}$ and $\lambda_R t$ corresponds to changing the time horizon from time T to time 1, giving the stated results when $T \neq 1$. □

It is only for very large claim sizes that the Pareto distribution is intended to be a good model, as discussed in Embrechts et al. (1997). This is formalized by introducing a threshold u, which in the discussion in Chapter 6 on Danish fire claims in Embrechts et al. (1997) is put to 10. The claim sizes which are larger than u are called *exceedances*. The Pareto distribution is fitted using only this part of the data. 109 of the 2167 claims in the data set are such exceedances.

In this section we let the distribution of the claim sizes be a weighted average of a standard exponential distribution and a standard Pareto distribution. The weight assigned to the Pareto distribution corresponds to the share of claims that are exceedances, $\frac{109}{2167}$.

The standardized Pareto distribution has a density given by

$$f_2(x) = (\hat{\alpha} - 1)^{\hat{\alpha}} \hat{\alpha} (\hat{\alpha} - 1 + x)^{-(1+\hat{\alpha})}, \; x > 0, \hat{\alpha} > 1, \tag{27}$$

and tail distribution function

$$\bar{F}_2(x) = \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + x} \right)^{\hat{\alpha}}.$$

This form of Pareto distribution is called 'standardized Pareto', since the expectation value is 1 for every $\hat{\alpha}$. This distribution is regularly varying with index $\hat{\alpha}$. For $\lambda_R = 0$ asymptotic formulas for the ruin probability are given below. Asymptotically the tail of the mixed distribution we use is dominated by the Pareto part, and thus is also regularly varying.

Corollary 1. *Assume that the claim sizes follow a mix of a Pareto distribution with density, as in* (27)*, and a light-tailed distribution. Assume that a weight* $0 < w \leq 1$ *is assigned to the Pareto distribution and a weight* $1 - w$ *is assigned to the light-tailed distribution, i.e. a distribution whose moment-generating function exists in a neighborhood around zero.*

(i) *Consider the Cramér-Lundberg model, with claim sizes following a mixed distribution as described above. In this model the ruin probability is asymptotically given by*

$$\psi(y, t) \sim \lambda w t \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + y} \right)^{\hat{\alpha}}.$$

(ii) *Consider a risk process with investment of the form given in* (2)*, with claim sizes following a mixed distribution as described above, and with* $\lambda_R = 0$*.*

$$\hat{\theta} = \frac{1}{2} \hat{\alpha}^2 \sigma_R^2 - \hat{\alpha} \left(\bar{r} - \frac{1}{2} \sigma_R^2 \right) \neq 0.$$

Then the ruin probability is asymptotically given by

$$\psi(y, t) \approx \frac{\lambda w}{\hat{\theta}} \left(e^{\hat{\theta} t} - 1 \right) \left(\frac{\hat{\alpha} - 1}{\hat{\alpha} - 1 + y} \right)^{\hat{\alpha}}. \tag{28}$$

m	128	512	2048	4196
n	105	408	1620	3237
	y=31			
Value	0.0755	0.0436	0.0376	0.0367
C.R	n.a.	2.42	2.15	2.08
	y=45			
Value	0.0298	0.0124	0.0100	0.0097
C.R	n.a.	2.98	2.32	2.17
	y=63			
Value	0.0101	0.0036	0.0031	0.0030
C.R	n.a.	4.09	2.61	2.30

Table 7: Convergence for the ruin probability in the Cramér-Lundberg model with $p = 216.7$ (corresponding to a safety loading of 0.1) and $\lambda = 197$. The claim distribution is a mixture of a standard exponential distribution and a standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$. The weight assigned to the Pareto distribution is $\frac{109}{2167}$.
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As expected, the numerical results show that ruin probabilities based on a heavy-tailed claim size distribution model are larger than when based on a light-tailed claim size distribution. The most striking differences between the results for the heavy-tail models, given in Table 8 compared with the results for light-tailed models, given in Table 5 are found for $T = 1$ (one year). With initial capital 63 (corresponding to 32 % of the expected annual claim cost) and assuming the Cramér-Lundberg model, the ruin probability in the heavy-tail case is 17.6 times larger than in the light-tail case.

For the stock models fitted to long-term trends the ruin probabilities are about the same or even lower than for the Cramér-Lundberg models, which do not include investment risk. On the other hand, when fitted to the period 2000-2011, given in Table 9, the ruin probabilities for the stock models start to grow quickly with increasing T, in particular for $T > 2$. Especially for large initial capitals and $T \in \{5, 10\}$, the ruin probabilities for the stock models are almost twice as high as the ruin probabilities with the Cramér-Lundberg model, given in Table 8.

For the case of regularly varying claim sizes and large values of the initial capital the formulas of Theorem 3 suggest the following: If $\theta > 0$ the ruin probabilities are higher than in the Cramér-Lundberg model. If $\theta < 0$ the ruin probabilities are lower than in the Cramér-Lundberg model. More precisely, when $\theta T << 0$ the ruin probability on the time horizon T is close to $-\frac{1}{\theta}\lambda\bar{F}(y)$. When $\theta T \approx 0$ the ruin probability grows approximately linearly as a function of T for fixed (high) initial capitals. When $\theta T >> 0$ the ruin probability grows exponentially with T. Our numerical experiments support this assertion. Moreover, as can be seen from Table 8 and Table 9, the asymptotic formula for the ruin probability seems to be quite accurate, at least as long as the ruin probability from that formula is less than 0.002.

Model	Cramér-Lundberg	Geometric Brownian Motion			
Data	n.a.	U.S. T-bills	U.S. T-bonds	U.S. Stocks	S&P 500
Period	n.a.	1928-2011 (annual)	1928-2011 (annual)	1928-2011 (annual)	7/1962-12/2003 (daily)
θ	0	-0.0082	-0.03004	-0.04026	-0.09595
			$T = 1$		
$y = 31$ Asym.	0.0367 (0.0095)	0.0393 (0.0095)	0.0359 (0.0094)	0.0351 (0.0093)	0.0337 (0.0091)
$y = 45$ Asym.	0.0097 (0.0046)	0.0107 (0.0046)	0.0095 (0.0045)	0.0095 (0.0045)	0.0089 (0.0044)
$y = 63$ Asym.	0.0030 (0.0024)	0.0032 (0.0023)	0.0030 (0.0023)	0.0030 (0.0023)	0.0028 (0.0022)
$y = 100$ Asym.	0.0010 (0.0009)	0.0010 (0.0009)	0.0010 (0.0009)	0.0010 (0.0009)	0.0009 (0.0009)
			$T = 2$		
$y = 31$ Asym.	0.0592 (0.0190)	0.0587 (0.0188)	0.0575 (0.0184)	0.0554 (0.0183)	0.0526 (0.0173)
$y = 45$ Asym.	0.0198 (0.0092)	0.0196 (0.0091)	0.0191 (0.0089)	0.0187 (0.0088)	0.0171 (0.0083)
$y = 63$ Asym.	0.0064 (0.0047)	0.0064 (0.0047)	0.0062 (0.0046)	0.0063 (0.0045)	0.0057 (0.0043)
$y = 100$ Asym.	0.0018 (0.0019)	0.0018 (0.0019)	0.0018 (0.0018)	0.0018 (0.0018)	0.0017 (0.0017)
			$T = 5$		
$y = 31$ Asym.	0.0767 (0.0047)	0.0758 (0.0465)	0.0734 (0.0441)	0.0699 (0.0430)	0.0653 (0.0377)
$y = 45$ Asym.	0.0308 (0.0229)	0.0303 (0.0224)	0.0290 (0.0213)	0.0278 (0.0207)	0.0248 (0.0182)
$y = 63$ Asym.	0.0120 (0.0118)	0.0117 (0.0115)	0.0111 (0.0109)	0.0110 (0.0107)	0.0094 (0.0094)
$y = 100$ Asym.	0.0036 (0.0047)	0.0036 (0.0046)	0.0034 (0.0044)	0.0034 (0.0043)	0.0029 (0.0037)
			$T = 10$		
$y = 31$ Asym.	0.0809 (0.0950)	0.0799 (0.0912)	0.0769 (0.0821)	0.0729 (0.0782)	0.0676 (0.0611)
$y = 45$ Asym.	0.0342 (0.0458)	0.0335 (0.0440)	0.0318 (0.0395)	0.0302 (0.0377)	0.0266 (0.0294)
$y = 63$ Asym.	0.0144 (0.0236)	0.0141 (0.0226)	0.0131 (0.0204)	0.0127 (0.0194)	0.0107 (0.0152)
$y = 100$ Asym.	0.0049 (0.0094)	0.0048 (0.0090)	0.0044 (0.0081)	0.0043 (0.0077)	0.0036 (0.0061)

Table 8: Calculated and asymptotic ruin probabilities for the classical Cramér-Lundberg model, plus four models with investment following geometric Brownian motion (GBM) fitted to long-term historical trends. The premium rate is $p = 216.7$, and the expected number of claims per year is $\lambda = 187$. This corresponds to a safety load of 0.1. $y = 63$ corresponds to an initial capital equal of 32% of the expected claim cost per year. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$, as used in Embrechts et al. (1997), and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. Asymptotic ruin probabilities are given in parentheses.

Data Period	U.S. T-bills 2000-2011 (annual)	U.S. T-bonds 2000-2011 (annual)	U.S. Stocks 2000-2011 (annual)	S&P 500 1/2000-11/2011 (daily)
θ	0.00583	-0.07559	0.12539	0.14040
$T = 1$				
$y = 31$ Asym.	0.0368 (0.0095)	0.0348 (0.0092)	0.0394 (0.0101)	0.0398 (0.0102)
$y = 45$ Asym.	0.0098 (0.0046)	0.0092 (0.0044)	0.0110 (0.0049)	0.0112 (0.0049)
$y = 63$ Asym.	0.0030 (0.0024)	0.0029 (0.0023)	0.0034 (0.0025)	0.0035 (0.0025)
$y = 100$ Asym.	0.0010 (0.0009)	0.0009 (0.0009)	0.0011 (0.0010)	0.0011 (0.0010)
$T = 2$				
$y = 31$ Asym.	0.0595 (0.0191)	0.0549 (0.0176)	0.0649 (0.0216)	0.0658 (0.0219)
$y = 45$ Asym.	0.0199 (0.0092)	0.0179 (0.0085)	0.0234 (0.0104)	0.0239 (0.0106)
$y = 63$ Asym.	0.0065 (0.0047)	0.0058 (0.0044)	0.0080 (0.0054)	0.0083 (0.0054)
$y = 100$ Asym.	0.0019 (0.0019)	0.0017 (0.0017)	0.0022 (0.0021)	0.0022 (0.0022)
$T = 5$				
$y = 31$ Asym.	0.0773 (0.0482)	0.0689 (0.0396)	0.0873 (0.0660)	0.0889 (0.0688)
$y = 45$ Asym.	0.0312 (0.0232)	0.0263 (0.0191)	0.0386 (0.0318)	0.0398 (0.0332)
$y = 63$ Asym.	0.0121 (0.0120)	0.0098 (0.0098)	0.0167 (0.0164)	0.0174 (0.0171)
$y = 100$ Asym.	0.0037 (0.0048)	0.0030 (0.0039)	0.0052 (0.0065)	0.0055 (0.0068)
$T = 10$				
$y = 31$ Asym.	0.0817 (0.0978)	0.0716 (0.0667)	0.0945 (0.1895)	0.0967 (0.2075)
$y = 45$ Asym.	0.0347 (0.0471)	0.0284 (0.0321)	0.0449 (0.0913)	0.0466 (0.1000)
$y = 63$ Asym.	0.0147 (0.0243)	0.0113 (0.0165)	0.0215 (0.0470)	0.0226 (0.0515)
$y = 100$ Asym.	0.0050 (0.0097)	0.0038 (0.0066)	0.0081 (0.0188)	0.0086 (0.0206)

Table 9: Calculated and asymptotic ruin probabilities for four models, with investment following geometric Brownian motion (GBM) fitted to data from 2000-2011. The premium rate $p = 216.7$ corresponds to a safety loading of 0.1. The expected number of claims per year is $\lambda = 187$. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$ (used in Embrechts et al. (1997)) and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. Asymptotic ruin probabilities are given in parentheses.

4.5 Jumps in the investment process

Ramezani and Zeng (2007) discuss maximum likelihood estimation for the Merton model and the Kou model, as well as the geometric Brownian model (GBM). In particular they fitted these models to the S&P for daily observations for the period 1962-2003. In the previous section we used estimates from Ramezani and Zeng (2007) for the GBM model.

In both the Merton model and the Kou model the returns follow a jump-diffusion process. The Merton model was introduced in Merton (1976) and consists of letting the jumps of the returns follow a log-normal distribution. The Kou model was first discussed in Ramezani and Zeng (1998) and in Kou (2002). In this model the jumps of the log-returns follow a double exponential distribution. The Merton and Kou models are discussed in more detail in Ramezani and Zeng (2007) and in Chapter 4 in Cont and Tankov (2004). Annualized and inflation-adjusted versions of the parameter estimates for the Merton model fitted to the S&P 500 data are in Table 10, where μ and σ are the parameters of the log-normal distribution. For the Merton model the θ defined in Theorem 3 is given by

$$\theta = \frac{1}{2}\alpha^2\sigma_R^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(\exp\left(-\alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right) - 1\right).$$

Similarly, annualized and inflation-adjusted versions of the parameter estimates from Ramezani and Zeng (2007) for the Kou model are given in Table 11. In Ramezani and Zeng (2007) it is assumed that the arrival times of "good" and "bad" news (leading to positive vs. negative jumps) follow two different independent Poisson processes. These have intensities λ_u and λ_d, respectively. In Table 11 the parameter estimate for λ_R corresponds to the annualized value of the sum of their estimates for λ_u and λ_d, while the parameter q refers to the fraction $\frac{\lambda_u}{\lambda_u+\lambda_d}$. η_1 is the parameter of the exponential distribution of the positive jumps in the log-returns process. η_2 is the parameter of the exponential distribution of the size of the negative jumps in the log-returns process. If $\eta_2 > \alpha$ then θ for the Kou model is given by

$$\theta = \frac{1}{2}\alpha^2\sigma_R^2 - \alpha\left(\bar{r} - \frac{1}{2}\sigma_R^2\right) + \lambda_R\left(\frac{q}{1+\frac{\alpha}{\eta_1}} + \frac{1-q}{1-\frac{\alpha}{\eta_2}} - 1\right).$$

With the parameter estimates from Ramezani and Zeng (2007) we get

$$\theta \approx -0.11.$$

Again this suggests a smaller ruin probability than with the Cramér-Lundberg model.

With these parameter estimates, values for the ruin probability in the Merton and Kou models are shown in Table 12. The table shows that, for the S&P 1962-2003 data, ruin probabilities do not differ much between models. This is not the case if we consider investment in a single stock and use parameter values from Ramezani and Zeng (2007), but this is a highly implausible strategy.

$\bar{r}$	σ_R	λ_R	μ	σ	θ
0.05674	0.127	18.5724	0.0005	0.0199	-0.06909
(0.0252)	(0.0252)	(2.6712)	(0.0009)	(0.0005)	n.a.

Table 10: Parameter estimates from Ramezani and Zeng (2007) for the Merton model fitted to daily returns of the S&P 500 1962-2003. Annualized standard errors are in parentheses. θ is defined in Theorem 3.

$\bar{r}$	σ_R	λ_R	q	η_1	η_2	θ
0.15754	0.07302	237.8376	0.4521	173.91	185.98	-0.1054
n.a.	n.a.	(41.1516)	n.a.	(0.36)	(0.38)	n.a.

Table 11: Parameter estimates from Ramezani and Zeng (2007) for the Kou model fitted to daily returns of the S&P 500 1962-2003. Annualized standard errors are in parentheses. θ is defined in Theorem 3.

Model	$T = 1$	$T = 2$	$T = 5$	$T = 10$
CL	0.0010	0.0018	0.0036	0.0049
GBM	0.0009	0.0017	0.0029	0.0036
Merton	0.0009	0.0017	0.0031	0.0036
Kou	0.0009	0.0017	0.0029	0.0039

Table 12: Ruin probabilities for the classical Cramér-Lundberg (CL) model and three investment models fitted to daily observations of S&P 500 7/1962-2003. The claim sizes follow a mixture of the standardized Pareto distribution with parameter $\hat{\alpha} = 2.01$ (used in Embrechts et al. (1997)) and a standard exponential distribution. The weight assigned to the Pareto distribution is $\frac{109}{2167}$. The initial capital is set to $y = 100$.

5 Conclusions

The numerical experiments suggest that using precalculated Gaussian quadrature rules is an efficient way of calculating the integrals in the numerical solution of the PIDE (8). The dominant source of error seems to be the differential terms rather than the numerical integration. From a numerical point of view the main problem is the following: For the finite-difference method to be numerically stable the differential terms often have to be approximated using one-sided finite differences rather than central differences. This gives a slower convergence, in particular for moderately sized initial capitals and models where the diffusion term is small.

As for the ruin probabilities for the fitted models, they can vary substantially depending on which time period is selected for the data. The ruin probabilities are also quite sensitive to the value of the θ defined in Theorem 3, and in particular the sign of θ is critical. If $\theta > 0$ (Table 9) the effect of investments on the ruin probabilities is moderate on short time horizons, but more pronounced for longer time horizons. Our numeric results suggest that the asymptotic result in Theorem 3, based on Theorem 4.1 in Hult and Lindskog (2011), is rather accurate for short time horizons. For long time horizons, in particular for $T > 5$) the initial capital needs to very high for the formula to be a good approximation.

A possible topic for future research is to estimate inflation from the claim sizes themselves. This might require new numerical methods. It might also be interesting to approximate the small claims with a diffusion process. This would increase the ellipticity and thus allow more nodes to be approximated with central differences. Since the computation time is proportional to $(\lambda + \lambda_R)T$ the ruin probability in such a model would be easier to compute efficiently.

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